Math 521 – Homework 11

Due Thursday, November 21, 2019 at 10:15am

Problem 1. For a field F, let $F(t)$ be the fraction field of the polynomial ring $F[t]$. Show that $\mathrm{Aut}(F(t)/F)$ is the set of fractional linear transformations given by $t \mapsto (at+b)/(ct+d)$ where $a, b, c, d \in F$ with $ad - bc \neq 0$.

Hint: use Gauss's Lemma with $R = F[y]$ *to show that for* $y = \frac{p(t)}{q(t)} \in F(t)$ *where* $\gcd(p, q) = 1$, $p(x) - yq(x)$ *is irreducible in* $F(y)[x]$ *and equal to the minimal polynomial of* t *over* $F(y)$.

Gauss's Lemma. *Let* R *be a unique factorization domain with field of fractions* $\mathrm{frac}(R)$*. If an element of* $R[x]$ *is irreducible in* $R[x]$*, then it is irreducible in* $\mathrm{frac}(R)[x]$.

Problem 2. Let $K = \mathbb{F}_2[\alpha]/\langle \alpha^3 + \alpha + 1 \rangle$. What is $\mathrm{Aut}(K/\mathbb{F}_2)$? Is K a Galois over $\mathbb{F}_2$?

Problem 3. Let K be the splitting field of $x^5 - 2$ over $\mathbb{Q}$.

(a) Show that $K = \mathbb{Q}(\sqrt[5]{2}, \omega)$ where $\omega \neq 1$ is a fifth root of unity, $\omega^5 = 1$.

(b) What is the group $\mathrm{Aut}(K/\mathbb{Q})$?

(c) Let $\sigma \in \mathrm{Aut}(K/\mathbb{Q})$ be an element of order 5. Find the fixed field of $\langle \sigma \rangle$. Is it Galois over $\mathbb{Q}$?

(d) Let $\tau \in \mathrm{Aut}(K/\mathbb{Q})$ be an element of order 4. Find the fixed field of $\langle \tau \rangle$. Is it Galois over $\mathbb{Q}$?

(e) Show that the action of $\mathrm{Aut}(K/\mathbb{Q})$ on the roots $\alpha_1, \ldots, \alpha_5$ of $x^5 - 2$ is doubly transitive and use this to compute the degree of the field extension $|\mathbb{Q}(\alpha_1 + \alpha_2) : \mathbb{Q}|$.