Tropical Geometry: The fundamental theorem for hypersurfaces $I = \langle f \rangle$

K = an alg. closed field with a nontrivial valuation (e.g. $K = \mathbb{C}\{\{t\}\} = \bigcup_{n \in \mathbb{Z}_{\geq 1}} \mathbb{C}((t^{1/n}))$)

Kapranov's Theorem: For $f = \sum_{\alpha \in A} c_\alpha x^\alpha \in K[x_1, \dots, x_n]$, the following sets coincide:

(1) $V(\operatorname{trop}(f)) = \{w : \bigoplus_{\alpha \in A} (\operatorname{val}(c_\alpha) \odot w^\alpha) \text{ is attained} \geq \text{twice}\}$

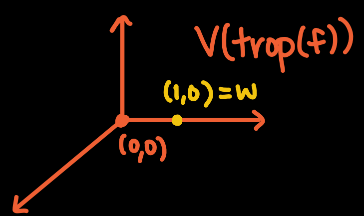
(2) $\{w \in \mathbb{R}^n : \operatorname{in}_w(f) \text{ is not a monomial}\}$

(3) the Euclidean closure of $\operatorname{val}(V(f)) \subseteq \mathbb{R}^n$.

Furthermore, if f is irreducible and $w \in V(\operatorname{trop}(f)) \cap (\Gamma_{val})^n$, then $\{y \in V(f) : \operatorname{val}(y) = w\}$ is Zariski-dense in $V(f)$.

Ex: $f = x_1 + x_2 - 1$

$\operatorname{trop}(f) = w_1 \oplus w_2 \oplus 0$



$\{(y_1, y_2) \in V(f) : \operatorname{val}(y) = w\}$
$= \{(y_1, 1 - y_1) : \operatorname{val}(y_1) = 1\}$

∞-many pts $\Rightarrow$ Zariski dense in $V(f)$.

Already discussed (1) = (2) and (3) $\subseteq$ (1). Need (2) $\subseteq$ (3)

Lemma 2.2.12: For $w \in \Gamma_{val}^n$ and $a \in (k^*)^n$, the set

$$Y = \{y \in (K^*)^n : \operatorname{val}(y_i) = w_i,\ \overline{t^{-w_i} y_i} = a_i \ \forall i = 1, \dots, n\}$$

is Zariski-dense in K^n. (i.e. only nonzero poly $h \in K[x_1, \dots, x_n]$ vanishing on this set is the zero polynomial)

(Proof by induction on n)

Let $h \in K[x_1,\dots,x_n]$ with $h \not\equiv 0$. Need to show $\exists y \in Y$ with $h(y) \neq 0$.

Let $\alpha \in (K^*)^n$ with $\bar{\alpha} = a$. Then $\alpha t^w = (\alpha_1 t^{w_1}, \dots, \alpha_n t^{w_n}) \in Y$.

Also, for any $v > w$, $\alpha t^w + t^v \in Y$.

(n=1) Set is infinite $\Rightarrow$ can choose $y \in Y$ s.t. $h(y) \neq 0$.

(n>1) Write $h = \sum_{j=0}^{d} h_j x_n^j$ with $h_j \in \underline{K[x_1,\dots,x_n]}$ with $h_d \not\equiv 0$. By induction, $\exists y \in (K^*)^{n-1}$ with $\mathrm{val}(y_i) = w_i$, $\overline{t^{-w_i} y_i} = a_i$ $\forall i = 1,\dots,n-1$ and $h_d(y) \neq 0$.

Then $h(y, x_n) \in \underline{K[x_n]}$ is not identically zero. By n=1 case, $\exists y_n \in K^*$ with $\mathrm{val}(y_n) = w_n$, $\overline{t^{-w_n} y_n} = a_n$ and $h(y, y_n) \neq 0$.

<u>Prop 3.1.5.</u> If $\mathrm{in}_w f(a) = 0$ for some $a \in (\Bbbk^*)^n$ and $\underline{w \in (\Gamma_{\mathrm{val}})^n}$, then $\exists y \in (K^*)^n$ with $f(y) = 0$, $\mathrm{val}(y) = w$, and $\overline{t^{-w} y} = a$.

If f is irred. then the set of such y is Zariski-dense in $V(f)$.

(Proof by induction on n) (n=1) $f = \sum_{i=0}^{d} c_i x^i \in K[x]$

K alg closed $\Rightarrow f = c_d \cdot \prod_{j=1}^{d} (x - r_j)$ for some $r_j \in K$.

