Math 521 – Homework 8

Due Thursday, October 31, 2019 at 10:15am

Problem 1. Let n be a square-free integer greater with $n > 3$ and consider $R = \mathbb{Z}[\sqrt{-n}]$.

(a) Show that the elements 2, $\sqrt{-n}$ and $1 + \sqrt{-n}$ are irreducible in R.

(b) Show that R is not a unique factorization domain.
Hint: show that either $\sqrt{-n}$ *or* $1 + \sqrt{-n}$ *is not prime in* R.

(c) Give an explicit ideal in R that is not principal.

Problem 2 (Gaussian integers and sums of integers squares)**.**

(a) Check that $a + ib \mapsto \overline{a + ib} = a - ib$ is a ring automorphism of $\mathbb{Z}[i]$.

(b) Show that for any prime $p \equiv 1 \bmod 4$ has a *unique* representation as $a^2 + b^2$ where $a < b \in \mathbb{Z}_{>0}$.

(c) Suppose that $p \neq q$ are both primes equal to 1 mod 4 show that $p \cdot q$ has two different representations as a sum of two squares $a^2 + b^2$ where $a \leq b \in \mathbb{Z}_{>0}$.
Hint: $\alpha\overline{\alpha} \cdot \beta\overline{\beta} = (\alpha\beta) \cdot (\overline{\alpha\beta}) = (\alpha\overline{\beta}) \cdot (\overline{\alpha}\beta)$

(d) Find two different representations of $9797 = 97 \cdot 101$ as a sum of two integer squares.