Tropical Geometry: Initial ideals & Gröbner complexes

Take K to be a field with a valuation $\mathrm{val}: K \to \mathbb{R} \cup \{\infty\}$.

Ex: $K = \Bbbk((t)) = \{ \sum_{k=N}^{\infty} a_k t^k : N \in \mathbb{Z}, a_k \in \Bbbk \}$

Q: Given $w \in \mathbb{R}^n$ and $f = \sum_{\alpha \in A} c_\alpha \underline{x}^\alpha \in K[x_1, \dots, x_n]$

when does there exist $y \in K^n$ with $f(y) = 0$ and $\mathrm{val}(y) = w$?

$(n=1)$ $f(x) = x^5 - x + t^2 \in \mathbb{C}((t))[x]$ $\quad y = at^w + \text{h.o.t.}$ $\quad (a \neq 0)$

$$f(y) = \underbrace{(a^5 t^{5w} + \text{h.o.t})}_{\mathrm{val} = 5w} - \underbrace{(at^w + \text{h.o.t.})}_{\mathrm{val} = w} + \underbrace{t^2}_{\mathrm{val} = 2}$$

For $f(y) = 0$, two min. val. terms need to cancel.

Necessary condition: $\min\{5w, w, 2\}$ attained twice

Cases: (1) $5w = w < 2$ and $a^5 - a = 0 \overset{?}{\rightsquigarrow} y = e^{2\pi i k/4} + \text{h.o.t.}$

(2) $5w > w = 2$ and $-a + 1 = 0 \overset{?}{\rightsquigarrow} y = t^2 + \text{h.o.t.}$

Compute $f(y) = \sum_\alpha c_\alpha \underline{y}^\alpha$. The valuation of each term is

$$\mathrm{val}(c_\alpha y_1^{\alpha_1} \cdots y_n^{\alpha_n}) = \mathrm{val}(c_\alpha) + \sum_{i=1}^{n} \alpha_i \,\mathrm{val}(y_i)$$

$\mathrm{val}(f(y)) \geq \mathrm{trop}(f)(\mathrm{val}(y))$ where $\mathrm{trop}(f)(\underline{w}) = \bigoplus_{\alpha \in A} (\mathrm{val}(c_\alpha) \odot \underline{w}^{\odot \alpha})$

Recall: $V(\mathrm{trop}(f)) = \{ w \in \mathbb{R}^n :$ the minimum $\bigoplus_{\alpha \in A} (\mathrm{val}(c_\alpha) \odot w^{\odot \alpha})$ is attained at least twice $\}$

Prop: If $\mathrm{val}(y) \notin V(\mathrm{trop}(f))$ then $\mathrm{val}(f(y)) = \mathrm{trop}(f)(\mathrm{val}(y))$.

Cor: If $y \in (K^*)^n$ with $f(y)=0$, then $\mathrm{val}(y) \in V(\mathrm{trop}(f))$.

Given $w \in V(\mathrm{trop}(f))$, how might we build a root $y \in (K^*)^n$ with $\mathrm{val}(y)=w$?

In Ex above we needed $w \in \{0,2\}$ and a condition on a!

More on valued fields (K, val)

The value group, Γ_{val}, is the image of K^* under val.

This is an additive subgroup of $\mathbb{R}$. (val gives a group hom. $(K^*, \times) \to (\mathbb{R}, +)$)

If $\Gamma_{\mathrm{val}} = \{0\}$ we call val the trivial valuation on K.

When val is nontrivial, we can rescale to assume $1 \in \Gamma_{\mathrm{val}}$.

Lemma (2.1.15) If K is algebraically closed, then the surjection $K^* \to \Gamma_{\mathrm{val}}$ splits. That is, $\exists$ a group hom. $\psi : (\Gamma_{\mathrm{val}}, +) \to (K^*, \times)$ with $\mathrm{val}(\psi(w)) = w \quad \forall w \in K^*$.

(See book for proof)

The valuation ring is $R = \{y \in K : \mathrm{val}(y) \geq 0\}$, which has a unique maximal ideal $\mathfrak{m} = \{y \in K : \mathrm{val}(y) > 0\}$.

The quotient $\Bbbk = R/\mathfrak{m}$ is the residue field.

Ex: $K = \Bbbk((t)) = \{\sum_{k=N}^{\infty} a_k t^k : N \in \mathbb{Z}, a_k \in \Bbbk\}$ (formal Laurent series)

$\mathrm{val}(\sum_k a_k t^k) = \min\{k : a_k \neq 0\}$ $\Gamma_{\mathrm{val}} = \mathbb{Z}$ $\Psi(w) = t^w$ splitting

$R = \{\sum_{k=N}^{\infty} a_k t^k : N \in \mathbb{Z}_{\geq 0}, a_k \in \Bbbk\}$ $\mathfrak{m} = \langle t \rangle$ $\Bbbk = R/\mathfrak{m}$

Ex: $K = \mathbb{Q}$ with p-adic valuation $\mathrm{val}(p^k \frac{a}{b}) = k$

$\Gamma_{\mathrm{val}} = \mathbb{Z}$ with splitting $\Psi(w) = p^w$.

$R = \{p^k \frac{a}{b} : k \geq 0, a,b \not\equiv 0 \bmod p\}$ $\mathfrak{m} = \langle p \rangle$ $\Bbbk = \mathbb{Z}/p\mathbb{Z}$

Ex: K any field with trivial valuation. $\Gamma_{\mathrm{val}} = \{0\}$, splitting $\Psi(0) = 1$.

