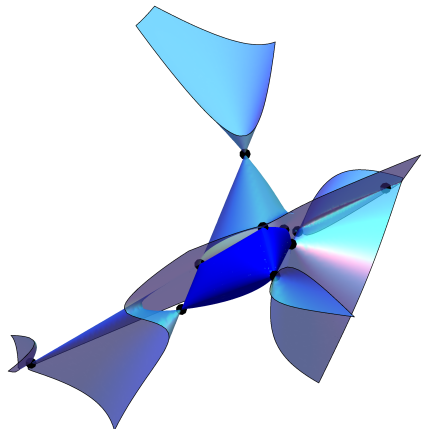


# Quartic Symmetroids and Spectrahedra



Cynthia Vinzant,  
University of Michigan

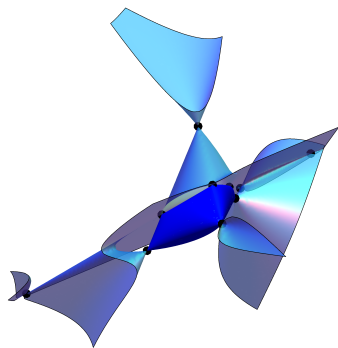
with  
John Christian Ottem,  
Kristian Ranestad, and  
Bernd Sturmfels

# Quartic Symmetroids

A **quartic symmetroid** is a surface  $\mathcal{V}(f) \subset \mathbb{P}^3(\mathbb{C})$  given by

$$f = \det(A(x)) = \det(x_0A_0 + x_1A_1 + x_2A_2 + x_3A_3)$$

where  $A_0, A_1, A_2, A_3$  are  $4 \times 4$  **symmetric** matrices.



## Fun facts:

- ▶  $\mathcal{V}(f)$  has 10 nodes (of rank 2)
- ▶ co-dimension 10 in  $\mathbb{P}(\mathbb{C}[x_0, x_1, x_2, x_3]_4)$
- ▶ studied by Cayley in a set of memoirs 1869 - 1871

# Real Spectrahedral Symmetroids

Here I'll talk about in surfaces  $\mathcal{V}(\det(A(x)))$  where

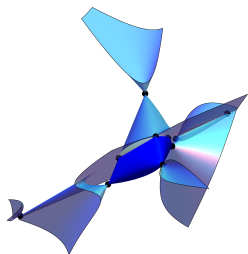
- ▶ the matrices  $A_0, A_1, A_2, A_3$  are **real** and
- ▶ their span contains a **positive definite** matrix.

**Motivation 1:** The convex sets  $\{x \in \mathbb{R}^4 : A(x) \succeq 0\}$  appear as feasible sets (spectrahedra) in semidefinite programming.

**Motivation 2:** Having a positive definite matrix puts interesting constraints on the surface  $\mathcal{V}_{\mathbb{R}}(f)$ .

For example ...

Friedland *et. al.* (1984) showed that in this case  $\mathcal{V}(\det(A(x)))$  has a *real* node.



# Linear spaces of matrices and spectrahedra

Let  $A_0, A_1, \dots, A_n$  be real symmetric  $d \times d$  matrices and

$$A(x) = x_0 A_0 + x_1 A_1 + \dots + x_n A_n.$$

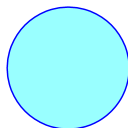
**Spectrahedron:**  $\{x \in \mathbb{R}^{n+1} : A(x) \text{ is positive semidefinite}\}$

*projectivize*  $\rightarrow \{x \in \mathbb{P}^n(\mathbb{R}) : A(x) \text{ is semidefinite}\}$

(bounded by the hypersurface  $\mathcal{V}(\det(A(x)))$ )

Example:

$$A(x) = \begin{pmatrix} x_0 + x_1 & x_2 \\ x_2 & x_0 - x_1 \end{pmatrix}$$

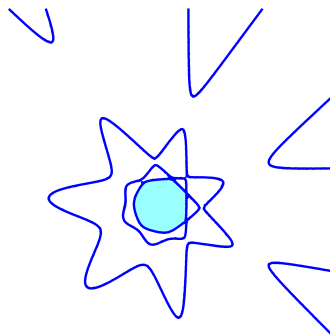
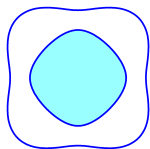


**Goal:** Understand the **algebraic** and **topological** properties of spectrahedra and their bounding polynomials.

# Polynomials bounding spectrahedra

Spectrahedra are bounded by **hyperbolic polynomials**,  $\det(A(x))$ .

