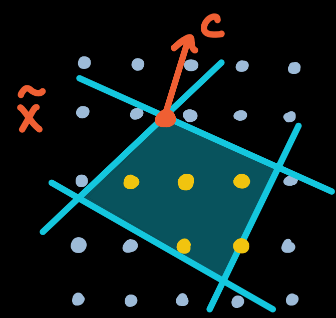
Math 409: Discrete Optimization

Today: Finish branch and bound



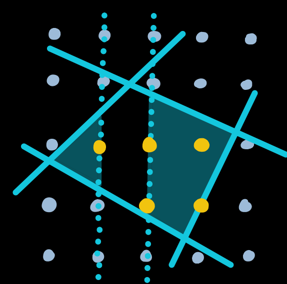
(IP) $\max\ c^Tx$ s.t. $Ax \le b,\ x \in \mathbb{Z}^n$

(LP) $\max\ c^Tx$ s.t. $Ax \le b$

Branch and bound (summary)

Find integer feasible solutions by

- solving LP subproblems $\max\{c^Tx : x \in P'\}$
- branching on a variable
  $P \to P \cap \{x_i \le \lfloor \alpha \rfloor\}$ and $P \cap \{x_i \ge \lceil \alpha \rceil\}$
- pruning by infeasibility, bounds, or optimality



Thm: If Branch & Bound algorithm terminates, then it outputs the true opt sol to (IP)

Idea: (Branching) $P' \cap \mathbb{Z}^n$ contained in $(P \cap \{x_i \le \lfloor \alpha \rfloor\}) \cup (P \cap \{x_i \ge \lceil \alpha \rceil\})$

$\Rightarrow$ if P' contains opt sol, one of branches will contain opt. sol.

(Bound) If $\exists x^* \in P \cap \mathbb{Z}^n$ with $c^Tx^* \ge \max\{c^Tx : x \in P'\}$ then either x^* is opt for (IP) or P' does not contain opt. sol. to (IP)

Choices in algorithm: (1) choice of branching index $\tilde{x}_i \notin \mathbb{Z}$

(2) choice of $P' \in \mathcal{S}$

Thm: If P is bounded, then Branch & Bound algorthm terminates. More precisely $P \subseteq [-R,R]^n$
$\Rightarrow$ any path in B&B tree has $\leq n(2R+2)$ edges
$\Rightarrow$ B&B tree has $\leq 2^{n(2R+2)}$ vertices.

Idea: When branching on a variable $P \to P \cap \{x_i \leq \lfloor \alpha \rfloor\}$, $P \to P \cap \{x_i \geq \lceil \alpha \rceil\}$
each new constraint did not hold on P.
Adding both $x_i \geq m$, $x_i \leq m$ fixes $x_i = m \in \mathbb{Z}$
$\Rightarrow$ opt sol $\tilde{x}$ has $\tilde{x}_i \in \mathbb{Z} \Rightarrow$ don't branch on x_i.
Only $2R+1$ possible values for $\lfloor \tilde{x}_i \rfloor, \lceil \tilde{x}_i \rceil$
$\Rightarrow$ branch on x_i at most $2R+2$ times

With any strategy, there are bad examples:

Lemma: Let $n \geq 4$ be even. For any strategy used, the branch and bound tree for

$$\text{(IP)} \quad \max\ x_0 \quad \text{s.t.} \quad \tfrac{1}{2}x_0 + \sum_{i=1}^{n} x_i = \tfrac{n}{2}, \quad 0 \leq x \leq 1, \quad x \in \mathbb{Z}^{n+1}$$

has $\geq 2^{n/3}$ leaves.

Idea: n even $\Rightarrow$ only integer sol. have $x_0 = 0$
Fractional sol. with $\tilde{x}_0 = 1$ survive through many branchings

(Proof) Since $x_i \in [0,1]$ for all feas. pts x, branching on x_i gives $P' \to P' \cap \{x_i = 0\}$ and $P' \cap \{x_i = 1\}$.

Consider node P' in B&B tree. Let $I_0 \subseteq [n]$, $I_1 \subseteq [n]$ denote the set of indices set to 0, 1 (resp.) in P'.

Claim: If $|I_0| + |I_1| \leq n/3$, the $\exists\ \tilde{x} \in P'$ with $\tilde{x}_0 = 1$.

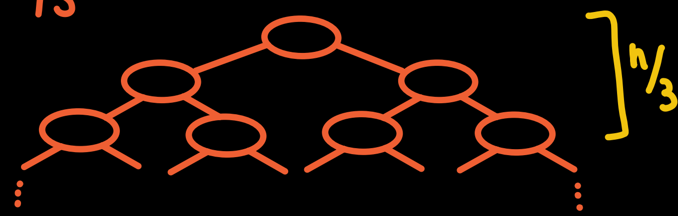
Take $\tilde{x}_0 = 0$, $\tilde{x}_i = 0$ for $i \in I_0$, $\tilde{x}_i = 1$ for $i \in I_1$, $\tilde{x}_j = \lambda$ otherwise

$$\frac{1}{2}\tilde{x}_0 + \sum_{i=1}^{n} \tilde{x}_i = \frac{1}{2} + |I_1| + (n - |I_0| - |I_1|)\lambda = n/2$$

Since $|I_0| \leq \frac{n}{3} \leq \frac{n-1}{2} \Rightarrow \frac{n-1}{2} - |I_1| \leq n - |I_0| - |I_1|$

$\Rightarrow \lambda \leq 1 \Rightarrow \tilde{x}$ feasible $\square$ claim

Branch at every node of depth $\leq n/3$



Any int. sol. has $x_0 = 0$.

While $\tilde{x}_0 = 1$, can't prune by bound, infeasibility, or optimality

$\Rightarrow\ \geq 2^{n/3}$ (pruned) leaves