MA 521 – Midterm

Fall 2019

Solutions

Name: ______________________________

Question	Points	Score
1	15	
2	15	
3	5	
4	5	
5	15	
6	15	
Total:	70	

Grading curve for midterm: For S = score out of 70,

$$\text{Curved score} = 70 - \frac{3}{4}(70 - S)$$

$$\text{Curved score percentage} = \left(70 - \frac{3}{4}(70 - S)\right) \cdot \frac{100}{70}.$$

1. Consider the group Q_8 of quaternions, which is a group of size 8 consisting of elements $\{1, -1, i, -i, j, -j, k, -k\}$ subject to the relations

$$i^2 = j^2 = k^2 = -1 \quad \text{and} \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Here $(-1)^2 = 1$ and $-x$ denotes $(-1)x$ for all $x \in G$.

(a) (5 points) What is the center of G?

Solution: For any $x \in G$ with $x \neq 1$, some power of x is -1, so $-1 \in Z(Q_8)$. So $\{1, -1\} \subseteq Z(Q_8)$.

For $x \neq \pm 1$, we see from the relations that x does not commute with $y \neq \pm 1, \pm x$. So $\{1, -1\} = Z(Q_8)$.

(b) (5 points) Let $\varphi : Q_8 \to S_4$ be the group homomorphism induced by the action of Q_8 on the left cosets of $\{1, -1\}$ by left multiplication, using the ordering

$$(1) \leftrightarrow \{1, -1\} \quad (2) \leftrightarrow \{i, -i\} \quad (3) \leftrightarrow \{j, -j\} \quad (4) \leftrightarrow \{k, -k\}.$$

Use cycle notation to write down the image of each element $i, j, k \in Q_8$ under φ. Do not justify your answers.

$$\varphi(i) = (12)(34) \quad \text{since } i \cdot \{1, -1\} = \{i, -i\}, \quad i \cdot \{i, -i\} = \{1, -1\}$$
$$i \cdot \{j, -j\} = \{k, -k\}, \quad i \cdot \{k, -k\} = \{j, -j\}$$

$$\varphi(j) = (13)(24) \quad \text{since } j \cdot \{1, -1\} = \{j, -j\}, \quad j \cdot \{i, -i\} = \{k, -k\}$$
$$j \cdot \{j, -j\} = \{1, -1\}, \quad j \cdot \{k, -k\} = \{i, -i\}$$

$$\varphi(k) = (14)(23) \quad \text{since } k \cdot \{1, -1\} = \{k, -k\}, \quad k \cdot \{i, -i\} = \{j, -j\}$$
$$k \cdot \{j, -j\} = \{i, -i\}, \quad k \cdot \{k, -k\} = \{1, -1\}$$

(c) (5 points) Using the same homomorphism $\varphi : Q_8 \to S_4$, what are the isomorphism types of the groups $\ker(\varphi)$, $Q_8/\ker(\varphi)$ and $\operatorname{im}(\varphi)$? Each should be written as a cyclic group or product of cyclic groups. Do not justify your answers.

$\ker(\varphi) \cong \mathbb{Z}/2\mathbb{Z}$ since $\ker(\varphi) = \{-1, 1\}$

$Q_8/\ker(\varphi) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ since $|Q_8/\ker(\varphi)| = 4$ and every element has order ≤ 2

$\operatorname{im}(\varphi) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ since $\operatorname{im}(\varphi) = \{id, (12)(34), (13)(24), (14)(23)\}$
(also $\operatorname{im}(\varphi) \cong Q_8/\ker(\varphi)$)

2. Continuing with the group Q_8, generated by elements i, j, k subject to the relations

$$i^2 = j^2 = k^2 = -1 \quad \text{and} \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

(a) (5 points) Show that $H = \{1, i, -1, -i\}$ is a normal subgroup of $G = Q_8$.

Solution: Note that $H = \langle i \rangle = \{1 = i^0 = i^4, i = i^1, -1 = i^2, -i = i^3\}$, so H is a subgroup (it is nonempty and closed under multiplication and inversion).

