Math 409: Discrete Optimization

Today: LP for matchings, total unimodularity

LP for max matchings in $G=(V,E)$

$e\in E \to$ variable x_e, constrained $0 \le x_e \le 1$ $\quad \begin{pmatrix} x_e = 1 \leftrightarrow e\in M \\ x_e = 0 \leftrightarrow e \notin M \end{pmatrix}$

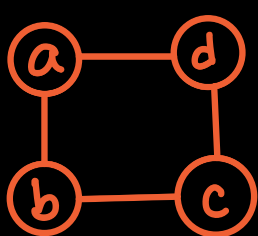
Constraints: $\deg_v(M) \le 1 \to \sum_{e \ni v} x_e \le 1 \quad \forall v \in V$

Maximize $|M|$ (or $\sum_{e\in M} c_e$ for some $c: E \to \mathbb{R}$) $\to \max \sum_{e\in E} c_e x_e$

$$(IP) \quad \max \sum_{e\in E} c_e x_e \quad \text{s.t.} \quad 0 \le x_e \le 1 \text{ for all } e\in E, \; x \in \mathbb{Z}^E$$
$$\sum_{e\ni v} x_e \le 1 \text{ for all } v \in V$$

Claim: This solves max weight matching on G

(Recover W from opt sol x^* by $W = \{e : x^*_e = 1\}$) You check!

Ex: 

Feas. region of LP relaxation:

$\{x \in \mathbb{R}^4 : x_{ab}, x_{bc}, x_{cd}, x_{ad} \in [0,1]$ and

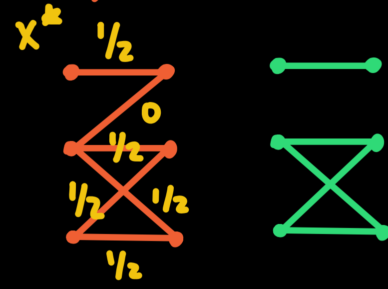
$x_{ab}+x_{ad} \le 1, \; x_{ab}+x_{bc} \le 1, \; x_{bc}+x_{cd} \le 1, \; x_{ad}+x_{cd} \le 1\}$

Vertices: $(x_{ab}, x_{bc}, x_{cd}, x_{ad}) = (0,0,0,0), (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$
$(1,0,1,0), (0,1,0,1)$

All integer!

Thm: For a bipartite graphs, any vertex x^* of the matching LP is integer. ($\Rightarrow P = P_I$ for $P =$ feas. region of matching LP)

(Proof) Suppose x^* is a vertex. Define $G_{x^*} = (V, E_{x^*})$ with $E_{x^*} = \{e \in E \text{ s.t. } 0 < x^*_e < 1\}$.



(Case 1) Suppose G_{x^*} has a cycle C

G_{x^*} bipartite $\Rightarrow$ C has even length: $C = (e_1, e_2, \dots, e_{2k})$

Define $w \in \mathbb{R}^E$ by $w_e = \begin{cases} 1 & \text{if } e = e_{2i} \in C \\ -1 & \text{if } e = e_{2i+1} \in C \\ 0 & \text{o.w.} \end{cases}$

Take $\varepsilon = \min\{x^*_e : e \in C\} \cup \{1 - x^*_e : e \in C\}$

Then $\underline{x^* + \varepsilon w \text{ and } x^* - \varepsilon w \text{ both feasible}} \Rightarrow x^*$ not a vertex

check $(x^* \pm \varepsilon w)_e \in [0,1]$ $\forall e$ and $\sum_{e \in v} (x^* \pm \varepsilon w)_e = \sum_{e \ni v} x^*_e$

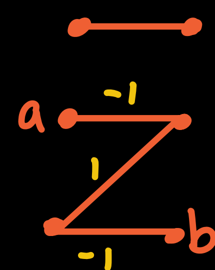
(Case 2) G_{x^*} is acyclic, E_f nonempty

$\Rightarrow$ any connected comp. of G_{x^*} is a tree

$\Rightarrow$ G_{x^*} has two vertices a, b of $\deg = 1$.

