Tropical Geometry: The fundamental theorem

K = an alg. closed field with a nontrivial valuation

(e.g. $K = \mathbb{C}\{\{t\}\} = \bigcup_{n\in\mathbb{Z}_{\geq 1}} \mathbb{C}((t^{1/n}))$)

Fundamental Theorem: For an ideal $I \subseteq K[x_1,\dots,x_n]$, the following sets coincide:

(1) $V(\mathrm{trop}(I)) = \bigcap_{f\in I} V(\mathrm{trop}(f))$

(2) $\{w\in\mathbb{R}^n : \mathrm{in}_w(I)$ does not contain a monomial$\}$

(3) the Euclidean closure of $\mathrm{val}(V(I)) \subseteq \mathbb{R}^n$.

Furthermore, if $V(I)$ is irreducible and $w\in V(\mathrm{trop}(I))\cap(\Gamma_{\mathrm{val}})^n$, then $\{y\in V(I) : \mathrm{val}(y)=w\}$ is Zariski-dense in $V(I)$.

Cor: $V(\mathrm{trop}(I))$ depends only on $V(I)$.

For a variety $X = V(I)$, we can define

$$\mathrm{trop}(X) = V(\mathrm{trop}(I)).$$

Proof steps for $(2)\subseteq(3)$: (went through last time for $I=\langle f\rangle$)

(1) monomial change of coordinates to "good" position to project away last coordinate

(2) choose any $a\in V(\mathrm{in}_w I)\cap(\mathbb{k}^*)^n$ and, by induction, choose $y\in V(I\cap K[x_1,\dots,x_{n-1}])\cap(K^*)^{n-1}$ with $\mathrm{val}(y_i)=w_i$,

$t^{-w_i} y_i = a_i$ for $i=1,\dots,n-1$. Use $n=1$ case to find y_n

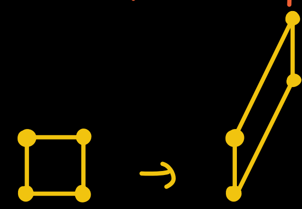
(3) Choice of arbitrary y and dimension count guarantees Zariski-denseness.

Ex: $f = t + 2x + 3y + 5txy \in \mathbb{C}\{\{t\}\}[x,y]$ $\qquad \phi(x,y) = (xy^3, y)$

$\tilde{f} = f(\phi(x,y)) = t + 2xy^3 + 3y + 5txy^4$

$w = (-1,-2) \in V(\mathrm{trop}(f)) \qquad in_w f = 3y + 5xy$

$\tilde{w} = (5,-2) \in V(\mathrm{trop}(\tilde{f})) \qquad in_{\tilde{w}} \tilde{f} = 3y + 5xy^4$



Newt(f) $\quad$ Newt(ϕ(f))

$(\alpha_1, \alpha_2) \mapsto (\alpha_1, \alpha_2 + 3\alpha_1)$

$a = (-\frac{3}{5}, 1)$ has $in_w(f)(a) = in_{\tilde{w}} \tilde{f}(a) = 0$

Choose any $x \in \mathbb{C}\{\{t\}\}$ with $\mathrm{val}(x) = 3$, $\overline{t^{-5}x} = -3/5$ $\quad (x = -\frac{3}{5}t^5 + \text{h.o.t.})$

Then $g(y) = \tilde{f}(x,y) \in \mathbb{C}\{\{t\}\}$ is nonzero

$$= 1 + (-\tfrac{6}{5}t^5 + \dots)y^3 + 3y + (-3t^6 + \dots)y^4$$

$in_{(-2)} g = 3y - 3y^4 = 3y(1 - y^3)$

$in_{(-2)} g(1) = 0 \Rightarrow \exists\, y \in \mathbb{C}\{\{t\}\}$ with $g(y) = 0$, $\mathrm{val}(y) = -2$, $\overline{t^2 y} = 1$

$\Rightarrow (x,y) = (-\frac{3}{5}t^5 + \dots, t^{-2} + \dots) \in V(\tilde{f})$

$\Rightarrow \phi(x,y) = (xy^3, y) = (-\frac{3}{5}t^{-1} + \dots, t^{-2} + \dots) \in V(f)$ has $\mathrm{val}(x,y) = (-1,-2) = w$

