Math 407 – Midterm

Autumn 2022

Time: 50 mins.

1. This exam is closed book and closed notes. You may not use any calculators, technological devices, or outside materials besides a writing utensil.
2. Answer all questions in the spaces provided. If you run out of room for an answer, continue on the back of the page.
3. Unless stated otherwise, justify your answers to receive full credit. Your answers do not have to be in complete sentences, but they do need to be understandable.

Name: ______________________________

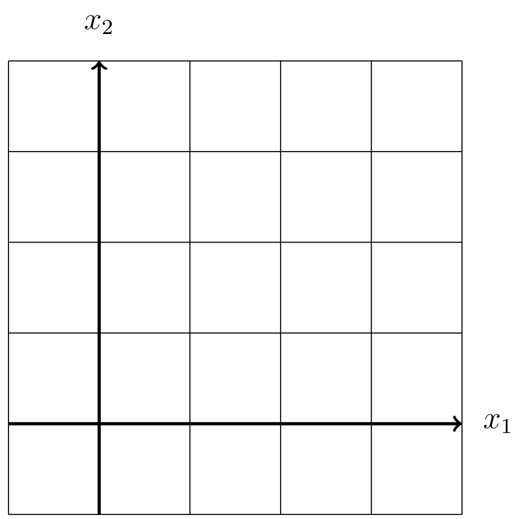
Student ID #: ______________________________

Question	Points	Score
1	18	
2	10	
3	10	
4	12	
5	10	
Total:	60	

1. (18 points) Parts (a) - (c) refer to the following linear program:

$$\max \quad 2x_1 + x_2 \quad \text{such that} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_1 + x_2 \leq 3, \quad x_1 - x_2 \leq 1, \quad x_1 \leq 2$$

(a) Plot the feasible region of this linear program and solve it graphically.
(An appropriately drawn and labeled line in your picture can serve as justification.)



optimal value =

optimal solution =

(b) Give a matrix A and vectors $\mathbf{b}$, $\mathbf{c}$ so that $\max \mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{b}$ is a reformulation of this linear program.

(c) List the vertices of the feasible region of the linear program in equational form and, for each vertex, list *every* feasible basis that corresponds to it.
(Use your picture in (a) to determine the vertices of the feasible region.)

2. (10 points) Here is a simplex tableau of an auxiliary linear program:

$$\begin{aligned} x_1 &= 1 + x_2 + x_3 + x_5 - x_7 \\ x_6 &= 2 - x_2 + x_3 + x_4 \\ z &= -2 + x_2 - x_3 - x_4 - x_7 \end{aligned}$$

(a) What is the number of variables and number of equations in the original linear program? (Do not justify your answer.)

variables =   # equations =

(b) Perform a pivot step on the tableau above and write down the resulting feasible basis and new simplex tableau.

(c) Is the original linear program feasible? If no, explain why. If yes, provide a basic feasible solution of the original linear program.

3. (10 points) Here is a simplex tableau of a linear program with a mystery parameter λ in the last row.

$$\begin{aligned} x_1 &= 9 + x_4 + x_5 - 2x_6 \\ x_2 &= 5 + x_5 - x_6 \\ x_3 &= 3 - x_6 \\ z &= 8 + \lambda x_5 - 2x_6 \end{aligned}$$

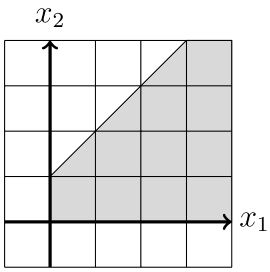
(a) For what values of $\lambda \in \mathbb{R}$ is this linear program bounded? (Justify your answer.)

(b) If λ satisfies your answer from (a), what is the optimal value and optimal solution in $\mathbb{R}^6$ of this linear program?

4. (12 points) Consider the linear program

$$\max\ \mathbf{c}^T\mathbf{x} \quad \text{s.t.} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_2 - x_1 \leq 1$$

whose feasible region is shown below.



In each part below, **circle all** choices of $\mathbf{c}^T$ that make the linear program have the requested property.

Do not justify your answers.

(a) The linear program is unbounded.

$\mathbf{c}^T = (-2, 1)$ $\quad$ $\mathbf{c}^T = (-1, 0)$ $\quad$ $\mathbf{c}^T = (-1, 1)$ $\quad$ $\mathbf{c}^T = (1, -1)$ $\quad$ $\mathbf{c}^T = (1, 3)$

(b) The linear program is feasible.

$\mathbf{c}^T = (-2, 1)$ $\quad$ $\mathbf{c}^T = (-1, 0)$ $\quad$ $\mathbf{c}^T = (-1, 1)$ $\quad$ $\mathbf{c}^T = (1, -1)$ $\quad$ $\mathbf{c}^T = (1, 3)$

(c) The linear program attains its optimal value at $(x_1, x_2) = (0, 1)$.

$\mathbf{c}^T = (-2, 1)$ $\quad$ $\mathbf{c}^T = (-1, 0)$ $\quad$ $\mathbf{c}^T = (-1, 1)$ $\quad$ $\mathbf{c}^T = (1, -1)$ $\quad$ $\mathbf{c}^T = (1, 3)$

(d) The linear program has more than one optimal solution.

$\mathbf{c}^T = (-2, 1)$ $\quad$ $\mathbf{c}^T = (-1, 0)$ $\quad$ $\mathbf{c}^T = (-1, 1)$ $\quad$ $\mathbf{c}^T = (1, -1)$ $\quad$ $\mathbf{c}^T = (1, 3)$

5. (10 points) In each part below, circle True or False. Do not justify your answers.

(a) Every nonempty polyhedron of the form $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$ has a vertex.

True False

(b) A linear program is bounded if and only if its feasible region is bounded.

True False

(c) The set of points achieving the optimal value of a bounded linear program is convex.

True False

(d) If $\mathbf{v}$ and $\mathbf{w}$ are basic feasible solutions of a linear program, then so is $\lambda\mathbf{v} + (1-\lambda)\mathbf{w}$ for all $0 \leq \lambda \leq 1$.

True False

(e) Every basic feasible solution of a linear program $\max \mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq 0$ with $A \in \mathbb{R}^{m\times n}$ has exactly $n - m$ coordinates equal to zero.

True False