Let P = unique a-b path in G_{x^*}; $P = (e_1, \dots, e_k)$

Define $w \in \mathbb{R}^E$ by $w_e = \begin{cases} 1 & \text{if } e = e_{2i} \in P \\ -1 & \text{if } e = e_{2i+1} \in P \\ 0 & \text{if } e \notin P \end{cases}$



For $\varepsilon = \min\{x_e : e \in P\} \cup \{1 - x_e : e \in P\}$, $\underline{x^* \pm \varepsilon w \text{ feasible}}$

$\Longrightarrow x^*$ not a vertex

Check: 1) $(x^* \pm \varepsilon w)_e$ in $[0,1]$

2) $\sum_{e \ni v} (x^* \pm \varepsilon w)_e = \sum_{e \ni v} (x^* \pm \varepsilon w)$ $\forall v \in V \setminus \{a, b\}$

3) For $v=a,b$, $\exists$ unique edge $\tilde{e} \ni v$ with $x^*_{\tilde{e}} \in (0,1)$

$$\Rightarrow \sum_{e \ni v} x^*_e = x^*_{\tilde{e}} + \sum_{\substack{e \ni v \\ e \neq \tilde{e}}} x^*_e \in [0,1] \Rightarrow \sum_{e \ni v} x^*_e = x^*_{\tilde{e}} \in (0,1)$$

must be integer $<1 \Rightarrow =0$

$$\Rightarrow \sum_{e \ni v} (x^* \pm \varepsilon w) = (x^* \pm \varepsilon w)_{\tilde{e}} \in [0,1]$$

(Case 3) $E_{x^*} = \emptyset \Rightarrow x^* \in \mathbb{Z}^E$

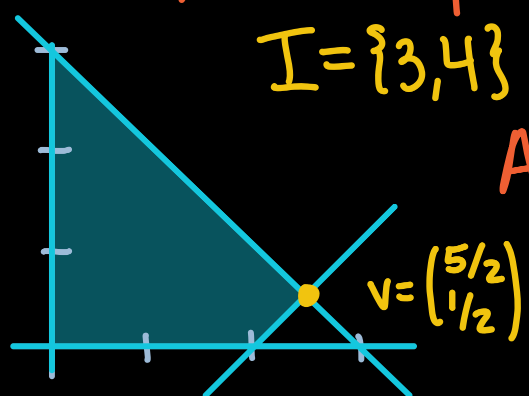
What is special about bipartite graphs?

Recall: A point v in $P = \{x \in \mathbb{R}^n : Ax \le b\}$ is a vertex of $P \Leftrightarrow \exists\, I \subseteq \{1,\dots,m\}$ with $|I| = n$ s.t. the $n \times n$ submatrix A_I with rows $\{a_i^T : i \in I\}$ has full rank n and $a_i^T v = b_i$ for all $i \in I$.

$\Rightarrow v = A_I^{-1} b_I$ (i.e. v is the unique sol. to $a_i^T x = b_i \; \forall i \in I$)

Ex: (P) max $2x_1 + x_2$ s.t. $-x_1 \le 0, -x_2 \le 0, \underbrace{x_1 + x_2 \le 3}_{v_1+v_2=3}, \underbrace{x_1 - x_2 \le 2}_{v_1-v_2=2}$

$I = \{3,4\}$



$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \quad A_{34} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad v = A_{34}^{-1} b_{34} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 1/2 \end{pmatrix}$$

For 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

If $a,b,c,d \in \mathbb{Z}$ and $\det(A) = \pm 1$ then entries of A^{-1} are also integer!

More generally...

Cramers Rule: Let $B \in \mathbb{R}^{n \times n}$ with $\text{rank}(B) = n$.
Then B^{-1} has entries

$$(B^{-1})_{ij} = (-1)^{i+j} \left(\frac{1}{\det(B)} \right) \det(M_{ji})$$

where M_{ji} is the $(n-1) \times (n-1)$ matrix obtained by removing the j^{th} row and i^{th} col of B.

Cor: If $B \in \mathbb{Z}^{n \times n}$ and $\det(B) = \pm 1$, then $B^{-1} \in \mathbb{Z}^{n \times n}$.

$\Longrightarrow$ for all $b \in \mathbb{Z}^n$, $B^{-1} b \in \mathbb{Z}^n$

Def: A matrix $A \in \mathbb{R}^{m \times n}$ is totally unimodular (TU) if every square submatrix has determinant $0, \pm 1$.

(1×1 submatrices $\Longrightarrow$ all entries of A are $0, \pm 1$)

Cor: If A is TU, for any $b \in \mathbb{Z}^m$, all vertices of $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ are integer.