Tropical Geometry: Tropical varieties and the (statement of the) fundamental theorem.

Recall: K a field, with $\mathrm{val}: K \to \mathbb{R} \cup \{\infty\}$.

For $f = \sum c_\alpha x^\alpha \in K[x_1, \dots, x_n]$ and $w \in \mathbb{R}^n$,

$$\mathrm{in}_w f = \sum_{\alpha \in \operatorname{argmin}(\mathrm{trop}(f)(w))} \overline{(c_\alpha t^{-\mathrm{val}(c_\alpha)})}\, x^\alpha \quad \text{in } \Bbbk[x_1, \dots, x_n].$$

Given an ideal $I \subseteq K[x_1, \dots, x_n]$, the <u>initial ideal</u> of I w.r.t. $w \in \mathbb{R}^n$ is $\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) \rangle$.

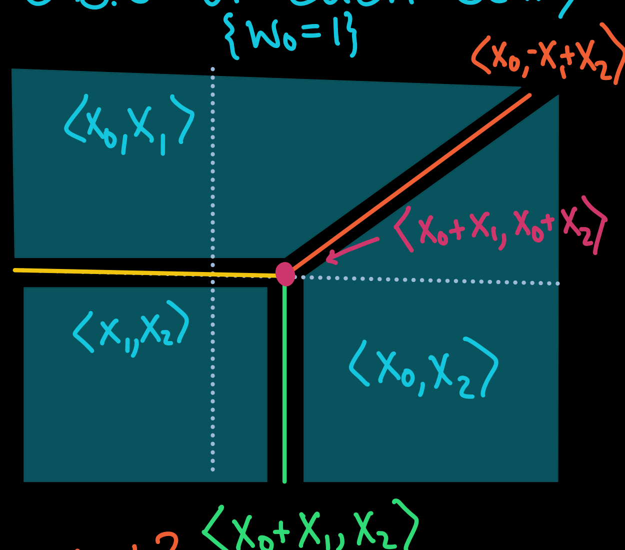
$\{g_1, \dots, g_s\} \subseteq I$ is a <u>Gröbner basis</u> for I w.r.t. $w \in \mathbb{R}^n$ if

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(g_1), \dots, \mathrm{in}_w(g_s) \rangle.$$

Using the existence of $\Sigma(I)$, one can show that any homog. ideal I has a <u>universal Gröbner basis</u>, i.e. a set $\{g_1, \dots, g_s\} \subseteq I$ that is a Gröbner basis for every $w \in \mathbb{R}^n$. (Take union of G.B.'s of each cell)

Ex: $I = \langle \underbrace{x_0 + x_1 + x_2}_{}^{\ell_1}, \overbrace{t x_0 + x_1 + t^2 x_2}^{\ell_2} \rangle$

UGB: $\{ (-1+t) x_1 + (t - t^2) x_2,$
$(1-t) x_0 + (1 - t^2) x_2,$
$(-t + t^2) x_0 + (-1 + t^2) x_1 \}$



Remark: What happens to $C_I[w]$ restricting $x_0 \to 1$?

The <u>tropical variety</u> of an ideal is

$$V(\mathrm{trop}(I)) = \bigcap_{f\in I} V(\mathrm{trop}(f)) = \{w \in \mathbb{R}^n : \mathrm{in}_w(I) \text{ does not contain a monomial}\}$$

Cor: $V(\mathrm{trop}(I))$ is the support of a Γ_{val}-rational polyhedral complex.

(Proof) Consider $I^{hom} = \langle f^{hom} : f \in I \rangle \subseteq K[x_0, \dots, x_n]$.
In Hwk 2, you'll show that $\mathrm{in}_{(0,w)} f^{hom} = x_0^k (\mathrm{in}_w f)^{hom}$ for some k. In particular, $\mathrm{in}_w f$ is a monomial iff $\mathrm{in}_{(0,w)} f^{hom}$ is.
$\Rightarrow V(\mathrm{trop}(I))$ is the support of the restriction of a a subcomplex of $\Sigma(I^{hom})$ to $\{w_0 = 0\}$.