Then $\mathrm{in}_w(f) = \overline{t^{-\mathrm{val}(c_d)} c_d} \prod_{j=1}^{d} \mathrm{in}_w(x - r_j)$ If $\mathrm{in}_w(f)(a) = 0$

then $\mathrm{in}_w(x - r_j)(a) = a - \overline{t^{-w} r_j} = 0$ for some j. Take $y = r_j$.

If f is irreducible, $f = c \cdot (x - r)$ $V(f) = \{r\} \subseteq K$.

If $\mathrm{in}_w(f)(a) = 0$ with $a \in \Bbbk^*$, then $w = \mathrm{val}(r)$, $a = \overline{t^{-w} r}$

and $\{y \in V(f) : \mathrm{val}(y) = w, \overline{t^{-w} y} = a\} = \{r\} = V(f)$.

$(n>1)$ (1) Reduce to case $\alpha_n \neq \beta_n$ for all $\alpha \neq \beta$ in A

Replace $f = \sum c_\alpha x^\alpha$ with $f(\phi(x))$ where $\phi(x) = (x_1 x_n^{u_1}, \dots, x_{n-1} x_n^{u_{n-1}}, x_n)$

with $u \in \mathbb{Z}^{n-1}$ "generic". Then $f(\phi(x)) = \sum c_\alpha x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n + \sum_{i=1}^{n-1} u_i \alpha_i}$.

Can choose u s.t. $\{\alpha_n + \sum_{i=1}^{n-1} u_i \alpha_i : \alpha \in A\}$ distinct. Eg. $u_i = \ell^i$ for $\ell \gg 0$.

Inverse (on $(K^*)^n$) : $\psi(x) = (x_1 x_n^{-u_1}, \dots, x_{n-1} x_n^{-u_{n-1}}, x_n)$

(2) Case: $\alpha_n \neq \beta_n$ for all $\alpha \neq \beta$ in A

Let $y \in (K^*)^{n-1}$ with $\mathrm{val}(y_i) = w_i$, $\overline{t^{-w_i} y_i} = a_i$ for $i = 1, \dots, n-1$.

Take $g(x_n) = f(y, x_n) \in K[x_n]$. By assumption on $\alpha_n \neq \beta_n$,

$g(x_n) = \sum_\alpha c_\alpha y_1^{\alpha_1} \cdots y_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n}$ is not the zero polynomial.

$$\text{Then } \mathrm{trop}(g)(w_n) = \bigoplus_{\alpha \in A} \left(\mathrm{val}(c_\alpha y_1^{\alpha_1} \cdots y_{n-1}^{\alpha_{n-1}}) \odot w_n^{\odot \alpha_n}\right)$$

$$= \bigoplus_{\alpha \in A} \left(\mathrm{val}(c_\alpha) \odot w^{\odot \alpha}\right) = \mathrm{trop}(f)(w)$$

$$\text{and } \mathrm{in}_{w_n}(g) = \sum_{\alpha : \alpha_n \in \operatorname{argmin}(\mathrm{trop}(g)\, w_n))} \overline{t^{-\mathrm{val}(c_\alpha y_1^{\alpha_1} \cdots y_{n-1}^{\alpha_{n-1}})} c_\alpha y_1^{\alpha_1} \cdots y_{n-1}^{\alpha_{n-1}}}\, x_n^{\alpha_n}$$

$$= \sum_{\alpha \in \operatorname{argmin}(\mathrm{trop}(f)(w))} \overline{t^{-\mathrm{val}(c_\alpha)} c_\alpha}\, a_1^{w_1} \cdots a_{n-1}^{w_{n-1}} x_n^{\alpha_n}$$

$$= \mathrm{in}_w f(a_1, \dots, a_{n-1}, x_n)$$

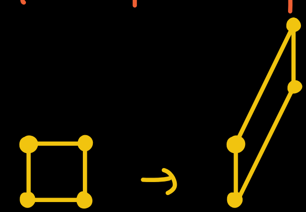
Since $\mathrm{in}_w f(a)=0$, $\mathrm{in}_{w_n} g(a_n)=0$. By $n=1$ case, $\exists\, y_n \in K$ with $g(y_n)=0$, $\mathrm{val}(y_n)=w_n$, and $\overline{t^{-w_n} y_n} = a_n$.

(Zariski-dense) Suppose f is irred. and consider $Y=\{y\in V(f) : \mathrm{val}(y)=w,\ \overline{t^{-w}y}=a\}$. By arguments above, then projection, $\pi(Y)$, of Y onto $(y_1,\dots,y_{n-1})$ is Z-dense in K^{n-1}. Thus $\dim(\overline{Y}^{Zar}) \geq \dim(\overline{\pi(Y)}^{Zar}) = n-1$. Since $V(f)$ is irred. of dim $n-1$, we see $\overline{Y}^{Zar} = V(f)$.

