Math 407 – Worksheet

Wednesday, November 16

Work in groups of two to four people. There are two problems on the front page and two problems on the back. You probably won't be able to make it through all four within the 50 minute class. I suggest trying Problem 1 or Problem 2 first. If you finish both problems on the front page, try the more challenging problems on the back.

Problem 1. Consider the linear program

$$\begin{aligned} \max \quad 3x_1 + x_2 \quad \text{s.t.} \quad & x_1 + x_2 + x_3 = 5 \\ & x_1 - x_2 + 2x_3 \geq 0 \\ & x_1 \geq 0 \\ & x_1 - x_2 \leq 2 \end{aligned}$$

(a) Consider the linear combination

$$y_1(x_1 + x_2 + x_3 = 5) + y_2(x_1 - x_2 + 2x_3 \geq 0) + y_3(x_1 \geq 0) + y_4(x_1 - x_2 \leq 2).$$

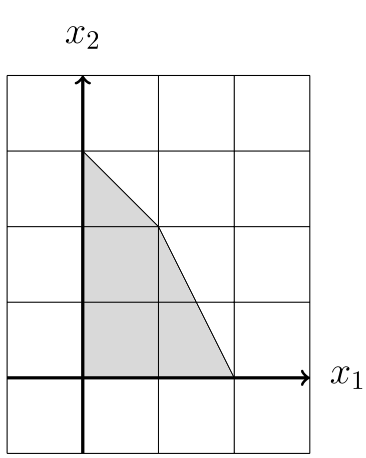
How should y_1, y_2, y_3, y_4 be so that this gives an upper bound on $3x_1 + x_2$. What upper bound does it give?

(b) Write the dual linear program that finds the best such upper bound on $3x_1 + x_2$.

(c) I claim that $\mathbf{x}^* = (4, 2, -1)^T$ is optimal for the primal and $\mathbf{y}^* = (2, -1, 0, 2)^T$ is optimal for the dual linear program. What needs to be checked to verify this?

Problem 2. Consider the (primal) linear program

$$\max \quad \mathbf{c}^T\mathbf{x} \quad \text{s.t.} \quad x_1 \geq 0, \ x_2 \geq 0, \ 2x_1 + x_2 \leq 4, \ x_1 + x_2 \leq 3$$



(a) For what values of $\mathbf{c}$ is the maximum of this LP attained at $\mathbf{x} = (1, 2)^T$? For what values of $\mathbf{c}$ is the maximum attained uniquely at $\mathbf{x} = (1, 2)^T$?

(b) Pick a cost vector $\mathbf{c}$ at which the maximum is attained uniquely at $\mathbf{x} = (1, 2)^T$, write the dual linear program and draw the feasible of the dual linear program in the (y_3, y_4)-plane.

(c) In the optimal solution $\mathbf{y}^* \in \mathbb{R}^4$ of the dual LP, what coordinates of y_j can be strictly positive?

(d) Repeat (a)-(c) for $\mathbf{x} = (0, 0)^T$ instead of $\mathbf{x} = (1, 2)^T$.

Problem 3 (The dual LP of a linear program in equational form)**.** Consider the primal linear program in equational form:

$$\text{(P)} \qquad \max \;\; \mathbf{c}^T\mathbf{x} \;\; \text{s.t.} \;\; A\mathbf{x} = \mathbf{b}, \;\; \mathbf{x} \geq 0$$

The dual linear program is

$$\text{(D)} \qquad \min \;\; \mathbf{b}^T\mathbf{y} \;\; \text{s.t.} \;\; A^T\mathbf{y} - \mathbf{s} = \mathbf{c}, \;\; \mathbf{s} \geq 0$$

where $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{s} \in \mathbb{R}^n$.

(a) Show that for any feasible point $(\mathbf{y}, \mathbf{s})$ of (D), $\mathbf{b}^T\mathbf{y}$ gives an upper bound on the primal linear program (P).

(b) Show that we can eliminate the variables $\mathbf{s}$ from (D) by replacing "$A^T\mathbf{y} - \mathbf{s} = \mathbf{c}, \;\; \mathbf{s} \geq 0$" with "$A^T\mathbf{y} \geq \mathbf{c}$"

Problem 4 (The dual of the dual)**.** Starting with the primal,

$$\text{(P)} \qquad \max \;\; \mathbf{c}^T\mathbf{x} \;\; \text{s.t.} \;\; A\mathbf{x} \leq \mathbf{b},$$

the dual linear program is

$$\text{(D)} \qquad \min \;\; \mathbf{b}^T\mathbf{y} \;\; \text{s.t.} \;\; A^T\mathbf{y} = \mathbf{c}, \quad \mathbf{y} \geq 0.$$

The dual problem is an LP in equational form!

(a) Rewrite (D) as

$$\text{(D)} \qquad -\max \;\; -\mathbf{b}^T\mathbf{y} \;\; \text{s.t.} \;\; A^T\mathbf{y} = \mathbf{c}, \quad \mathbf{y} \geq 0.$$

(b) Use Problem 3 to write the dual linear program to (D) as

$$\text{(DD)} \qquad -\min \;\; \mathbf{c}^T\mathbf{x} \;\; \text{s.t.} \;\; A\mathbf{x} \geq -\mathbf{b}$$

(c) Replace $\mathbf{x}$ with $-\mathbf{x}$ in the above expression and simplify to show that (DD) is equivalent to the primal problem (P).