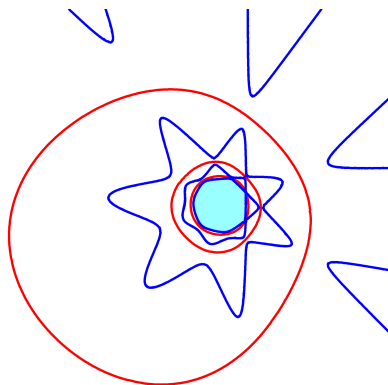
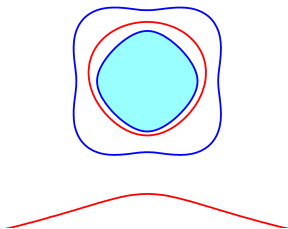
A polynomial  $f$  is **hyperbolic** with respect to a **point**  $p$  if every real line through  $p$  meets  $\mathcal{V}(f)$  in only real points.



**Theorem (Helton-Vinnikov 2007).** A polynomial  $f \in \mathbb{R}[x_0, x_1, x_2]_d$  bounds a spectrahedron if and only if  $f$  is hyperbolic.

# Spectrahedra and interlacers

The diagonal  $(d - 1) \times (d - 1)$  minors of  $A(x)$  interlace the determinant  $\det(A(x))$ .



**Theorem (Plaumann-V. 2013).** The matrix  $A(x)$  is definite at some point if and only if its minors interlace the determinant.

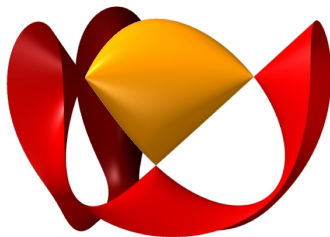
# Determinantal surfaces and 3-dim'l spectrahedra ( $n = 3$ )

The variety of rank- $(d - 2)$  matrices in  $\mathbb{C}_{sym}^{d \times d}$  has

codimension 3 and degree  $\binom{d+1}{3}$ .

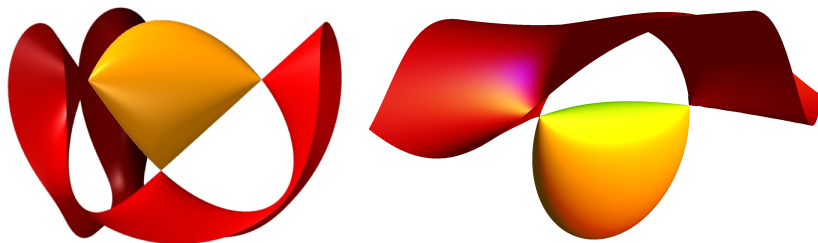
Generically, the  $\text{span}_{\mathbb{C}}\{A_0, A_1, A_2, A_3\}$  meets this variety **transversely** and contains  $\binom{d+1}{3}$  matrices of rank  $d - 2$ .

The complex surface  $\mathcal{V}(\det(A(x)))$  bounding a three-dimensional spectrahedron has  $\binom{d+1}{3}$  nodes.



# Three-dimensional spectrahedra bounded by cubics

Over  $\mathbb{C}$  there are (generically) 4 nodes of rank one.

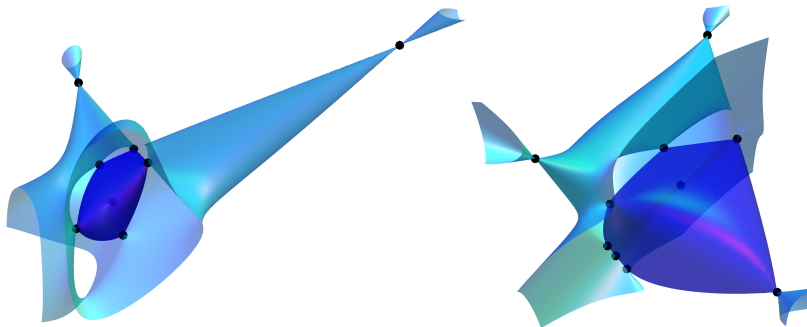


Either 2 or 4 of them are real and lie on the spectrahedron.



