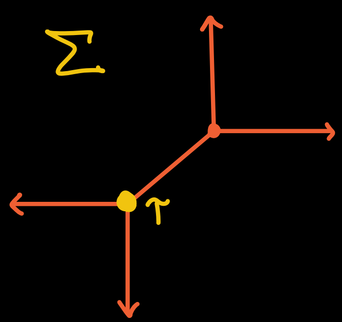
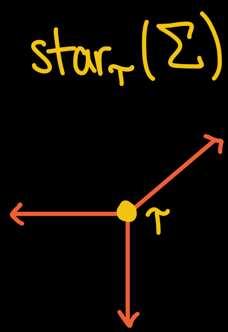
Tropical Geometry: The structure theorem

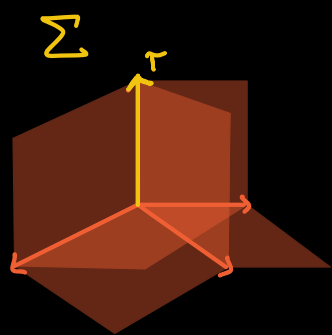
Let Σ be a polyhedral complex. The <u>star</u> of $\tau \in \Sigma$ is a polyhedral fan with cones defined by $\sigma \in \Sigma$ with $\tau \subseteq \sigma$:

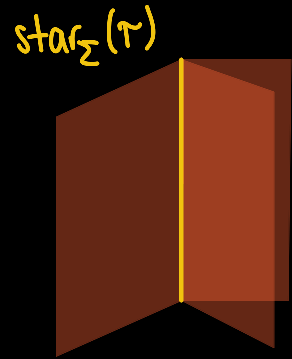
cone corres. to σ: $\{\lambda(x-y) : \lambda \geq 0,\ x \in \sigma,\ y \in \tau\}$

$= \{v \in \mathbb{R}^n : w + \varepsilon v \in \sigma \text{ for some } \varepsilon > 0, \text{ where } w \in \tau\}$









<u>Prop</u>: Let $I \subseteq K[x_1, \ldots, x_n]$ and let Σ be the polyhedral complex (obtained from the Gröbner complex of I^{hom}) with support $V(\text{trop}(I))$. Then for $\tau \in \Sigma$ and $w \in \text{relint}(\tau)$,

$$\text{star}_\Sigma(\tau) = V(\text{trop}(\text{in}_w I))$$

$$= \{v \in \mathbb{R}^n : \text{in}_v(\text{in}_w I)) \text{ does not contain a monomial}\}$$

Ex: $I = \langle t + x + y + xy \rangle$

$\tau = \{(0,0)\}$ $\quad w = (0,0)$

$\text{in}_w(I) = \langle x + y + xy \rangle$

pics above!

Ex: $I = \langle 1 + x + y + z \rangle$

$\tau = \mathbb{R}_{\geq 0}((0,0,1))$ $\quad w = (0,0,1)$

$\text{in}_w(I) = \langle 1 + x + y \rangle$

K = an alg. closed field with a nontrivial valuation

Structure Thm: Let X be an irred. variety of dim d. Then $\text{trop}(X)$ is the support of a balanced, weighted Γ_{val}-rational polyhedral complex pure of dim d that is connected in codim 1.

"pure of dim d" = all maximal cones have dim d.

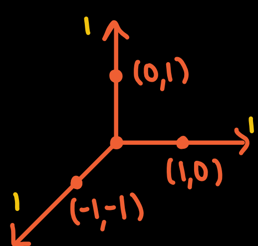
"weighted" = all maximal cones σ have weight $m(\sigma) \in \mathbb{Z}_+$

"balanced" = ?

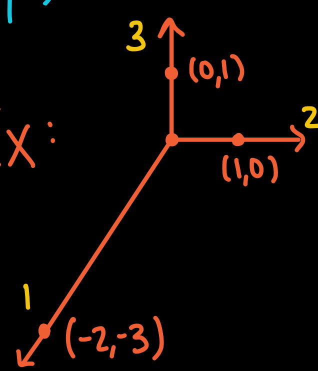
"connected in codim 1": graph on maximal cells σ with edges $\{\{\sigma_1, \sigma_2\}$ where σ_1, σ_2 share a $(d-1)$-face$\}$ is connected.

If Σ is a 1-dim'l rational polyhedral fan, each maximal cell is a ray $\mathbb{R}_{\geq 0} \cdot v_i$ with $v_i \in \mathbb{Z}^n$. (Can choose v_i to be the first int. pt. on ray.) Then Σ is <u>balanced</u> if $\sum_i m_i v_i = 0$.

Ex:



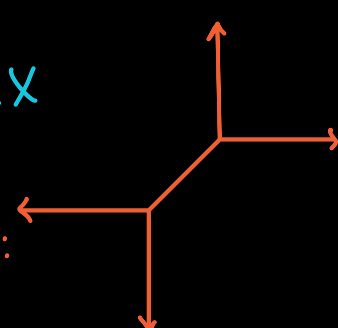
Ex:



Σ = 1-dim'l Γ_{val}-rational polyhedral complex

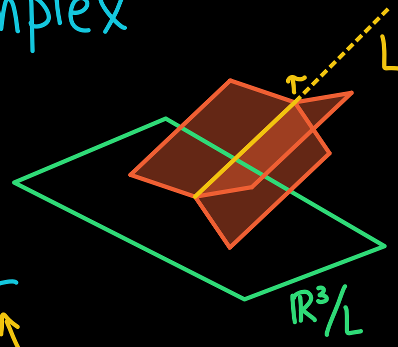
Σ is <u>balanced</u> if $\text{star}_\tau(\Sigma)$ is balanced for all 0-dim'l cells of Σ.

