Math 425H – Homework 10

Due Friday, November 15, 2019 at 11:45am

Problem 1 (Abbott Exercise 6.4.5 (Antidifferentiation))**.** Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$.

(a) Show that

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

is defined on $(-R, R)$ and satisfies $F'(x) = f(x)$.

(b) Antiderivatives are not unique. If g is an arbitrary function satisfying $g'(x) = f(x)$ on $(-R, R)$, find a power series representation for g.

Problem 2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable on $\mathbb{R}$ and for all $x \in (-R, R)$,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

(a) Show that if $\sum a_n x^n$ converges at $x = R$, then it must converge to $f(R)$.

(b) Give an example where $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for all $x \in (-1, 1)$, but the series does not converge at either end point $x = \pm 1$.

(c) Find the limit of the series

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \frac{1}{11} + \dots$$

That is, $\sum_{n=1}^{\infty} \frac{\varphi(n)}{n}$ where $\varphi(3k) = 0$, $\varphi(3k+1) = 1$, $\varphi(3k+2) = -1$ for every $k \in \mathbb{N}$. It may be useful to integrate

$$\frac{(1-x)}{1-x^3} = \frac{1}{x^2+x+1} = \frac{4/3}{1 + \left((1+2x)/\sqrt{3}\right)^2}.$$

Problem 3. Let's prove a slightly weaker version of Lagrange's Remainder Theorem (without using the original). Suppose that f is differentiable $N+1$ times on an interval $(-R, R)$.

(a) Show that if g, h are differentiable on an interval $[0, x]$ with $g(0) = h(0)$ and $g'(t) \leq h'(t)$ for all $t \in [0, x]$, then $g(t) \leq h(t)$ for all $t \in [0, x]$.

(b) Show that for $x \in (0, R)$, if $|f^{N+1}(t)| \leq M$ for all $t \in [0, x]$ then

$$\left| f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n \right| \leq \frac{M x^{N+1}}{(N+1)!}.$$