# Three-dimensional spectrahedra bounded by quartics

Over  $\mathbb{C}$  there are generically 10 nodes of rank two.

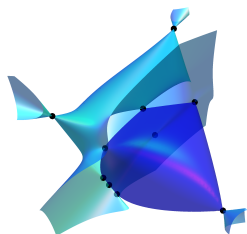


There are two flavors of real node (**on** or **off** the spectrahedron).  
What configurations are possible?

# Theorem (Degtyarev-Itenberg, 2011)

There is a (transversal) quartic spectrahedron with  $\alpha$  nodes on its boundary and  $\beta$  nodes on its real surface if and only if

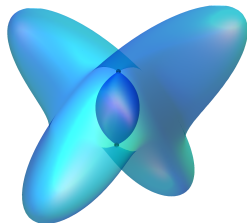
$$\alpha, \beta \text{ are even} \quad \text{and} \quad 2 \leq \alpha + \beta \leq 10.$$



$$\alpha = 8$$
$$\beta = 2$$



$$\alpha = 0$$
$$\beta = 10$$



$$\alpha = 2$$
$$\beta = 0$$

# Back to the classics (Cayley's Symmetroids)

**Idea of Cayley:** Look at the projection of  $\mathcal{V}(f)$  from a node  $p$ .

This projection  $\pi_p : \mathcal{V}(f) \rightarrow \mathbb{P}^2$  from a node  $p$  is a double cover of  $\mathbb{P}^2$  whose **branch locus** is a **sextic curve**.

**Why?** If  $p = [1 : 0 : 0 : 0]$  then

$$f = a \cdot x_0^2 + b \cdot x_0 + c \quad \text{where} \quad a, b, c \in \mathbb{R}[x_1, x_2, x_3].$$

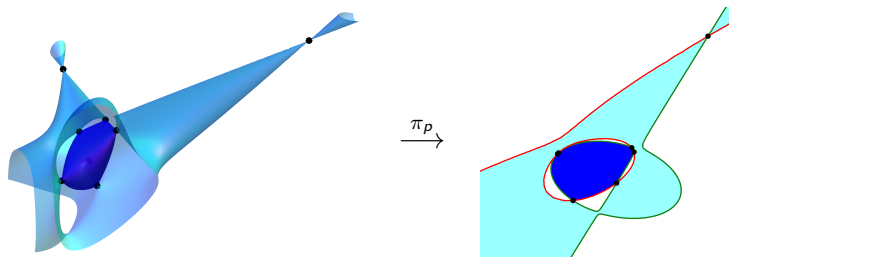
The branch locus of  $\pi_p$  is  $\mathcal{V}(b^2 - 4ac)$ .

# Projection from a node

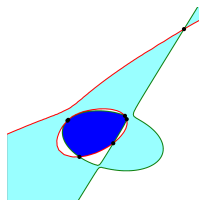
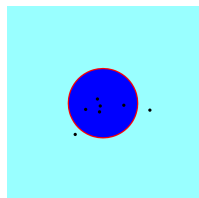
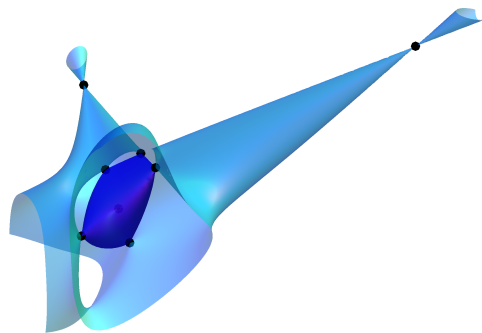
## Theorem (Cayley 1869-71)

A quartic  $f \in \mathbb{C}[x_0, x_1, x_2, x_3]_4$  with node  $p$  is a symmetroid if and only if the branch locus of  $\pi_p$  is the product of two cubics,  $b_1 \cdot b_2$ .

Moreover the images of the other 9 nodes are  $\mathcal{V}(b_1) \cap \mathcal{V}(b_2)$ .



# The view from a node on or off the spectrahedron



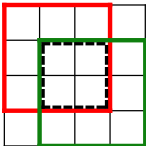
For  $p \in \text{Spec}$ ,  $b_1 = \overline{b_2}$ .

The image  $\pi_p(\text{Spec})$  is the conic  $\{a \geq 0\}$ .

