Large Example of the Dual Simplex Method

UW Math 407, Fall 2022

Below is a large example of the dual simplex method, carried through until an optimal solution is found. Afterwards, is a side-by-side comparison of using the usual simplex method on the dual LP.

Large Example

Original LP: $\max -x_1 - x_2 - x_3$ such that $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$ where

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} -1 \\ -2 \\ -1 \\ -3 \end{pmatrix}$$

From inspection, we see that the basis $B = \{4, 5, 6, 7\}$ is dual feasible, so we can start the dual simplex method from there:
$B = \{4, 5, 6, 7\}$, $\mathcal{T}(B)$:

$$\begin{aligned} x_4 &= -1 - x_1 + x_2 \\ x_5 &= -2 - x_1 + x_2 + x_3 \\ x_6 &= -1 + x_1 - x_3 \\ x_7 &= -3 + x_2 \\ z &= -x_1 - x_2 - x_3 \end{aligned}$$

This basis is not feasible. We see that any of x_4, x_5, x_6, x_7 can leave the basis. We choose x_4 to leave. We then add the highest nonnegative scalar multiple λ of the "$x_4 =$" row to the last row to eliminate some variable. We compute

$$\lambda x_4 + z = -\lambda + (-\lambda - 1)x_1 + (\lambda - 1)x_2 - x_3.$$

For the coefficients of x_1, x_2, x_3 to be ≤ 0, the tightest constraint comes from the coefficient of x_2, so x_2 enters the basis, giving
$B = \{2, 5, 6, 7\}$, $\mathcal{T}(B)$:

$$\begin{aligned} x_2 &= 1 + x_1 + x_4 \\ x_5 &= -1 + x_3 + x_4 \\ x_6 &= -1 + x_1 - x_3 \\ x_7 &= -2 + x_1 + x_4 \\ z &= -1 - 2x_1 - x_3 - x_4 \end{aligned}$$

This basis is also not feasible. Any of x_5, x_6, x_7 can leave. Let's choose x_5 to leave. We add the highest nonnegative scalar multiple λ of the "$x_5 =$" row to the last row to eliminate some variable. We compute

$$\lambda x_5 + z = (-\lambda - 1) - 2x_1 + (\lambda - 1)x_3 + (\lambda - 1)x_4.$$

For the coefficients of x_1, x_3, x_4 to be ≤ 0, the tightest constraint of $\lambda \leq 1$ comes from the coefficient of x_3 and x_4. We can choose either to enter. Let's choose x_3 to enter, giving

$B = \{2, 3, 6, 7\}$, $\mathcal{T}(B)$:

$$\begin{aligned} x_2 &= 1 + x_1 + x_4 \\ x_3 &= 1 - x_4 + x_5 \\ x_6 &= -2 + x_1 + x_4 - x_5 \\ x_7 &= -2 + x_1 + x_4 \\ z &= -2 - 2x_1 - x_5 \end{aligned}$$

This basis is also not feasible. Either of x_6 or x_7 can leave. Let's choose x_6 to leave. We add the highest nonnegative scalar multiple λ of the "$x_6 =$" row to the last row to eliminate some variable. We compute

$$\lambda x_6 + z = (-2\lambda - 2) + (\lambda - 2)x_1 + \lambda x_4 + (-\lambda - 1)x_5$$

For the coefficients of x_1, x_4, x_5 to be ≤ 0, the tightest constraint of $\lambda \leq 0$ comes from the coefficient of x_4, so x_4 to enters, giving
$B = \{2, 3, 4, 7\}$, $\mathcal{T}(B)$:

$$\begin{aligned} x_2 &= 3 + x_5 + x_6 \\ x_3 &= -1 + x_1 - x_6 \\ x_4 &= 2 - x_1 + x_5 + x_6 \\ x_7 &= x_5 + x_6 \\ z &= -2 - 2x_1 - x_5 \end{aligned}$$

This basis is still not feasible. Only x_3 can leave. We add the highest nonnegative scalar multiple λ of the "$x_3 =$" row to the last row to eliminate some variable. We compute

$$\lambda x_3 + z = (-\lambda - 2) + (\lambda - 2)x_1 - x_5 - \lambda x_6$$

For the coefficients of x_1, x_5, x_6 to be ≤ 0, the tightest constraint of $\lambda \leq 2$ comes from the coefficient of x_1, so x_1 to enters, giving
$B = \{1, 2, 4, 7\}$, $\mathcal{T}(B)$:

$$\begin{aligned} x_1 &= 1 + x_3 + x_6 \\ x_2 &= 3 + x_5 + x_6 \\ x_4 &= 1 - x_3 + x_5 \\ x_7 &= x_5 + x_6 \\ z &= -4 - 2x_3 - x_5 - 2x_6 \end{aligned}$$

This basis $B = \{1, 2, 4, 7\}$ is now both feasible an dual feasible, so it corresponds to an optimal solution, namely $\mathbf{x}^* = (1, 3, 0, 1, 0, 0, 0)$.

