Math 521 – Homework 4

Due Thursday, September 19, 2019 at 10:15am

Problem 1. The action of a group G on a set A is called *doubly transitive* if for any pairs of elements $a \neq b \in A$ and $c \neq d \in A$ there exists a group element $g \in G$ so that

$$g \cdot a = c \quad \text{and} \quad g \cdot b = d.$$

(a) Show that for $|A| \geq 3$, the action of G on A is doubly transitive if and only if for every $a \in A$, the action of the stabilizer G_a on $A \backslash \{a\}$ is transitive.
(b) Prove that the action of S_n on $\{1, 2, \ldots, n\}$ is doubly transitive.
(c) Show that the action of D_8 on the vertices $\{1, 2, 3, 4\}$ of the square is not doubly transitive.

Problem 2. Recall that for any group G, $\mathrm{Aut}(G)$ is the group of automorphisms of G. For any $g \in G$, let φ_g denote the function $\varphi_g : G \to G$ given by $\varphi_g(x) = gxg^{-1}$, and let $\mathrm{Inn}(G)$ denote the set of all such functions

$$\mathrm{Inn}(G) \ = \ \{\varphi_g : g \in G\}.$$

This is called the *inner automorphism group* of G.

(a) Show that $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ (That is, for any $g \in G$, the function $\varphi_g : G \to G$ given by $\varphi_g(x) = gxg^{-1}$ is an automorphism of g.)
(b) Show that $\mathrm{Inn}(G) \leq \mathrm{Aut}(G)$.
(c) Show that $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$ by showing that for any $\sigma \in \mathrm{Aut}(G)$, $\sigma \varphi_g \sigma^{-1} = \varphi_{\sigma(g)}$.

The quotient group $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is known as the *outer automorphism group* of G.

Problem 3 (DF 4.5.3)**.** Use Sylow's Theorem to prove Cauchy's for non-abelian groups. (Note that we only used Cauchy's Theorem for abelian groups in the proof of Sylow's theorem, so this line of reasoning is not circular.) As a reminder:

Theorem (Cauchy's Theorem)**.** *If G is a finite group and p is a prime number dividing $|G|$, then G contains an element of order p.*

Problem 4.

(a) [DF 4.5.31] For $p = 2$, 3, and 5 find $n_p(A_5)$ and $n_p(S_5)$.
(b) Whenever $n_p(G) > 1$ in part (a), exhibit a conjugacy between two distinct Sylow p-subgroups of G. (That is, pick $P \neq Q \in \mathrm{Syl}_p(G)$ and find $g \in G$ so that $Q = gPg^{-1}$.)