Math 407 – Self-Diagnostic Test – Solutions

Problems (1)-(5) below refer to the following system of equations:

$$\begin{aligned} 2x_2 + 2x_3 + x_4 + x_5 &= b_1 \\ x_1 + x_2 + x_3 + x_4 + 2x_5 &= b_2 \\ -x_1 + x_2 + x_3 - x_5 &= b_3 \end{aligned}$$

where $\mathbf{b} = (b_1, b_2, b_3)^T \in \mathbb{R}^3$.

(1) Find a matrix A that expresses the linear system above in the form $A\mathbf{x} = \mathbf{b}$.

$$A = \begin{pmatrix} 0 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ -1 & 1 & 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

(2) Find a basis for each of the following subspaces:

(a) the row span of A

Performing row operations on A to put it in reduced row-echelon form, we see that

$$\text{rowspan}(A) = \text{rowspan} \begin{pmatrix} 1 & 0 & 0 & 1/2 & 3/2 \\ 0 & 1 & 1 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From this, we see that A has rank two and that one basis for the rowspan is $\{(1, 0, 0, 1/2, 3/2), (0, 1, 1, 1/2, 1/2)\}$.

Other solutions:$\{(0, 2, 2, 1, 1), (1, 1, 1, 1, 2)\}$ or any $\{v, w\}$ where v, w are linearly independent vectors in the rowspan of A.

(b) the column span of A

Since A has rank two, its column span has dimension two and any two linearly independent vectors in the column span form a basis. We can take the first two columnns: $\{(0, 1, -1)^T, (2, 1, 1)^T\}$.

(c) the right kernel of A, $\{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$

Row operations do not change the right kernel of A. So

$$\ker(A) = \ker \begin{pmatrix} 1 & 0 & 0 & 1/2 & 3/2 \\ 0 & 1 & 1 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A vector $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)^T$ belongs to the right kernel if and only if $x_1 + \frac{1}{2}x_4 + \frac{3}{2}x_5 = 0$ and $x_2 + x_3 + \frac{1}{2}x_4 + \frac{1}{2}x_5 = 0$. Using x_3, x_4, x_5 to parametrize the kernel, we see that one basis for the right kernel is

$$\{(0, -1, 1, 0, 0), (-1/2, -1/2, 1, 0, 0), (-3/2, -1/2, 0, 0, 1)\}.$$

(d) the left kernel of A, $\{\mathbf{y} : \mathbf{y}^T A = \mathbf{0}\}$

Since A has rank two, its column span is two-dimensional and so its left kernel is one-dimensional. Using the first two columns, we see that a vector $\mathbf{y} = (y_1, y_2, y_3)^T$

belongs to the left kernel if and only if $y_2 - y_3 = 0$ and $2y_1 + y_2 + y_3 = 0$. One basis for this one-dimensional linear space is $\{(-1, 1, 1)^T\}$

(3) Describe the possible dimensions of the set of solutions to the system of equations $A\mathbf{x} = \mathbf{b}$ (without knowing the value of $\mathbf{b}$) and give an example $\mathbf{b}$ for each.
Since the right kernel of A is three-dimensional, if there any $\tilde{\mathbf{x}}$ with $A\tilde{\mathbf{x}} = \mathbf{b}$, then the set of solutions is three-dimensional, namely $\{\tilde{\mathbf{x}} + \mathbf{v} : A\mathbf{v} = \mathbf{0}\}$. There is such a solution $\tilde{\mathbf{x}}$ if and only if $\mathbf{b}$ belongs to the column span of A, e.g. $\mathbf{b} = (0, 1, -1)^T$.

If $\mathbf{b}$ does not belong to the column span, the the set of solutions is empty. For example, $\mathbf{b} = (2, 2, 1)^T$ does not belong to the column span and so the solution set to $A\mathbf{x} = \mathbf{b}$ is empty.

(4) Give a parametrization of $\{\mathbf{x} : A\mathbf{x} = \mathbf{b}\}$ for $\mathbf{b} = (3, 1, 2)^T$.
Performing row operations on the 3×6 matrix $(A|\mathbf{b})$ we find that

$$(A|\mathbf{b}) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1/2 & 3/2 & -1/2 \\ 0 & 1 & 1 & 1/2 & 1/2 & 3/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

One parametrization of the solution set is

$$\{(-1/2 - t/2 - 3u/2, -3/2 - s - t/2 - u/2, s, t, u) : s, t, u \in \mathbb{R}\}$$

(5) Let U be an invertible 3×3 matrix. Describe the row span, column span, right kernel, and left kernel of the matrix UA and their relation to your answers from (2).
The row span of UA is the same as the row span of A and the right kernel of UA is the same as the right kernel of A.

The column span of UA equals $\{U\mathbf{v} : \mathbf{v} \in \text{columnspan}(A)\}$, which will be a 2-dimensional linear subspace of $\mathbb{R}^3$. Similarly, the left kernel of UA equals the set of vectors $U^{-T}\mathbf{y}$ where $\mathbf{y}$ belongs to the left kernel of A. So the left kernel of A is a one-dimensional linear subspace spanned by the vector $U^{-T}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

Problems (6)-(7) refer to the plane in $\mathbb{R}^3$ parametrized by

$$\{(5+t, 1+s+t, 2s+t) : s, t \in \mathbb{R}\}.$$

(6) Find the defining equation of this plane.
The coefficient vector of the equation is must be orthogonal to the coefficient vectors of the parameters s and t, $(0,1,2)$ and $(1,1,1)$, which gives $(1,-2,1)$. Using the point $(5,1,0$, we see that this plane is the set of (x,y,z) satisfying $x-2y+z=3$.

(7) Find the point in this plane closest to $(1,0,3)$.
The line between $(1,0,3)$ and the closest point on the plane is orthogonal to the plane. So this line can be parametrizes as $\{(1,0,3)+\lambda(1,-2,1) : \lambda \in \mathbb{R}\}$ To find the unique value of λ for which this lies on the plane, we plug $(x,y,z)=(1+\lambda,-2\lambda,3+\lambda)$ into the equation $x-2y+z=3$, which gives $\lambda=-1/6$. The closest point is therefore

$$(1,0,3)-1/6(1,-2,1)=(5/6,2/6,17/6).$$