Math 409: Discrete Optimization

Today: total unimodularity (cont')

Recall:

Def: A matrix $A \in \mathbb{R}^{m \times n}$ is **totally unimodular** (TU) if every square submatrix has determinant $0, \pm 1$.

Prop: If A is TU, $b \in \mathbb{Z}^m$ and $\ell, u \in \mathbb{Z}^n$ then all vertices of each of the following are integer:

1) $\{x \in \mathbb{R}^n : Ax \leq b \ x \geq 0\}$
2) $\{x \in \mathbb{R}^n : Ax \leq b, \ \ell \leq x \leq u\}$
3) $\{x \in \mathbb{R}^n : Ax = b, \ x \geq 0\}$
4) $\{x \in \mathbb{R}^n : Ax = b, \ \ell \leq x \leq u\}$

Prop: The node-edge incidence matrix of any directed graph is totally unimodular.

Applications to Network Flow

$G = (V, E)$ directed graph, $s, t \in V$, $c : E \to \mathbb{Z}_{\geq 0}$

Equations defining s-t flows $f : E \to \mathbb{R}_{\geq 0}$, $f(e) = x_e$

$$\sum_{e \in \delta^{in}(v)} x_e = \sum_{e \in \delta^{out}(v)} x_e \quad \forall v \in V \setminus \{s, t\}$$

Encode into node edge incidence matrix $A \in \{0, \pm 1\}^{V \times E}$

$$A_{ve} = \begin{cases} 1 & \text{if } e \in \delta^{in}(v) \\ -1 & \text{if } e \in \delta^{out}(v) \\ 0 & \text{o.w.} \end{cases}$$

$e = (w, v)$ for some w $\quad w \xrightarrow{e} v$

$e = (v, w)$ for some w $\quad v \xrightarrow{e} w$

For $x \in \mathbb{R}^E$, $Ax \in \mathbb{R}^V$ with $(Ax)_v = \sum_{e \in \delta^{in}(v)} x_e - \sum_{e \in \delta^{out}(v)} x_e$ ← netflow into v

<u>Cor</u>: For integer edge capacities $c: E \to \mathbb{Z}_{\geq 0}$, the set of s-t flows equals the polytope

$$P_{flow} = \{ x \in \mathbb{R}^E : \tilde{A}x = 0,\ 0 \leq x \leq c \}$$

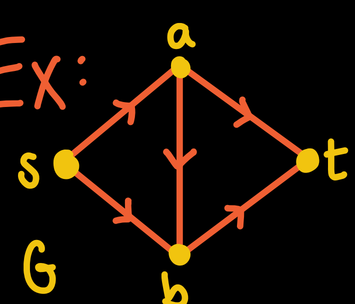
where $\tilde{A}$ is the $V \setminus \{s,t\} \times E$ submatrix of A.

Moreover, all vertices of P_{flow} are integer.

(Proof) A TU $\Rightarrow$ $\tilde{A}$ TU $\Rightarrow$ integer vertices

↑ by Cor about $\{x \in \mathbb{R}^n : Ax = b,\ \ell \leq x \leq u\}$

Ex:



$$A = \begin{array}{c} \\ s \\ a \\ b \\ t \end{array} \begin{array}{c} \begin{array}{ccccc} sa & sb & ab & at & bt \end{array} \\ \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ \boxed{\begin{matrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \end{matrix}} \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

$\tilde{A}$

$$Ax = \begin{pmatrix} -x_{sa} - x_{sb} \\ \boxed{\begin{matrix} x_{sa} - x_{ab} - x_{at} \\ -x_{sb} + x_{ab} + x_{bt} \end{matrix}} \\ x_{at} + x_{bt} \end{pmatrix}$$

$\tilde{A}x$

<u>Cor</u>: For $c: E \to \mathbb{R}_{\geq 0}$, the max value of an s-t flow is acheived by an integer flow $f: E \to \mathbb{Z}_{\geq 0}$

(objective function $= \sum_{e \in \delta^{out}(s)} x_e - \sum_{e \in \delta^{in}(s)} x_e$, linear in x_e)