Ex:



Σ = d-dim'l Γ_{val}-rational polyhedral complex

Σ is <u>balanced</u> if, for every cell $\tau \in \Sigma$ of dim $d-1$, $\mathrm{star}_\tau(\Sigma)$ is balanced in $\mathbb{R}^n/L$ where the affine span of τ is $w+L$



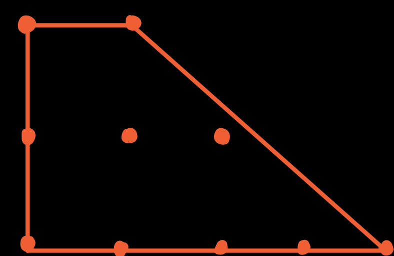
need to take some care with lattice pts!

Multiplicities $m(\sigma)$ of maximal cones σ?

Last time: For $w \in \mathrm{relint}(\sigma)$, $V(\mathrm{in}_w I)$ is the orbit of finitely many pts $a \in (\mathbb{k}^*)^n$ under $(\mathbb{C}^*)^d$.

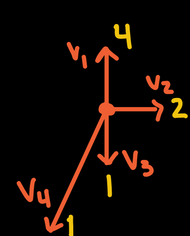
Define $m(\sigma)$ to be the number of these pts (counting multiplicities given by $\mathrm{in}_w I$).

Ex: $f = 4 - 5x^2 + x^4 - 4y + y^2 - xy^2$



$V(\mathrm{trop}(f)) = \mathbb{R}_{\geq 0} v_1 \cup \mathbb{R}_{\geq 0} v_2 \cup \mathbb{R}_{\geq 0} v_3 \cup \mathbb{R}_{\geq 0} v_4$

ray $\mathbb{R}_{\geq 0} w$	$\mathrm{in}_w(f)$	$V(\mathrm{in}_w f) \cap (\mathbb{C}^*)^2$	
$v_1 = (0,1)$	$4 - 5x^2 + x^4$	$t^{(0,1)} \cdot \{(\pm 2, 1), (\pm 1, 1)\}$	$m_1 = 4$
$v_2 = (1,0)$	$4 - 4y + y^2$	$t^{(1,0)} \cdot (1,2)$	$m_2 = 2$
$v_3 = (0,-1)$	$y^2(1-x)$	$t^{(0,-1)}(1,1)$	$m_3 = 1$
$v_4 = (-2,-3)$	$x(x^3 - y^2)$	$t^{(-2,-3)}(1,1)$	$m_4 = 1$



$\sum m_i v_i = 0$

balanced!

NEXT TIME: Orbits/homogenity

Let $v \in \mathbb{R}^n$. A polynomial $f = \sum_{\alpha} c_\alpha x^\alpha \in \Bbbk[x_1, \dots, x_n]$ (trivial valuation!) is homog. w.r.t. v of v-degree d if $v^T\alpha = d$ $\forall \alpha$ with $c_\alpha \neq 0$.

An ideal $J \subseteq \Bbbk[x_1,\dots,x_n]$ is homog. w.r.t. v if $J = \langle f_1, \dots, f_r \rangle$ where each f_i is homog. w.r.t. v.

Prop: If $J \subseteq \Bbbk[x_1,\dots,x_n]$ and $\mathrm{in}_v J = J$ for some $v \in \mathbb{R}^n$ then J is homog. w.r.t. v. If $a \in V(J)$ then for any $t \in \Bbbk$, $t^v \cdot a = (t^{v_1} a_1, \dots, t^{v_n} a_n) \in V(J)$.

Last time: If J doesn't contain a monomial and J is homog. w.r.t. all $v \in L$ where L is a d-dim'l $\mathbb{Q}$-linear space then $\dim(V(J)) \geq d$. ← applied to $J = \mathrm{in}_w I$

Take a basis $v^{(1)}, \dots, v^{(d)} \in \mathbb{Z}^n$ for L and $a \in V(J) \cap (\Bbbk^*)^n$.
For $t_1, \dots, t_d \in \Bbbk^*$, $t_1^{v(1)} \cdots t_d^{v(k)} \cdot a$ belongs to $V(J)$.
$\Rightarrow \dim(V(I)) \geq d$.

Ex: $f = x^2 y - z^3$ homog w.r.t. $v \in L = \mathrm{span}\{\overset{v^{(1)}}{(1,1,1)}, \overset{v^{(2)}}{(0,3,1)}\}$
$a = (1,1,1) \in V(f) \Rightarrow t_1^{v^{(1)}} t_2^{v^{(2)}} \cdot a = (t_1, t_1 t_2^3, t_1 t_2) \in V(f)$.

Cor: If $w \in V(\mathrm{trop}(I))$ and $\mathrm{in}_v(\mathrm{in}_w I) = \mathrm{in}_w I$ for all v in a d-dim'l subspace L then $\dim V(\mathrm{in}_w I) \geq d$.