Math 521 – Homework 6

Due Thursday, October 3, 2019 at 10:15am

Problem 1. Let $\mathcal{I}$ be a nonempty index set and for each $i \in \mathcal{I}$, let R_i be a ring.

(a) [DF 7.1.19] Prove that the direct product $\prod_i R_i$ is a ring under componentwise addition and multiplication.

(b) Assuming that each ring R_i has an identity $1_i \neq 0_i$, show that $(1_i)_{i\in\mathcal{I}}$ is an identity in $\prod_{i\in\mathcal{I}} R_i$ and describe the set of units in $\prod_{i\in\mathcal{I}} R_i$ in terms of the units of each R_i.

(c) Describe the zero-divisors of $\prod_{i\in\mathcal{I}} R_i$ in terms of the zero-divisors of R_i.

Problem 2 (DF 7.3.5)**.** Describe all ring homomorphisms from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$. In each case describe the kernel and the image.

Problem 3 (DF 7.3.34)**.** Let R be a ring with identity $1 \neq 0$. Let I, J be ideals of R. Define the *sum* of I and J to be

$$I + J = \{a + b : a \in I, b \in J\}$$

and the *product* to be the set of all finite sums of products of elements of I and J:

$$IJ \;=\; \left\{ \sum_{i=1}^{n} a_i b_i : n \in \mathbb{N}, a_i \in I, b_i \in J \right\}.$$

(a) Prove that $I + J$ is the smallest ideal of R containing both I and J. (Here "smallest" means "inclusion minimal".)

(b) Prove that IJ is an ideal contained in $I \cap J$.

(c) Give an example where $IJ \neq I \cap J$.

(d) Prove that if R is a commutative and if $I + J = R$, then $IJ = I \cap J$.