Side-by-side view of primal and dual

The dual linear program is

$$\min -y_1 - 2y_2 - y_3 - 3y_4 \ \text{ s.t. } \begin{pmatrix} 1 & 1 & -1 & 0 \\ -1 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} - \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \end{pmatrix} \geq 0.$$

As in HW6, Problem 2, we can eliminate the variables $\mathbf{y}$ and rewrite this as

$$\min -s_4 - 2s_5 - s_6 - 3s_7 \quad \text{s.t.} \quad \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \end{pmatrix} \mathbf{s} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \ \text{ and } \mathbf{s} \geq 0.$$

So that we can translate to maximization, let's rewrite this as

$$-\max s_4 + 2s_5 + s_6 + 3s_7 \quad \text{s.t.} \quad \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \end{pmatrix} \mathbf{s} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \ \text{ and } \mathbf{s} \geq 0.$$

The basis $\{1, 2, 3\}$ is clearly feasible. Let us look at a side-by-side progression of the dual simplex method on the original problem and the usual simplex method on this linear program in $\mathbf{s}$:

$\mathcal{T}(\{4,5,6,7\})$:

$$\begin{aligned} x_4 &= -1 - x_1 + x_2 \\ x_5 &= -2 - x_1 + x_2 + x_3 \\ x_6 &= -1 + x_1 - x_3 \\ x_7 &= -3 + x_2 \\ z &= -x_1 - x_2 - x_3 \end{aligned}$$

x_4 leaves, x_2 enters

$\mathcal{T}(\{1,2,3\})$:

$$\begin{aligned} s_1 &= 1 + s_4 + s_5 - s_6 \\ s_2 &= 1 - s_4 - s_5 - s_7 \\ s_3 &= 1 - s_5 + s_6 \\ z &= s_4 + 2s_5 + s_6 + 3s_7 \end{aligned}$$

s_4 enters, s_2 leaves

$\mathcal{T}(\{2,5,6,7\})$:

$$\begin{aligned} x_2 &= 1 + x_1 + x_4 \\ x_5 &= -1 + x_3 + x_4 \\ x_6 &= -1 + x_1 - x_3 \\ x_7 &= -2 + x_1 + x_4 \\ z &= -1 - 2x_1 - x_3 - x_4 \end{aligned}$$

x_5 leaves, x_3 enters

$\mathcal{T}(\{1,3,4\})$:

$$\begin{aligned} s_1 &= 2 - s_2 - s_6 - s_7 \\ s_3 &= 1 - s_5 + s_6 \\ s_4 &= 1 - s_2 - s_5 - s_7 \\ z &= 1 - s_2 + s_5 + s_6 + 2s_7 \end{aligned}$$

s_5 enters, s_3 leaves

$\mathcal{T}(\{2,3,6,7\})$:

$x_2 = 1 + x_1 + x_4$

$x_3 = 1 - x_4 + x_5$

$x_6 = -2 + x_1 + x_4 - x_5$

$x_7 = -2 + x_1 + x_4$

$z = -2 - 2x_1 - x_5$

x_6 leaves, x_4 enters

$\mathcal{T}(\{2,3,4,7\})$:

$x_2 = 3 + x_5 + x_6$

$x_3 = -1 + x_1 - x_6$

$x_4 = 2 - x_1 + x_5 + x_6$

$x_7 = x_5 + x_6$

$z = -2 - 2x_1 - x_5$

x_3 leaves, x_1 enters

$\mathcal{T}(\{1,2,4,7\})$:

$x_1 = 1 + x_3 + x_6$

$x_2 = 3 + x_5 + x_6$

$x_4 = 1 - x_3 + x_5$

$x_7 = x_5 + x_6$

$z = -4 - 2x_3 - x_5 - 2x_6$

Done!

$\mathcal{T}(\{1,4,5\})$:

$s_1 = 2 - s_2 - s_6 - s_7$

$s_4 = -s_2 + s_3 - s_6 - s_7$

$s_5 = 1 - s_3 + s_6$

$z = 2 - s_2 - s_3 + 2s_6 + 2s_7$

s_6 enters, s_4 leaves

$\mathcal{T}(\{1,5,6\})$:

$s_1 = 2 - s_3 + s_4$

$s_5 = 1 - s_2 - s_4 - s_7$

$s_6 = -s_2 + s_3 - s_4 - s_7$

$z = 2 - 3s_2 + s_3 - 2s_4$

s_3 enters, s_1 leaves

$\mathcal{T}(\{3,5,6\})$:

$s_3 = 2 - s_1 + s_4$

$s_5 = 1 - s_2 - s_4 - s_7$

$s_6 = 2 - s_1 - s_2 - s_7$

$z = 4 - s_1 - 3s_2 - s_4$

Done!