Math 521 – Homework 7

Due Thursday, October 24, 2019 at 10:15am

Problem 1. Let $R = F[x]$ where F is a field and let $p_1, \dots, p_n \in F$ be distinct field elements (i.e. $p_i \neq p_j$ for $i \neq j$). For each $i = 1, \dots, n$, let

$$I_i = \{f \in F[x] : f(p_i) = 0\} \quad \text{and } J = \cap_{i=1}^n I_i = \{f \in F[x] : f(p_i) = 0 \text{ for all } i = 1, \dots, n\}.$$

(a) Show that $I_i = \langle x - p_i \rangle$ for each $i = 1, \dots, n$.

(b) Show that $F[x]/I_i \cong F$ for each $i = 1, \dots, n$.

(c) Using the steps above, prove that $R/J \cong F^n$. (Here F^n denotes $F \times \cdots \times F$).

Problem 2. Let $R = \mathbb{Z}[\sqrt{-2}] = \mathbb{Z}[x]/\langle x^2 + 2 \rangle$.

(a) Show that R is a Euclidean Domain with $N(a + b\sqrt{-2}) = a^2 + 2b^2$.
(*Check out the proof that $\mathbb{Z}[i]$ is a Euclidean Domain in Ex. (3) on pg. 271 of §8.1.*)

(b) Find the greatest common divisor of 6 and $2 + 7\sqrt{-2}$ in R.

Problem 3 (Maintaining PID-ness)**.**

(a) Prove that a quotient of a principal ideal domain by a prime ideal is again a principal ideal domain.

(b) Prove that if R is a principal ideal domain and D is a multiplicatively closed subset of R with $0 \notin D$, then the ring of fractions $D^{-1}R$ is also a principal ideal domain.