Math 521 – Homework 5

Due Thursday, September 26, 2019 at 10:15am

Problem 1 (DF 5.1.12)**.** Let I be any nonempty index set and let G_i be a group for each $i \in I$. The *restricted direct product* or *direct sum* of the groups G_i is the set of elements of the direct product which are the identity in all but finitely many components, that is, the set of elements $(a_i)_{i\in I} \in \prod_{i\in I} G_i$ such that $a_i = 1_i$ for all but a finite number of $i \in I$, where 1_i is the identity of G_i.

(a) Prove that the restricted direct product is a subgroup of the direct product.
(b) Prove that the restricted direct produce is normal in the direct product.
(c) Let $I = \mathbb{Z}_+$, let p_i denote the ith prime integer, and let $G_i = \mathbb{Z}/p_i\mathbb{Z}$ for all $i \in \mathbb{Z}_+$. Show that every element of the restricted direct product of the G_i's has finite order but the direct product $\prod_{i\in Z_+} G_i$ has elements of infinite order. Show that in this example, the restricted direct product is the torsion subgroup of $\prod_{i\in Z_+} G_i$.

Problem 2 (≈DF 5.5.8)**.**

(a) Show that (up to isomorphism), there are exactly two abelian groups of order 75.
(b) Show that the automorphism group of $\mathbb{Z}/5\mathbb{Z}\times\mathbb{Z}/5\mathbb{Z}$ is isomorphic to $\mathrm{GL}_2(\mathbb{F}_5)$, where $\mathbb{F}_5$ is the field $\mathbb{Z}/5\mathbb{Z}$. What is the size of this group?
(c) Show that there exists a non-abelian group of order 75.
(d) Show that there is no non-abelian group of order 75 with an element of order 25.

In fact, up to isomorphism, there are only three groups of order 75 so this the complete list (but you do not need to show this).

Problem 3 (≈DF 5.5.22)**.** Consider the following subgroups of $\mathrm{GL}_n(\mathbb{R})$:

$$G = \{A \in \mathrm{GL}_n(\mathbb{R}) : A_{ij} = 0 \text{ for all } i > j\}$$
$$U = \{A \in G : A_{ii} = 1 \text{ for all } i\}$$
$$D = \{A \in G : A_{ij} = 0 \text{ for all } i \neq j\}$$

(You do not need to show that these are subgroups of $\mathrm{GL}_n(\mathbb{R})$)

(a) Show that G is isomorphic to the semidirect product $U \rtimes D$.
(b) For $n = 2$, show that U is isomorphic to the group $\mathbb{R}$ under addition and D is isomorphic to $\mathbb{R}^* \times \mathbb{R}^*$ under multiplication. (Here $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$.)
(c) Describe explicitly the group homomorphism from $\mathbb{R}^* \times \mathbb{R}^*$ to $\mathrm{Aut}(\mathbb{R}, +)$ induced by the semidirect product above.