Tropical Geometry: Initial ideals & Gröbner complexes

Take K to be a field with a valuation $\mathrm{val}: K \to \mathbb{R} \cup \{\infty\}$.

Recall: For $f \in K[x_1, \dots, x_n]$ and $w \in \mathbb{R}^n$, the <u>initial form</u> of f w.r.t. w is

$$\mathrm{in}_w f = \sum_{\alpha \in \operatorname{argmin}(\mathrm{trop}(f)(w))} \overline{\left(c_\alpha t^{-\mathrm{val}(c_\alpha)}\right)} x^\alpha = \overline{t^{-\mathrm{trop}(f)(w)} f(t^{w_1} x_1, \dots, t^{w_n} x_n)}$$

when $w \in \Gamma^n$

in $\Bbbk[x_1, \dots, x_n]$.

Given an ideal $I \subseteq K[x_1, \dots, x_n]$, the <u>initial ideal</u> of I w.r.t. $w \in \mathbb{R}^n$ is $\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) \rangle$.

We say that $\{g_1, \dots, g_s\} \subseteq I$ is a <u>Gröbner basis</u> for I w.r.t. $w \in \mathbb{R}^n$ if $\mathrm{in}_w(I) = \langle \mathrm{in}_w(g_1), \dots, \mathrm{in}_w(g_s) \rangle$.

always exist by Noetherianity!

Remark: If $y \in V(I) \cap (K^*)^n$ with $w = \mathrm{val}(y)$, then $\overline{(t^{-w_1} y_1, \dots, t^{-w_n} y_n)} \in V(\mathrm{in}_w(I)) \cap (\Bbbk^*)^n$.

$\Rightarrow \mathrm{in}_w(I)$ does not contain a monomial.

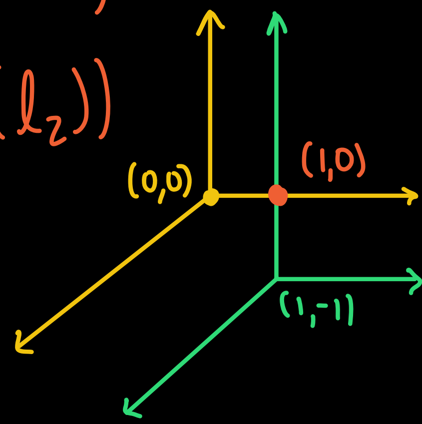
Ex: $I = \langle \underbrace{1 + x_1 + x_2}_{\ell_1}, \underbrace{t + x_1 + t^2 x_2}_{\ell_2} \rangle$

$w = (-1, -5)$ $\quad \mathrm{in}_w(\ell_1) = x_2, \quad \mathrm{in}_w(\ell_2) = x_2$

$\ell_2' = \ell_2 - t^2 \ell_1 = (t - t^2) + (1 - t^2) x_1 \qquad \mathrm{in}_w(\ell_2') = x_1$

$\{\ell_1, \ell_2\}$ is not a G.B. w.r.t. w but $\{\ell_1, \ell_2'\}$ is.

For $\mathrm{in}_w(I)$ to not contain a monomial, need $w \in V(\mathrm{trop}(\ell_1)) \cap V(\mathrm{trop}(\ell_2))$



$\mathrm{trop}(\ell_1) = 0 \oplus w_1 \oplus w_2$

$\mathrm{trop}(\ell_2) = 1 \oplus w_1 \oplus (2 \odot w_2)$

$w \in V(\mathrm{trop}(\ell_1)) \cap V(\mathrm{trop}(\ell_2))$

$\Rightarrow w = (1,0) = \mathrm{val}\left(\frac{-t}{(1+t)}, \frac{-1}{(1+t)}\right)$ (unique sol. to $\ell_1 = \ell_2 = 0$)

$\mathrm{in}_{(1,0)} = \langle 1+x_2, 1+x_1 \rangle$ $\quad t^{(-1,0)} \cdot \left(\frac{-t}{1+t}, \frac{-1}{1+t}\right) = (-1,-1) \in V(\mathrm{in}_{(1,0)}(I))$.