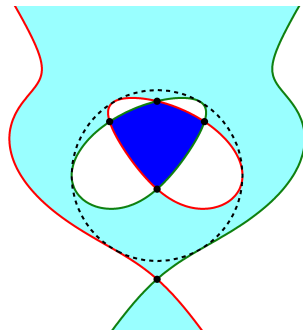
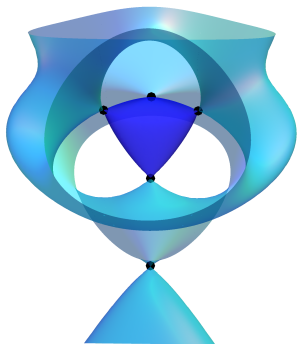
For  $p \notin \text{Spec}$ ,  $b_1, b_2$  are real and hyperbolic.

The image  $\pi_p(\text{Spec})$  is the intersection of cubic ovals.

# The view from a node: interlacing branch locus

$$\text{If } p = [1 : 0 : 0 : 0] \text{ and } A(x) = x_0 \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} +$$


then the branch cubics  $b_1, b_2$  are diagonal minors of  $A(0, x_1, x_2, x_3)$ .

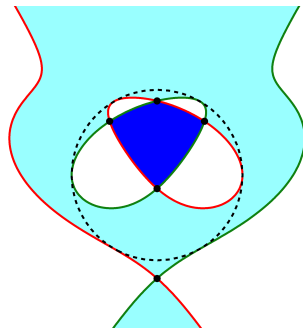
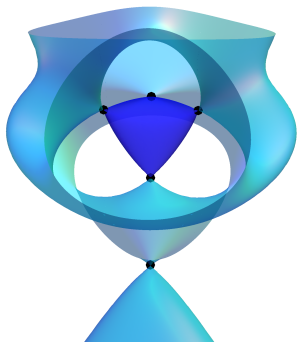


# The view from a node: interlacing branch locus

The image of the spectrahedron is the intersection of cubic ovals.

→ There are an even number of spectrahedral nodes.

To understand the other direction of the Degtyarev-Itenberg Theorem ...