<u>Tropical hypersurfaces</u>

$$f = \sum_{\alpha\in A} c_\alpha \underline{x}^\alpha \qquad \mathrm{trop}(f)(\underline{w}) = \bigoplus_{\alpha\in A} (\mathrm{val}(c_\alpha) \odot \underline{w}^{\odot\alpha})$$

The height function $h(\alpha) = \mathrm{val}(c_\alpha)$ induces a $\underline{\qquad}$ polyhedral subdivision of $\mathrm{Newt}(f)$ with cells $\overline{C_f[w]}$ where

$$C_f[w] = \{v \in \mathbb{R}^n : \mathrm{in}_w(f) = \mathrm{in}_v(f)\}$$
$$= \{v \in \mathbb{R}^n : \mathrm{face}_{(w,1)} P = \mathrm{face}_{(v,1)} P\}$$

where $P = \mathrm{conv}\{(\alpha, \mathrm{val}(c_\alpha)) : \alpha \in A\}$

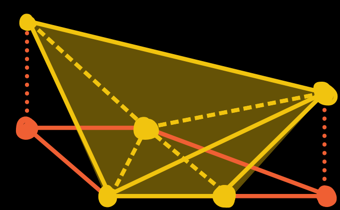
The cells $C_f[w]$ contained in $V(\mathrm{trop}(f))$ are those for which $\min\{\mathrm{val}(c_\alpha) \odot \underline{w}^{\odot \alpha}\}$ is attained at $\alpha \neq \beta \in A \Rightarrow C_f[w] \subseteq \{v : \mathrm{val}(c_\alpha) \odot v^{\odot\alpha} = \mathrm{val}(c_\beta) \odot v^{\odot\beta}\}$

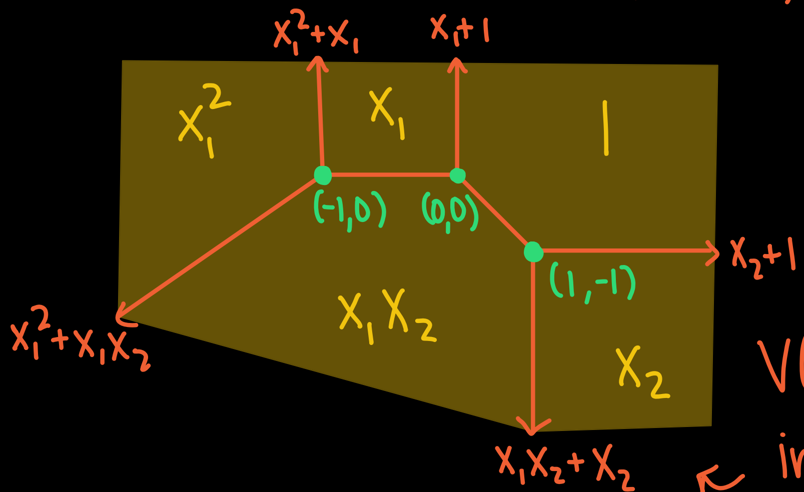
$\Rightarrow C_f[w]$ has dim $\le n-1$.

In general $\dim C_f[w] = n - \dim \mathrm{face}_{(w,1)} P$.

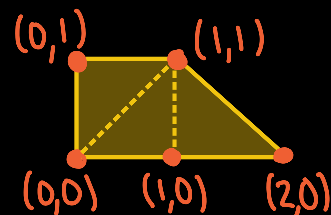
Ex: $f = 1 + x_1 + t x_2 + x_1 x_2 + t x_1^2$

$P = \mathrm{conv}\{(0,0,0), (1,0,0), (0,1,1), (1,1,0), (2,0,1)\}$





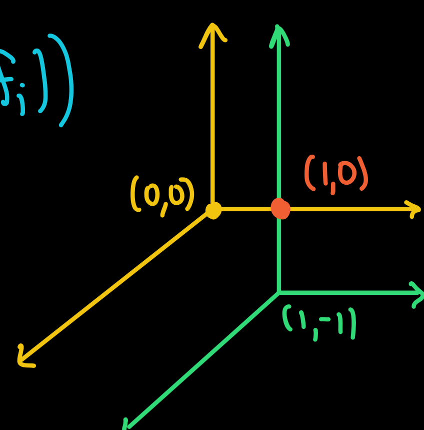
Newt(f)



$V(\mathrm{trop}(f))$ is the union of cells in this complex of dim ≤ 1.