$R = K$, $\mathfrak{m} = \{0\}$ $\Rightarrow$ $\Bbbk = K$

Initial forms

Assume $\exists$ a splitting Ψ. We use t^w to denote $\Psi(w)$.

For $y \in R$, let $\bar{y} \in \Bbbk$ denote its image in $\Bbbk = R/\mathfrak{m}$.

Then for $y \in K^*$, $\mathrm{val}(t^{-\mathrm{val}(y)} y) = 0$ and $\overline{(t^{-\mathrm{val}(y)} y)} \in \Bbbk^*$.

$\hookrightarrow t^{-\mathrm{val}(y)} y \notin \mathfrak{m}$

Let $f \in K[x_1, \dots, x_n]$ and $w \in \mathbb{R}^n$. The <u>initial form</u> of f w.r.t. w is

$$\mathrm{in}_w f = \sum_{\alpha \in \mathrm{argmin}(\mathrm{trop}(f))} \overline{(c_\alpha t^{-\mathrm{val}(c_\alpha)})} x^\alpha = \overline{t^{-\mathrm{trop}(f)(w)} f(t^{w_1} x_1, \dots, t^{w_n} x_n)}$$

when $w \in \Gamma^n$

in $\Bbbk[x_1, \dots, x_n]$.

Ex: $f = x^5 - x + t^2$, $w = 2$, $\mathrm{in}_w(f) = \overline{t^{-2} f(t^2 x)} = \overline{t^8 x^5 - x + 1} = -x + 1$

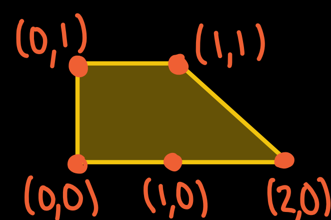
Ex: $f = \frac{9}{5}x^2 + x + \frac{7}{6} \in \mathbb{Q}[x]$ with 3-adic val. $w = -1$

$$\mathrm{in}_w(f) = \overline{3(f(\tfrac{1}{3}x))} = \overline{\tfrac{3}{5}x^2 + x + \tfrac{7}{2}} = x - 1 \in \mathbb{Z}/3\mathbb{Z}[x]$$

Ex (trivial valuation) $f = \sum_{\alpha \in A} c_\alpha \underline{x}^\alpha$ $\quad$ $\mathrm{in}_w(f) = \sum_{\alpha \in \mathrm{face}_w(P)} c_\alpha \underline{x}^\alpha$

where $P = \mathrm{conv}\{\alpha : c_\alpha \neq 0\}$ = the <u>Newton polytope</u> of f.

e.g. $f = 1 + 5x_1 + 6x_2 - 4x_1x_2 + x_1^2$ $\qquad$ Newt(f)



$w = (0,1)$ $\quad$ $\mathrm{in}_w(f) = 1 + 5x_1 + x_1^2$

$w = (-1,-1)$ $\quad$ $\mathrm{in}_w(f) = -4x_1x_2 + x_1^2$

$w = (-2,-1)$ $\quad$ $\mathrm{in}_w(f) = x_1^2$

Prop: If $y \in V(f) \cap (K^*)^n$ and $w = \mathrm{val}(y)$, then

$\overline{(t^{-w_1}y_1, \dots, t^{-w_n}y_n)} \in V(\mathrm{in}_w(f)) \cap (\Bbbk^*)^n$. $\qquad$ ($\Rightarrow w = V(\mathrm{trop}(f))$)

(Proof) Note that $t^{-w} \cdot y = (t^{-w_1}y_1, \dots, t^{-w_n}y_n) \in R^n$ and

$g(\underline{x}) = t^{-\mathrm{trop}(f)(w)} f(t^w \cdot x) \in R[x_1, \dots, x_n]$. Then

$$0 = t^{-\mathrm{trop}(f)(w)} f(y) = g(t^{-w} \cdot y)$$

Taking image mod $\mathfrak{m}$ gives the result.

Given an ideal $I \subseteq K[x_1,\dots,x_n]$, the initial ideal of I w.r.t. $w \in \mathbb{R}^n$ is $\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) \rangle$.

Remark: If $y \in V(I) \cap (K^*)^n$ with $w = \mathrm{val}(y)$, then $\overline{(t^{-w_1}y_1,\dots,t^{-w_n}y_n)} \in V(\mathrm{in}_w(I)) \cap (\Bbbk^*)^n$.

$\Rightarrow$ $\mathrm{in}_w(I)$ does not contain a monomial.

Given an ideal $I \subseteq K[x_1,\dots,x_n]$ and $w \in \mathbb{R}^n$, define

$$C_I[w] = \{ v \in \mathbb{R}^n : \mathrm{in}_v(I) = \mathrm{in}_w(I) \}$$

Special case: $I = \langle f \rangle$, trivial valuation.

$$C_{\langle f \rangle}[w] = \{ v \in \mathbb{R}^n : \mathrm{face}_v(\mathrm{Newt}(f)) = \mathrm{face}_w(\mathrm{Newt}(f)) \}$$

$\Sigma = \{ \overline{C_{\langle f \rangle}[w]} : w \in \mathbb{R}^n \}$ is the inner normal fan of $\mathrm{Newt}(f)$.

Ex: $f = 1 + 5x_1 + 6x_2 - 4x_1x_2 + x_1^2$

Newt(f)