Alternatively: if $\exists\, g\in K[x_1,\dots,x_n]$ vanishing on Y and $g\notin\langle f\rangle$, then $\exists$ nonzero $h\in\langle f,g\rangle \cap K[x_1,\dots,x_n]$ also vanishing on $\pi(Y)=\{(y_1,\dots,y_{n-1})\in(K^*)^{n-1} : \mathrm{val}(y_i)=w_i,\ \overline{t^{-w_i}y_i}=a_i\}$, contradicting Lemma 2.2.12. $\square$

Ex: $f = t+2x+3y+5txy \in \mathbb{C}\{\{t\}\}[x,y]$ $\qquad \phi(x,y)=(xy^3,x)$

$\tilde{f} = f(\phi(x,y)) = t+2xy^3+3y+5txy^4$



Newt(f) $\qquad$ Newt(ϕ(f))

$(\alpha_1,\alpha_2) \mapsto (\alpha_1, \alpha_2+3\alpha_1)$

$w=(-1,-2)\in V(\mathrm{trop}(f))$ $\qquad \mathrm{in}_w f = 3y+5xy$

$\hookrightarrow \tilde{w}=(5,-2)\in V(\mathrm{trop}(\tilde{f}))$ $\qquad \mathrm{in}_{\tilde{w}} \tilde{f} = 3y+5xy^4$

$a=(-\frac{3}{5},1)$ has $\mathrm{in}_w(f)(a) = \mathrm{in}_{\tilde{w}}\tilde{f}(a)=0$

Choose any $x\in\mathbb{C}\{\{t\}\}$ with $\mathrm{val}(x)=3$, $\overline{t^{-5}x} = -3/5$ $\ (x=-\frac{3}{5}t^5 +$ h.o.t.$)$

Then $g(y)=\tilde{f}(x,y)\in\mathbb{C}\{\{t\}\}$ is nonzero

$= 1 + (-\frac{6}{5}t^5 + \ldots)y^3 + 3y + (-3t^6 + \ldots)y^4$

$\mathrm{in}_{(-2)} g = 3y - 3y^4 = 3y(1-y^3)$

$\mathrm{in}_{(-2)} g(1) = 0 \Rightarrow \exists\, y \in \mathbb{C}\{\{t\}\}$ with $g(y)=0$, $\mathrm{val}(y) = -2$, $\overline{t^2 y} = 1$

$\Rightarrow (x,y) = (-\frac{3}{5}t^5 + \ldots, t^{-2} + \ldots) \in V(\tilde{f})$

$\Rightarrow \phi(x,y) = (xy^3, y) = (-\frac{3}{5}t^{-1} + \ldots, t^{-2} + \ldots) \in V(f)$ has $\mathrm{val}(x,y) = (-1,-2) = w$

$\overline{t^{-w}(x,y)} = (-\frac{3}{5}, 1)$

(Proof of Kapranov's Thm) (2) $\subseteq$ (3)

We've shown that $V(\mathrm{trop}(f)) \cap (\Gamma_{\mathrm{val}})^n = \mathrm{val}(V(f))$.

Also $V(\mathrm{trop}(f))$ is the support of a Γ_{val}-rational polyhedral complex. Moreover, since K is alg. closed, $\overline{\Gamma_{\mathrm{val}}} = \mathbb{R}$.

The result then follows from:

Claim: If P is a Γ_{val}-rational polyhedron, $\overline{P \cap (\Gamma_{\mathrm{val}})^n} = P$.

(Proof) If P is full dim'l, then $P = \overline{P \cap \mathbb{Q}^n} \subseteq \overline{P \cap \Gamma_{\mathrm{val}}^n} \subseteq P$.

If not, let $L = \{\sum_{i=1}^{k} \lambda_i v_i : \sum \lambda_i = 1, v_i \in P\}$ be the affine span of P.

We know $P = \{x \in \mathbb{R}^n : a_1^T x \le b_1, \ldots, a_m^T x \le b_m\}$ for some $a_1, \ldots, a_m \in \mathbb{Q}^n$, $b_1, \ldots, b_m \in \Gamma_{\mathrm{val}} \Rightarrow L = \{x \in \mathbb{R}^n : a_i^T x = b_i \text{ for } i \in I\}$ for some $i \in I$.

By row reducing the matrix with rows $(a_i^T : i \in I)$ over $\mathbb{Q}$, one can show that $L \cap (\Gamma_{\mathrm{val}})^n \cong (\Gamma_{\mathrm{val}})^{\dim(L)}$. Then P is full-dim'l in L and the result follows from the full dim'l case.