Midterm Preparation – MA521 – Fall 2019

Chapter 1. (DF §1.1-1.7)

- Definition of a group, types of group (finite, abelian)
- Order of group, and group elements
- Examples: $\mathbb{Z}/n\mathbb{Z}$, D_{2n}, S_n, $\mathrm{GL}_n(\mathbb{F})$
- Presentation of a group and relations,
- Group homomorphisms, isomorphisms, kernel
- Group action on a set, orbits, stabilizers

Chapter 2. (DF §2.1-2.5)

- Subgroup definition, criterion
- Special examples: centralizers, normalizers, stabilizers, kernels
- Intersections and generation of subgroups
- lattice of subgroups

Chapter 3. (DF §3.1-3.5)

- Normal subgroups, quotient group
- Cosets, Lagrange's Theorem
- Isomorphism Theorems
- Simple groups, composition series
- The alternating group A_n

Chapter 4. (DF §4.1-4.5)

- Group action terminology: kernel, stabilizer, faithful, transitive, orbit
- Cayley's Theorem
- Conjugacy classes, the class equation
- Conjugacy in S_n via cycle decomposition
- Automorphism group of a group
- Sylow p-subgroups, Sylow theorems

Chapter 5. (DF §5.1, 5.4, 5.5)

- Direct products, recognizing direct products
- Semi-direct products, constructing and recognizing semi-direct products
- Applications to groups of small sizes

Chapter 7. (DF §7.1-7.3)

- Definition of a ring types of ring (commutative, with an identity)
- Units, zero-divisors
- Division ring, field, integral domain
- Subring, Ideal
- Examples: Polynomial rings, matrix rings
- Ring homomorphism, isomorphism, kernels
- Ideals (left-sided, right-sided, two-sided)
- First Isomorphism Theorem for Rings

Some practice problems

(1) When is the the map $G \to G$ given by left multiplication by an element $h \in G$ (i.e. $\varphi(g) = hg$) a group homomorphism?

(2) For $A = \{a, b, c, d\}$, S_A acts on the set of partitions of A into two sets of size two $\{\{\{a, b\}, \{c, d\}\}, \{\{a, c\}, \{b, d\}\}, \{\{a, d\}, \{b, c\}\}\}$ by

$$\sigma \cdot \{\{x, y\}, \{z, w\}\} = \{\{\sigma(x), \sigma(y)\}, \{\sigma(z), \sigma(w)\}\}$$

This induces a group homomorphism $\varphi : S_A \to S_3$. What is the kernel of this action? For each element of S_3 find an element of S_A that maps to it.

(3) Describe a subgroup of S_6 that is isomorphic to D_{12}.

(4) Show that there exists a non-abelian group of order 39.

(5) True or false: For a finite group G and any divisor d of $|G|$, there exists a subgroup $H \leq G$ with $|H| = d$.

(6) True or false: The intersection of two normal subgroups is a normal subgroup.

(7) What are the conjugacy classes in S_4?

(8) State the Orbit-Stabilizer theorem.

(9) Show that any group of order 35 is abelian.

(10) True or false: Every simple group has prime order.

(11) True or false: There is no simple group of order $605 = 5 \cdot 11^2$.

(12) Suppose $\varphi : G \to \mathbb{Z}/p^2\mathbb{Z}$ is a surjective group homomorphism, where p is prime. What are the possible orders of elements of $G/\ker(\varphi)$?

(13) Describe cosets of $\mathbb{Z}/7\mathbb{Z}$ the semidirect product $G = \mathbb{Z}/7\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ and how the group acts on then by left multiplication.

(14) Let G be a group of order 30. Show that G has a normal subgroup that is cyclic of order 15.

(15) Show that if P is a finite p-group and $\langle 1 \rangle \neq N \trianglelefteq P$, then $N \cap Z(P) \neq \langle 1 \rangle$.

(16) Give an example of an infinite group G and subgroup $H \leq G$ so that $|G : H|$ is finite.

(17) * For any non-abelian group of order p^3 where p is prime, $G/Z(G)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

(18) Let G be a group with a subgroup H of index n in G. Show that there exists a group homomorphism $\varphi : G \to S_n$ with $\ker(\varphi) \subseteq H$.

(19) A subgroup $H \leq G$ is called *characteristic* in G if for any automorphism σ of G, $\sigma(H) = H$. Show that any normal Sylow p-subgroup is characteristic.

(20) What is the order of $(33 + 60\mathbb{Z})$ in $\mathbb{Z}/60\mathbb{Z}$?

(21) True or false: S_5 is isomorphic to a direct product of A_5 and $\mathbb{Z}/2\mathbb{Z}$.

(22) True of false: A_5 is isomorphic to D_{60}.

(23) True or false: If G is finite and p is the smallest prime dividing $|G|$, then G contains a normal subgroup of index p.

(24) * Is every non-simple group isomorphic to a semidirect product of two proper subgroups?

(25) * Suppose that R, S are rings with (respective) identities $1_R \neq 0$ and $1_S \neq 0$ and let $\varphi : R \to S$ be a ring homomorphism. Show that if $\varphi(1_R) \neq 1_S$ then $\varphi(1_R)$ is a zero-divisor in S.

(26) Show that if $I_1 \subseteq I_2 \subseteq \dots$ are ideals of a ring R, then so is $\cup_{n=1}^{\infty} I_n$.

(27) Which of the following are ideals in $\mathbb{Z} \times \mathbb{Z}$?: $\{(a, a) : a \in \mathbb{Z}\}$, $\{(2a, 2b) : a, b \in \mathbb{Z}\}$

*Problems 17, 24, 25 are good practice but longer than anything that will show up on the exam.