Initial ideals can capture useful information about the original ideals. For example:

Thm (Cor 2.4.9) For homogeneous ideals I, the Hilbert function of I and $\mathrm{in}_w I$ agree:

$$\dim \mathrm{span}_K\{I_d\} = \dim \mathrm{span}_{\mathbb{K}}\{\mathrm{in}_w I\}$$

Gröbner complexes

Given an ideal $I \subseteq K[x_1, \dots, x_n]$ and $w \in \mathbb{R}^n$, define

$$C_I[w] = \{v \in \mathbb{R}^n : \mathrm{in}_v(I) = \mathrm{in}_w(I)\}$$

Thm 2.5.3. If I is a homogeneous ideal then $\forall w$, $\overline{C_I[w]}$ is a Γ_{val}-rational polyhedron and

$\Sigma(I) = \{\overline{C_I[w]} : w \in \mathbb{R}^n\}$ is a polyhedral complex.

This is the <u>Gröbner complex</u> of I.

"Γ_{val}-rational" means defined by $a_1^T v \le b_1, \dots, a_m^T v \le b_m$ where $a_1, \dots, a_m \in \mathbb{Q}^n$ and $b_1, \dots, b_m \in \Gamma_{val}$.

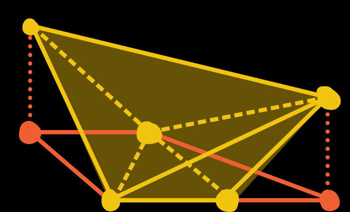
Special case: $I = \langle f \rangle \quad f = \sum_{\alpha} c_\alpha \underline{x}^\alpha$

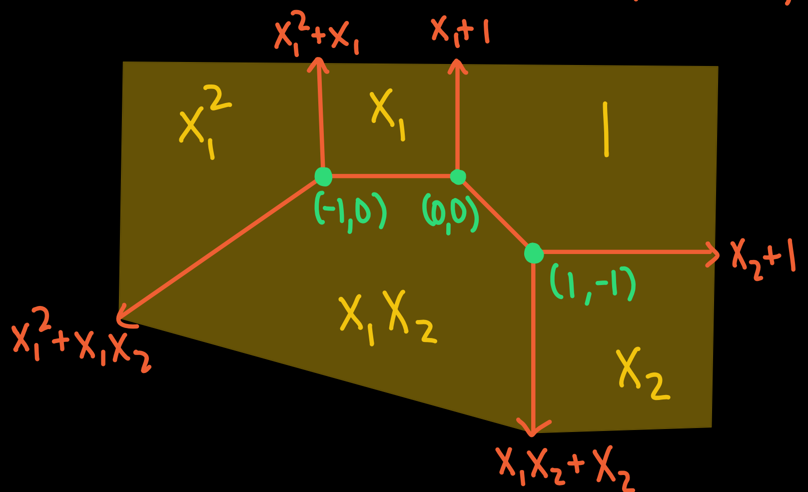
$$\mathrm{in}_w f = \mathrm{in}_v(f) \iff \{\alpha : \mathrm{trop}(f)(w) = \mathrm{val}(c_\alpha) \odot w^{\odot \alpha}\}$$
$$= \{\alpha : \mathrm{trop}(f)(v) = \mathrm{val}(c_\alpha) \odot w^{\odot \alpha}\}$$
$$\iff \mathrm{face}_{(1,w)} P = \mathrm{face}_{(1,v)} P$$

where $P = \mathrm{conv}\{(\mathrm{val}(c_\alpha), \alpha) : \alpha \in A\}$

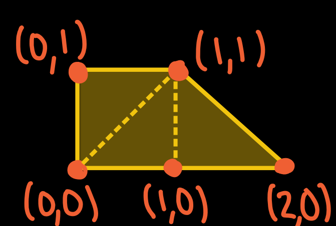
Ex: $f = 1 + x_1 + t x_2 + x_1 x_2 + t x_1^2$

$P = \mathrm{conv}\{(0,0,0), (1,0,0), (0,1,1), (1,1,0), (2,0,1)\}$





Newt(f)



This is <u>dual</u> to the (lower) regular subdivision of Newt(f) induced by heights $h(\alpha) = \mathrm{val}(c_\alpha)$

(Constant coefficients) When I is generated by polynomials $f = \sum c_\alpha \underline{x}^\alpha$ with $\mathrm{val}(c_\alpha) = 0 \ \forall \alpha$ then the Gröbner complex is a polyhedral fan known as the Gröbner fan.

For $I = \langle f \rangle$, $\Sigma(I)$ is the inner normal fan of $\mathrm{Newt}(f)$.

Ex: $f = 1 + x_1 + x_2 + x_1x_2 + x_1^2$