

MA/AMA 514

Today: Matchings ; bipartite graphs (§3.2, 3.4)

§3.2 M -augmenting paths

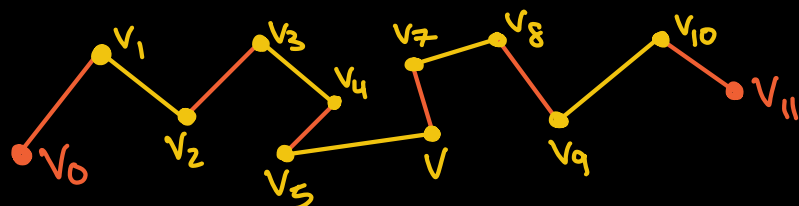
Recall: A matching of a graph $G=(V,E)$ is a subset $M \subseteq E$ s.t. $e \cap e' = \emptyset$ for all $e \neq e' \in M$.

Let $\nu(G) = \max\{|M| : M \text{ is a matching in } G\}$.

Def: Let M be a matching in $G=(V,E)$.

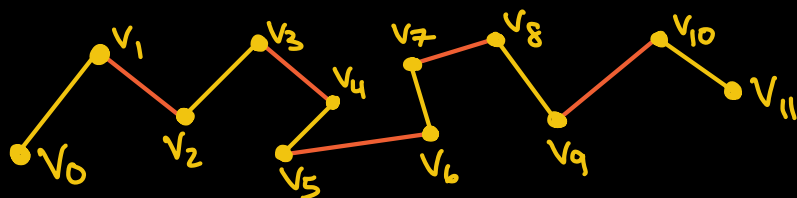
A path $P=(v_0, \dots, v_{2t+1})$ in G is M -augmenting if

$v_1 v_2, v_3 v_4, \dots, v_{2t-1} v_{2t} \in M$ and $v_0, v_{2t+1} \notin \bigcup_{e \in M} e$.



Then $M' = M \Delta E(P)$ is a matching in G .

$$(\quad = M \setminus \{v_1 v_2, \dots, v_{2t-1} v_{2t}\} \cup \{v_0 v_1, v_2 v_3, \dots, v_{2t} v_{2t+1}\})$$



Thm 3.2 If M is a matching in G , either $|M| = \nu(G)$ or there exists an M -augmenting path.

(Proof) If there is an M -augmenting path, then M' is a matching with $|M'| > |M|$, so $|M|$ not maximal.

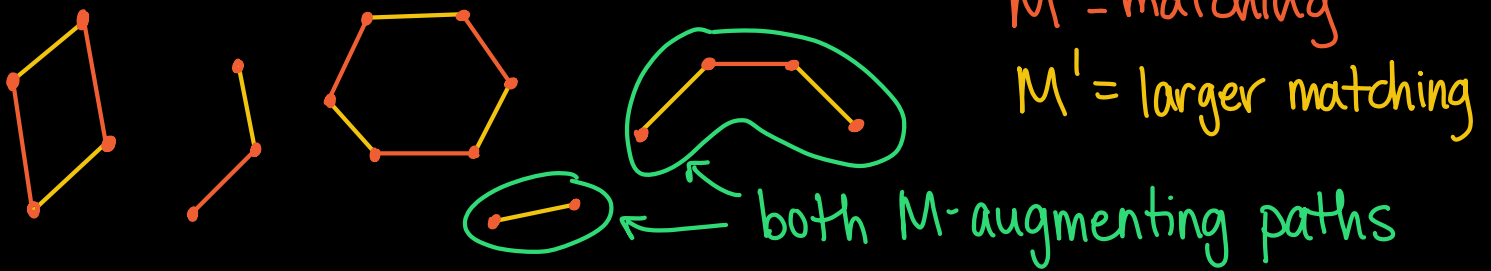
Conversely, suppose M' is a matching with $|M'| > |M|$.

Consider $G' = (V, M \Delta M')$. Every $v \in V$ has degree ≤ 2 in G' .

$\Rightarrow$ each connected component of G' is a path or a circuit

Since $|M'| > |M|$, some connected component of G' has more edges from M' than from M .

This component must be an M -augmenting path!

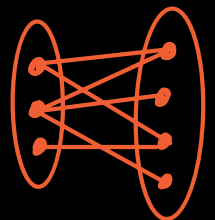


Challenge: Finding M -augmenting paths

This becomes easier in bipartite graphs.

A graph $G = (V, E)$ is bipartite if there is a partition of the vertex set $V = V_1 \cup V_2$ with no edges between vertices in the same part

That is, $|e \cap V_i| \leq 1$ for all $e \in E$, $i = 1, 2$.



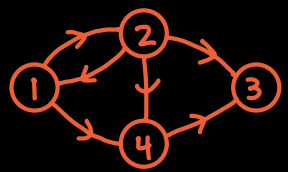
§3.4 Algorithm for max. matchings in bipartite graphs

We'll use directed graphs to do this

A directed graph is a pair $D=(V,A)$

V = "vertices" (finite set)

A = "arcs" = ordered pairs of vertices (v,w) 

Ex:  $V=\{1,2,3,4\}$ $A=\{(1,2),(2,1),(1,4),(2,4),(2,3),(4,3)\}$

A path in P consists of arcs $(v_0,v_1),(v_1,v_2),\dots,(v_{k-1},v_k)$

with $v_i \neq v_j$ for $i \neq j$. Paths (and shortest paths)

between $v,w \in V$ can be computed in poly time. (See §1.1)

Algorithm:

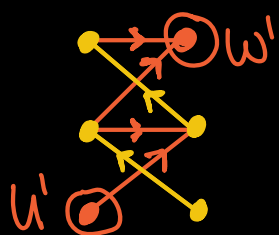
Input: Bipartite graph $G=(V,E)$, matching M

Output: Larger matching M' (if one exists)

Write $V=U \cup W$.

For $e=\{u,w\} \in E$ with $u \in U, w \in W$, direct edge e

$w \rightarrow u$ if $e \in M$ and $u \rightarrow w$ if $e \notin M$.



Let U', W' denote the set of vertices of U, W (resp.) uncovered by M

Claim: A path $P \subseteq E$ is M -augmenting iff it gives a directed path from some $u' \in U'$ to some $w' \in W'$.

(Proof) If P is M -augmenting, its endpoints lie in $U' \cup W'$.

$\text{Length}(P)$ odd $\Rightarrow$ one end pt lies in U' , one in W'

Alternating edges lie in $M \Rightarrow$ all edges directed $u' \rightarrow w'$

Conversely, if P is a directed path from $u' \in U'$ to $w' \in W'$ then $\text{length}(P)$ is odd and because of edge directions edges must alternate between M and $E \setminus M$.

One can find such a directed path P in $O(|E|)$ time (if it exists). Take $M' = P \Delta M$ (a bigger matching!)

Starting from $M = \emptyset$, repeat $O(|V|)$ times to get max matching.

Improved alg. by Hopcroft, Karp (1973)

has running time $O(|V|^{1/2} |E|)$.

See "Blossom algorithm" for generalization to non bipartite graphs!