# $(A_0 A_1 A_2 A_3)$ giving different types of spectrahedra

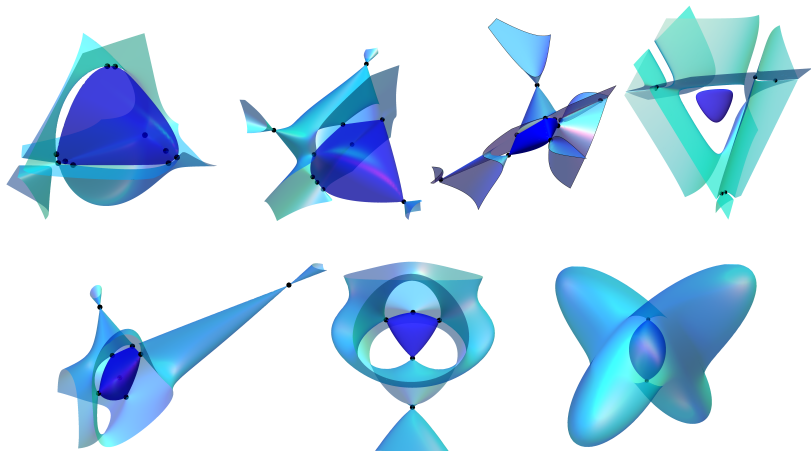
$(2, 2)$ :	$\begin{bmatrix} 3 & 4 & 1 & -4 \\ 4 & 14 & -6 & -10 \\ 1 & -6 & 9 & 2 \\ -4 & -10 & 2 & 8 \end{bmatrix}$	$\begin{bmatrix} 11 & 0 & 2 & 2 \\ 0 & 6 & -1 & 4 \\ 2 & -1 & 6 & 2 \\ 2 & 4 & 2 & 4 \end{bmatrix}$	$\begin{bmatrix} 17 & -3 & 2 & 9 \\ -3 & 6 & -4 & 1 \\ 2 & -4 & 13 & 10 \\ 9 & 1 & 10 & 17 \end{bmatrix}$	$\begin{bmatrix} 9 & -3 & 9 & 3 \\ -3 & 10 & 6 & -7 \\ 9 & 6 & 18 & -3 \\ 3 & -7 & -3 & 5 \end{bmatrix}$
$(4, 4)$ :	$\begin{bmatrix} 18 & 3 & 9 & 6 \\ 3 & 5 & -1 & -3 \\ 9 & -1 & 13 & 7 \\ 6 & -3 & 7 & 9 \end{bmatrix}$	$\begin{bmatrix} 17 & -10 & 4 & 3 \\ -10 & 14 & -1 & -3 \\ 4 & -1 & 5 & -4 \\ 3 & -3 & -4 & 6 \end{bmatrix}$	$\begin{bmatrix} 8 & 6 & 10 & 10 \\ 6 & 18 & 6 & 15 \\ 10 & 6 & 14 & 9 \\ 10 & 15 & 9 & 22 \end{bmatrix}$	$\begin{bmatrix} 8 & -4 & 8 & 0 \\ -4 & 10 & -4 & 0 \\ 8 & -4 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
$(6, 6)$ :	$\begin{bmatrix} 10 & 8 & 2 & 6 \\ 8 & 14 & 0 & 2 \\ 2 & 0 & 5 & 7 \\ 6 & 2 & 7 & 11 \end{bmatrix}$	$\begin{bmatrix} 11 & -6 & 10 & 9 \\ -6 & 10 & -5 & -5 \\ 10 & -5 & 14 & 11 \\ 9 & -5 & 11 & 9 \end{bmatrix}$	$\begin{bmatrix} 6 & 2 & 6 & -5 \\ 2 & 9 & 2 & 0 \\ 6 & 2 & 6 & -5 \\ -5 & 0 & -5 & 5 \end{bmatrix}$	$\begin{bmatrix} 8 & 6 & 2 & -2 \\ 6 & 9 & 9 & 6 \\ 2 & 9 & 13 & 12 \\ -2 & 6 & 12 & 13 \end{bmatrix}$
$(8, 8)$ :	$\begin{bmatrix} 5 & 3 & -3 & -4 \\ 3 & 6 & -3 & -2 \\ -3 & -3 & 6 & 4 \\ -4 & -2 & 4 & 4 \end{bmatrix}$	$\begin{bmatrix} 19 & 10 & 12 & 17 \\ 10 & 14 & 10 & 7 \\ 12 & 10 & 10 & 11 \\ 17 & 7 & 11 & 17 \end{bmatrix}$	$\begin{bmatrix} 5 & 1 & 3 & -3 \\ 1 & 5 & -7 & -1 \\ 3 & -7 & 22 & 7 \\ -3 & -1 & 7 & 10 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 4 & 4 \\ 2 & 2 & 4 & 8 \end{bmatrix}$
$(10, 10)$ :	$\begin{bmatrix} 18 & 6 & 6 & -6 \\ 6 & 2 & 2 & -2 \\ 6 & 2 & 2 & -2 \\ -6 & -2 & -2 & 4 \end{bmatrix}$	$\begin{bmatrix} 4 & -6 & 6 & 4 \\ -6 & 13 & -9 & -8 \\ 6 & -9 & 9 & 6 \\ 4 & -8 & 6 & 5 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 4 & 0 & 6 \\ -3 & 0 & 9 & 0 \\ 0 & 6 & 0 & 9 \end{bmatrix}$	$\begin{bmatrix} 9 & -3 & 0 & 0 \\ -3 & 10 & 9 & -6 \\ 0 & 9 & 9 & -6 \\ 0 & -6 & -6 & 4 \end{bmatrix}$
$(2, 0)$ :	$\begin{bmatrix} 20 & 6 & -14 & -4 \\ 6 & 18 & 3 & -12 \\ -14 & 3 & 17 & -2 \\ -4 & -12 & -2 & 8 \end{bmatrix}$	$\begin{bmatrix} 54 & -27 & 16 & 12 \\ -27 & 18 & -2 & -15 \\ 16 & -2 & 20 & -10 \\ 12 & -15 & -10 & 21 \end{bmatrix}$	$\begin{bmatrix} 42 & -8 & 9 & -3 \\ -8 & 10 & 5 & -11 \\ 9 & 5 & 29 & 7 \\ -3 & -11 & 7 & 29 \end{bmatrix}$	$\begin{bmatrix} 0 & 9 & 3 & -3 \\ 9 & -9 & -6 & 6 \\ 3 & -6 & -3 & 3 \\ -3 & 6 & 3 & -3 \end{bmatrix}$