A finite generating set $T = \{f_1, \dots, f_s\}$ is a <u>tropical basis</u> for I if

$$V(\mathrm{trop}(I)) = \bigcap_{i=1}^{s} V(\mathrm{trop}(f_i))$$



Ex (above) $\{\ell_1, \ell_2\}$ are a tropical basis for I.

Thm 2.6.6 Every ideal $I \subseteq K[x_1,\dots,x_n]$ has a finite tropical basis.

(How to compute?)

Remark: $\mathrm{val}(V(I)) \subseteq V(\mathrm{trop}(I))$.

Why? $y \in V(I)$, $\mathrm{val}(y) = w$

$\Rightarrow a = (t^{-w_1} y_1, \dots, t^{-w_n} y_n) \in (\Bbbk^*)^n$ belongs to $V(\mathrm{in}_w I)$

$\Rightarrow \mathrm{in}_w(I)$ does not contain a monomial.

Fundamental Thm of Tropical Geometry

Let K be an algebraically closed field with a nontrivial valuation and $I \subseteq K[x_1,\dots,x_n]$ an ideal. The following sets coincide:

(1) $V(\mathrm{trop}(I)) = \bigcap_{f \in I} V(\mathrm{trop}(f))$

(2) $\{w \in \mathbb{R}^n : \mathrm{in}_w(I)$ does not contain a monomial$\}$

(3) the Euclidean closure of $\mathrm{val}(V(I)) \subseteq \mathbb{R}^n$.

Furthermore, if $V(I)$ is irreducible and $w \in V(\mathrm{trop}(I)) \cap (\Gamma_{\mathrm{val}})^n$, then $\{y \in V(I) : \mathrm{val}(y) = w\}$ is Zariski-dense in $V(I)$.

For hypersurfaces, proven by Kapranov in 1990's.
The general statement was proven by Payne in 2009.

We will also see a "structure theorem" that $\dim(V(\text{trop}(I))) = \dim(V(I) \cap (K^*)^n)$ and that $V(\text{trop}(I))$ is connected in codim 1.

Algebraically closed fields with a valuation:

Any valuation on K extends to a valuation on $\overline{K}^{alg}$.

(Idea: For $\alpha \in K^{alg} \setminus K$, take min poly. $f_\alpha(x) \in K[x]$ and assign $\text{val}(\alpha) \in V(\text{trop}(f))$.

Q: Is $\mathbb{C}((t))$ alg. closed? No. $x^2 - t$ has no root.

$\text{trop}(x^2 - t) = 2w \oplus 1 \Rightarrow$ assign $\text{val}(\sqrt{t}) = 1/2$

The Puiseux series over a field $\Bbbk$ is

$$\Bbbk\{\{t\}\} = \bigcup_{n \in \mathbb{Z}_{\geq 0}} \Bbbk((t^{1/n})) = \left\{ \sum_{k=N}^{\infty} a_{k/n} t^{k/n} : N \in \mathbb{Z},\ n \in \mathbb{Z}_{\geq 0},\ a_{k/n} \in \Bbbk \right\}.$$

This is a valuated field with $\text{val}(\sum a_q t^q) = \min\{q : a_q \neq 0\}$.

Thm 2.1.5: If $\Bbbk$ is an alg. closed field of char 0 then $\Bbbk\{\{t\}\}$ is alg. closed.

Remark (Lemma 2.1.12) Let K be an alg. closed field with a nontrivial valuation. The Γ_{val} is dense in $\mathbb{R}$.

(Proof) We can assume $1 \in \Gamma_{val}$, i.e. $1 = val(t)$ for some $t \in K$.
For all n, $x^n - t$ has a root $t^{1/n} \in K$ with $val(t^{1/n}) = 1/n$.
Since Γ_{val} is an add. subgrp of $\mathbb{R}$, $\mathbb{Q} \subseteq \Gamma_{val} \Rightarrow \overline{\Gamma_{val}} = \mathbb{R}$.