Math 409: Discrete Optimization

Today: Max matching, Min vertex cover

Bipartite graphs

König's Thm: For $G=(V,E)$ bipartite,

$$\max\{|M| : M \subseteq E \text{ matching}\} = \min\{|U| : U \subseteq V \text{ vertex cover}\}$$

can be found in poly time $O(|V|\cdot|E|)$.

Non-bipartite graphs

$$\max\{|M| : M \subseteq E \text{ matching}\} \leq \min\{|U| : U \subseteq V \text{ vertex cover}\}$$

Why? Every $e \in M$ covered by some $u \in U$.
Since M is a matching, map $e \to u$ is injective.

This "$\leq$" can be strict. E.g. $G = K_3$

Blossom algorithm computes max size matching in poly. time $O(|V|^5)$. ($\to O(|V|^3)$ with careful implementation)

Computing min vertex cover is NP-hard!

In fact, NP-hard to compute within a factor of 1.4!

Today: Computing within a factor of 2 in poly. time. That is, β s.t. $\beta \leq |U^*| \leq 2\beta$

Approximation Algorithm 1: Fractional Rounding

Input: $G=(V,E)$ undirected (not necessarily bipartite)

Goal: min vertex cover $S \subseteq V$ (with $|e \cap S| \geq 1 \ \forall e \in E$)

Formulation as an IP:

$$\min \sum_{v \in V} y_v \quad \text{s.t.} \quad 0 \leq y \leq 1, \quad y_u + y_v \geq 1 \ \forall \{u,v\} \in E, \quad y \in \mathbb{Z}^V$$

Let y^*_{IP} = opt. sol. for (IP) $= \mathbb{1}_S$ for min vertex cover $S \subseteq V$

y^*_{LP} = " " LP relaxation

(Rounding) Define $\tilde{y} \in \{0,1\}^V$ by $\tilde{y}_v = \begin{cases} 1 & \text{if } (y^*_{LP})_v \geq 1/2 \\ 0 & \text{if } (y^*_{LP})_v < 1/2 \end{cases}$

Claim 1: $\tilde{y}$ is feasible for (IP) (i.e. $\tilde{y} = \mathbb{1}_S$, $S \subseteq V$ a vertex cover)

(Proof) For $\{u,v\} \in E$, $(y^*_{LP})_u + (y^*_{LP})_v \geq 1$ $\Rightarrow$ at least one $\geq 1/2$

$\Rightarrow \tilde{y}_u + \tilde{y}_v \geq 1$

Claim 2: $\mathbb{1}^T y^*_{LP} \underset{(1)}{\leq} \mathbb{1}^T y^*_{IP} \underset{(2)}{\leq} \mathbb{1}^T \tilde{y} \underset{(3)}{\leq} 2 y^*_{LP}$

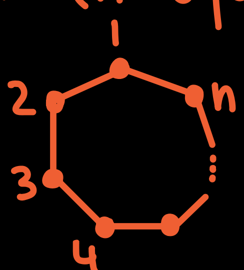
(Proof) (1) minimizing over larger set

(2) $\tilde{y}$ feasible for (IP), $m^*_{IP} = \min \mathbb{1}^T y$ over all feas. y

(3) $\sum_{v \in V} \tilde{y}_v \leq \sum_{v \in V} 2 (y^*_{LP})_v = 2 \mathbb{1}^T y^*_{LP}$

Ex (n-cycle) $\min \sum_{i=1}^{n} y_i$ s.t. $0 \leq y \leq 1$, $y_i + y_{i+1} \geq 1 \ \forall i = 1, \dots, n-1$

$y_1 + y_n \geq 1$



$y^*_{LP} = (1/2, \dots, 1/2)$ $\quad \mathbb{1}^T y^*_{LP} = n/2$

round

$\tilde{y} = (1, \dots, 1)$ $\quad \mathbb{1}^T \tilde{y} = n$

Approximation Algorithm 2: maximal matching

Input: $G=(V,E)$
Output: a vertex cover of G

(1) Initialize $U=\emptyset$, $E'=E$
(2) While $E'\neq\emptyset$
    (3) Choose $e=\{u,v\}\in E'$
    (4) Update $U=U\cup\{u,v\}$, $E'=E'\setminus(\delta(u)\cup\delta(v))$
(5) Output U

Claim: Output U is a vertex cover with
$$\tfrac{1}{2}|U|\leq|U^*|\leq|U|$$

(Proof) Every edge in E starts in E' and is only removed once covered by U
$\Rightarrow$ U is a vertex cover $\Rightarrow |U^*|\leq|U|$

Let M denote the set of edges chosen in (3).
Note that M is a matching of G $\Rightarrow |M|\leq|U^*|$
Moreover, by construction $|U|=2|M|$
$\Rightarrow \tfrac{1}{2}|U|=|M|\leq|U^*|$