Application: Min Cost Max Flow

Given prices $p: E \to \mathbb{R}_{\geq 0}$ with
$p(e)$ = cost of sending one unit of flow on e,
cost of flow $f: E \to \mathbb{R}_{\geq 0}$ is $\sum_{e \in E} p(e) f(e)$

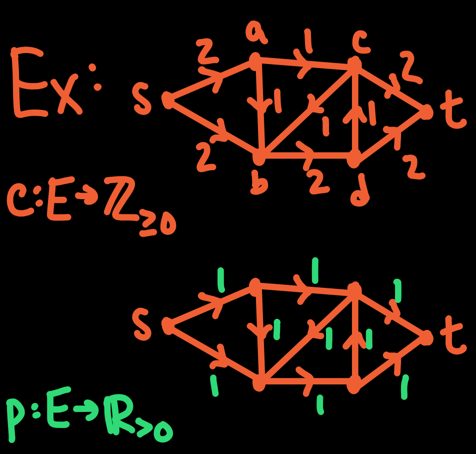
> **MIN COST MAX FLOW**
>
> **Input**: digraph $G=(V,E)$, $s,t \in V$, $\underbrace{c: E \to \mathbb{Z}_{\geq 0}}_{\text{capacities}}$, $\underbrace{p: E \to \mathbb{R}}_{\text{prices}}$
>
> **Goal**: Find max value s-t flow of minimum cost

We can solve this in polynomial time with LP:

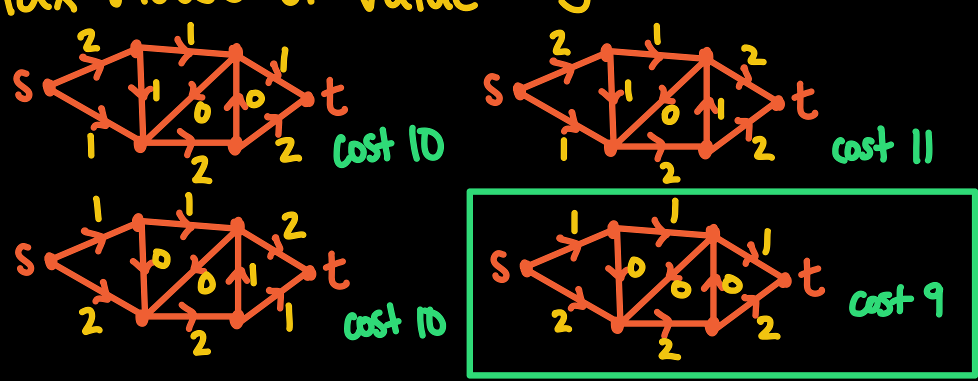
1) Compute max value m of an s-t flow
2) Find $\min \sum_{e \in E} p(e) x_e$ s.t. $0 \leq x \leq c$, $Ax = \underbrace{m(\mathbb{1}_t - \mathbb{1}_s)}_{\substack{t^{th}\text{ entry} = m \\ s^{th}\text{ entry} = -m \\ \text{rest} = 0}}$

Moreover, since A is TU, the optimal value is acheived by an integer-valued flow $f: E \to \mathbb{Z}_{\geq 0}$.

Ex:



Max flows of value = 3



min cost max flow

Encoded as an LP:

min $x_{sa} + x_{sb} + \dots + x_{dt}$ s.t.

$0 \le x_{sa} \le 2,\ 0 \le x_{sb} \le 2,\ \dots,\ 0 \le x_{dt} \le 2$ and

$$\begin{array}{cc} & \begin{array}{cccc} sa & sb & \cdots & dt \end{array} \\ \begin{array}{c} s \\ a \\ b \\ c \\ d \\ t \end{array} & \underbrace{\begin{pmatrix} -1 & -1 & & 0 \\ 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & & -1 \\ 0 & 0 & & 1 \end{pmatrix}}_{\text{full node-edge adjacency matrix } A} \end{array} \begin{pmatrix} x_{sa} \\ x_{sb} \\ \vdots \\ x_{dt} \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix}$$