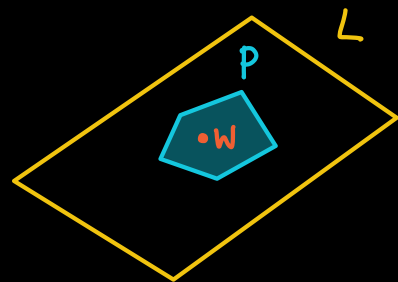
$\overline{t^{-w}(x,y)} = (-\frac{3}{5}, 1)$

Useful Lemma (3.2.10) Let X be a d-dim'l subvariety of $(K^*)^n$ defined by an ideal $I \subseteq K[x_1, \dots, x_n]$. Every cell of the Gröbner complex $\Sigma(I^{hom}) \cap \{w_0=0\}$ contained in $V(trop(I))$ has dim $\leq d$.

Proof will use:

Lemma (Cor 2.4.10) Let I be a homog. ideal. For any $w, v \in \mathbb{R}^n$, for all sufficiently small $\varepsilon > 0$, $in_{w+\varepsilon v}(I) = in_v(in_w I)$.

(Proof of 3.2.10) Let P be a maximal cell of $\Sigma(I^{hom}) \cap \{w_0=0\}$ and take w in the relative interior of P.



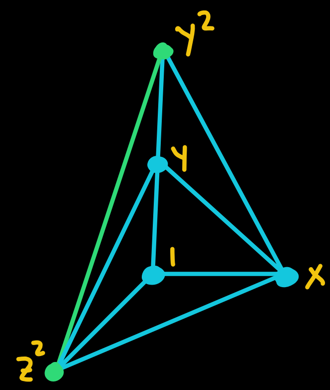
Then the affine span of P is $w+L$ where $L \subseteq \mathbb{R}^n$ is a subspace of $k = \dim(P)$.

Since $w \in relint(P)$, $in_w(I) = in_{w+\varepsilon v}(I) = in_v(in_w I)$ for all $v \in L$ and sufficiently small $\varepsilon > 0$. Choose a generating set G of $in_w I$ so that no elt' of G is a sum of other elt' of $in_w I$ with fewer monomials. $\Longrightarrow$ $in_v f = f$ $\forall$ $v \in L$ (otherwise $in_v(f) \in in_v(in_w I) = in_w(I)$, as is $f - in_v f$, both with fewer monomials). If $a \in V(in_w I) \cap (\mathbb{k}^*)^n$ then $t^v \cdot a = (t^{v_1} a_1, \dots, t^{v_n} a_n)$ also belongs to $V(in_w I) \cap (\mathbb{k}^*)^n$ for any $t \in \mathbb{k}^*$. Let $v^{(1)}, \dots, v^{(k)} \in \mathbb{Z}^n$ be a basis for L. Then for any $t_1, \dots, t_k \in \mathbb{k}^*$, $t_1^{v(1)} \cdots t_k^{v(k)} \cdot a$ belongs to $V(in_w(I)) \cap (\mathbb{k}^*)^n$ $\Longrightarrow$ $k \leq \dim(V(in_w I)) \leq \dim(X)$

$in_{(0,w)} I^{hom}$ and I^{hom} have the same Hilbert series

$\Rightarrow$ varieties in $\mathbb{P}^n(K)$ have the same dimension!

Ex: $I = \langle f \rangle$ $f = 1 + x + y + ty^2 + z$



$P = \{(w_x, w_y, w_z) \in \mathbb{R}^3 : 2w_y + 1 = w_z \leq 0, w_x, w_y\}$

$w = (0, -2, -3)$ $in_w f = y^2 + z$

$aff(P) = w + L$ with $L = Span_{\mathbb{R}}\{(1,0,0), (0,1,2)\}$

$V(in_w f) = V(y^2 + z)$ invariant under map

$(x,y,z) \mapsto t_1^{(1,0,0)} \cdot t_2^{(0,1,2)} (x,y,z) = (t_1 x, t_2 y, t_2^2 z)$ for $t_1, t_2 \in (\mathbb{C}^*)^n$

(In fact, $V(y^2+z) \cap (\mathbb{C}^*)^3$ is the orbit of a single pt $(x,y,z) = (1,1,-1)$ under this action of $(\mathbb{C}^*)^2$.)

Thm 3.3.8 Let X be an irreducible variety in $(K^*)^n$ of dim d. Then $trop(X)$ is the support of a pure d-dim'l Γ_{val}-rational polyhedral complex.