Since $|Q_8 : H| = 2$ is the smallest prime dividing $|Q_8| = 2^3$, H is normal in Q_8

(b) (5 points) Let $\sigma : Q_8 \to \mathrm{Aut}(H)$ be the group homomorphism induced by the action of Q_8 on H by conjugation. What is $\sigma(j)$?

Solution: $\sigma(j)$ is a function from H to H given by $\sigma(j)(h) = jhj^{-1} = jh(-j)$.

Specifically

$$\sigma(j)(1) = j(1)(-j) = 1 \qquad \sigma(j)(i) = j(i)(-j) = ij^2 = -i$$
$$\sigma(j)(-1) = j(-1)(-j) = -1 \qquad \sigma(j)(-i) = j(-i)(-j) = j^2(-i) = i$$

(c) (5 points) Is Q_8 isomorphic to a semidirect product of proper subgroups? Explain.

Solution: No.

If Q_8 were to be isomorphic to a semidirect product of proper subgroups, there would be two subgroups $A, B \leq Q_8$ with $A, B \neq \{1\}$ but $A \cap B = \{1\}$.

However any proper subgroup $A \leq Q_8$ contains -1. Indeed if $A \neq \{1\}$ then take some element $x \in A \backslash \{1\}$. Then either $x = -1$ or $x^2 = -1$.

So there are not two proper subgroups of $A, B \leq Q_8$ with $A \cap B = \{1\}$.

3. (5 points) Give generators for two Sylow-2 subgroups of the symmetric group S_6 and an element $\sigma \in S_6$ that conjugates one to the other.
(Hint: each is isomorphic to $D_8 \times \mathbb{Z}/2\mathbb{Z}$.)

Solution: Since $|S_6| = 6! = 2^4 \cdot 3^2 \cdot 5$, as Sylow-2 subgroup has size $2^4 = 16$.

Consider $P = \langle (1234), (13), (56) \rangle \subset S_6$. Since (56) commutes with the elements $\langle (1234), (12) \rangle$, but does not belong to it, we see that

$$P \;\cong\; \langle (1234), (13) \rangle \times \langle (56) \rangle \;\cong\; D_8 \times \mathbb{Z}/2\mathbb{Z}.$$

Conjugating by $\sigma = (12)$ gives another Sylow-2 subgroup

$$Q \;=\; (12)P(12) \;=\; \langle (2134), (23), (56) \rangle.$$

4. (5 points) Suppose $\varphi : G \to H$ is a group homomorphism where G has size 20 and H has size 30. What are the possible values of $|\ker(\varphi)|$ and $|\mathrm{im}(\varphi)|$? Give an example for each of your answers.

Solution: Let $k = |\ker(\varphi)|$. The first isomorphism theorem

$$|\mathrm{im}(\varphi)| = |G/\ker(\varphi)| = 20/k.$$

Since $\ker(\varphi) \leq G$, k divides 20. Since $\mathrm{im}(\varphi) \leq H$, $|\mathrm{im}(\varphi)| = 20/k$. divides $|H| = 30$. In particular, since $20 = 2^2 \cdot 5$ and $30 = 2 \cdot 3 \cdot 5$, 2 divides k.

Then $k|20$ gives the options $k = 2, 2 \cdot 2 = 4, 2 \cdot 5 = 10, 2^2 \cdot 5 = 20$. We can achieve each option with $G = \mathbb{Z}/20\mathbb{Z}$ and $H = \mathbb{Z}/30\mathbb{Z}$. Note that a homomorphism $\varphi : \mathbb{Z}/20\mathbb{Z} \to \mathbb{Z}/30\mathbb{Z}$ is uniquely determined by $\varphi(\overline{1})$.

$(\lvert\ker(\varphi)\rvert, \lvert\mathrm{im}(\varphi)\rvert = (k, 20/k)$	Example
$(2, 10)$	$\varphi(\overline{1}) = \overline{3}$
$(4, 5)$	$\varphi(\overline{1}) = \overline{6}$
$(10, 2)$	$\varphi(\overline{1}) = \overline{15}$
$(20, 1)$	$\varphi(\overline{1}) = \overline{0}$

5. Consider the sets of $n \times n$ matrices with real entries with positive determinant:

$$\mathcal{P} = \{A \in M_n(\mathbb{R}) : \det(A) > 0\}.$$

(a) (5 points) Show that $\mathcal{P}$ is a subgroup of $\mathrm{GL}_n(\mathbb{R})$.

