Math 521 – Homework 10

Due Thursday, November 14, 2019 at 10:15am

Problem 1. For each field F and polynomial $f(x) \in F[x]$, find the splitting field K of $f(x)$ over F and the degree of the extension $[K : F]$.

(a) $F = \mathbb{Q}$, $f(x) = x^4 + 4x^2 + 1$
(b) $F = \mathbb{F}_2$, $f(x) = x^3 + x^2 + 1$
(c) $F = \mathbb{F}_2(t)$, $f(x) = x^3 + t$

Problem 2. Suppose that K_1 and K_2 are two fields containing F and contained in some field K. (That is $F \subseteq K_i \subseteq K$) Suppose that K_1 is the splitting field of some polynomial $f(x) \in F[x]$ over F and K_2 is the splitting field for some polynomial $g(x) \in F[x]$ over F.

(a) Let K_1K_2 denote the smallest subfield of K containing both K_1 and K_2. Show that K_1K_2 is the splitting field of some polynomial $p(x) \in F[x]$ over F.

(b) Suppose that $E \supseteq F$ is a finite extension of F so that every irreducible polynomial that has a root in E factors into linear factors in $E[x]$. Show that E is the splitting field of some polynomial $h(x) \in F[x]$ over F.
(*Hint: write* $E = F(\alpha_1, \ldots, \alpha_n)$ *and consider the minimal polynomials of* α_i)

(c) [Extra credit] Show that $K_1 \cap K_2$ is the splitting field of some polynomial $q(x) \in F[x]$ over F.

Problem 3. Show that for any prime $p \geq 2$, the algebraic closure of $\mathbb{F}_p$ is infinite.