Solution: Since all matrices in $\mathcal{P}$ have determinant $\neq 0$, $\mathcal{P} \subset \mathrm{GL}_n(\mathbb{R})$. Note that $\mathcal{P}$ is non-empty since it contains the identity matrix. Also, given $A, B \in \mathcal{P}$, $\det(A) > 0$ and $\det(B) > 0$, so

$$\det(AB^{-1}) \;=\; \det(A)\det(B^{-1}) \;=\; \frac{\det(A)}{\det(B)} \;>\; 0$$

so $AB^{-1} \in \mathcal{P}$.

(b) (5 points) Is $\mathcal{P}$ a subring of the ring $M_n(\mathbb{R})$ of $n \times n$ matrices? Justify your answer.

Solution: No.

The additive identity of $M_n(\mathbb{R})$ is the zero matrix. Since $\mathcal{P}$ does not contain the zero matrix, it is not an additive subgroup of $M_n(\mathbb{R})$.

(c) (5 points) Given an example of a zero divisor in $M_2(\mathbb{R})$ and an example of a unit in $M_2(\mathbb{R})$ other than the identity matrix. Justify your answer.

Solution: Zero divisors: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
Both are nonzero, but

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Non-identity units: $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$

Both are not the identity matrix, but

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

6. True or false. Circle one. Do not justify your answers.

   (a) (3 points) If $A \trianglelefteq B$ and $B \trianglelefteq G$, then $A \trianglelefteq G$.

   Solution: False. For example, in $G = D_8$, $\langle s \rangle \trianglelefteq \langle s, r^2 \rangle$ and $\langle s, r^2 \rangle \trianglelefteq D_8$, but $\langle s \rangle \ntrianglelefteq D_8$.

   (b) (3 points) A group homomorphism $\varphi : G \to H$ induces an injective group homomorphism $G/\ker(\varphi) \to H$.

   Solution: True. Whenever $N \trianglelefteq G$ is contained in $\ker(\varphi)$, the map $\tilde{\varphi} : G/N \to H$ given by $\tilde{\varphi}(gN) = \varphi(g)$ is a well-defined group homomorphism. When $N = \ker(\varphi)$, $\tilde{\varphi}(gN) = e_H$ implies that $g \in N$, so $gN = N$ is the identity in G/N and $\tilde{\varphi}$ is injective.

   (c) (3 points) A_5 is isomorphic to a semidirect product of a group of size 6 and 10.

   Solution: False. If A_5 were isomorphic to a semidirect product, say $G \rtimes H$, then G would be a normal subgroup of A_5. Since A_5 is simple, the only possibilities are $|G| = 1$ and $|G| = |A_5| = 60$.

   (d) (3 points) Every group of size 15 is abelian.

   Solution: True Any group of size 15 has subgroups A, B of size 5 and 3 respectively. Then $A \trianglelefteq G$, $A \cap B = \{e\}$ and $AB = G$, so G is isomorphic to $A \rtimes_\varphi B$ for some $\varphi : B \to \mathrm{Aut}(A)$. However since $A \cong \mathbb{Z}/5\mathbb{Z}$, $|\mathrm{Aut}(A)| = 4$. Since $|B| = 3$, φ must be trivial, meaning $G \cong A \times B$. Since A, B are abelian, G must be abelian as well.

   (e) (3 points) The intersection $\cap_{i \in \mathcal{I}} I_i$ of an arbitrary collection of ideals I_i is an ideal.

   Solution: True. Note that the intersection of an arbitrary collection of subgroups is a subgroup, so $\cap_{i \in \mathcal{I}} I_i$ is an additive subgroup of $(R, +)$.

   Also, for $a \in \cap_{i \in \mathcal{I}} I_i$ and $r \in R$, for any i, $a \in I_i$, so $ar, ra \in I_i$. This holds for all i, so $ar, ra \in \cap_{i \in \mathcal{I}} I_i$.