$(4, 2)$ :	$\begin{bmatrix} 9 & -4 & 1 & 1 \\ -4 & 5 & -3 & -2 \\ 1 & -3 & 3 & 1 \\ 1 & -2 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 6 & 1 & 3 & 4 \\ 1 & 5 & 5 & 2 \\ 3 & 5 & 6 & 2 \\ 4 & 2 & 2 & 8 \end{bmatrix}$	$\begin{bmatrix} 8 & 2 & -6 & 4 \\ 2 & 5 & 1 & 3 \\ -6 & 1 & 6 & -2 \\ 4 & 3 & -2 & 3 \end{bmatrix}$	$\begin{bmatrix} -4 & 4 & -2 & 2 \\ 4 & 0 & 0 & -2 \\ -2 & 0 & 0 & 1 \\ 2 & -2 & 1 & -1 \end{bmatrix}$
$(6, 4)$ :	$\begin{bmatrix} 6 & -1 & 5 & 5 \\ -1 & 2 & 1 & -3 \\ 5 & 1 & 6 & 2 \\ 5 & -3 & 2 & 9 \end{bmatrix}$	$\begin{bmatrix} 5 & -5 & 5 & -3 \\ -5 & 6 & -5 & 5 \\ 5 & -5 & 5 & -3 \\ -3 & 5 & -3 & 9 \end{bmatrix}$	$\begin{bmatrix} 6 & -3 & 5 & 2 \\ -3 & 5 & -3 & 2 \\ 5 & -3 & 9 & -4 \\ 2 & 2 & -4 & 9 \end{bmatrix}$	$\begin{bmatrix} 0 & -2 & -2 & 0 \\ -2 & 1 & 2 & 1 \\ -2 & 2 & 3 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$
$(8, 6)$ :	$\begin{bmatrix} 4 & 0 & 4 & -2 \\ 0 & 5 & -2 & 5 \\ 4 & -2 & 8 & -4 \\ -2 & 5 & -4 & 6 \end{bmatrix}$	$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 3 & 6 & -1 & -4 \\ -1 & -1 & 6 & -3 \\ -1 & -4 & -3 & 3 \end{bmatrix}$	$\begin{bmatrix} 6 & 2 & 0 & 1 \\ 2 & 8 & 4 & -2 \\ 0 & 4 & 8 & -2 \\ 1 & -2 & -2 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -3 & 0 & 0 \\ -3 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 5 \end{bmatrix}$
$(10, 8)$ :	$\begin{bmatrix} 5 & -1 & -1 & 4 \\ -1 & 6 & -3 & 5 \\ -1 & -3 & 2 & -4 \\ 4 & 5 & -4 & 9 \end{bmatrix}$	$\begin{bmatrix} 8 & 0 & 0 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ -4 & -1 & 0 & 3 \end{bmatrix}$	$\begin{bmatrix} 6 & 5 & 1 & -2 \\ 5 & 9 & -3 & -4 \\ 1 & -3 & 6 & 4 \\ -2 & -4 & 4 & 4 \end{bmatrix}$	$\begin{bmatrix} 8 & 0 & 0 & -4 \\ 0 & 8 & 4 & 4 \\ 0 & 4 & 2 & 2 \\ -4 & 4 & 2 & 4 \end{bmatrix}$



# Combinatorial types of quartic spectrahedra (11-20)

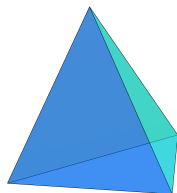
(4, 0) :	$\begin{bmatrix} 21 & 10 & 1 & -6 \\ 10 & 10 & 0 & -1 \\ 1 & 0 & 2 & -3 \\ -6 & -1 & -3 & 6 \end{bmatrix}$	$\begin{bmatrix} 0 & 6 & -6 & 2 \\ 6 & 3 & 0 & -4 \\ -6 & 0 & -3 & 5 \\ -2 & -4 & 5 & -3 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 2 & -1 & -1 & 5 \end{bmatrix}$	$\begin{bmatrix} 0 & 3 & -1 & 1 \\ 3 & -3 & 8 & -5 \\ -1 & 8 & -5 & 4 \\ 1 & -5 & 4 & -3 \end{bmatrix}$
(6, 2) :	$\begin{bmatrix} 7 & -1 & 5 & 2 \\ -1 & 5 & -1 & 5 \\ 5 & -1 & 4 & 1 \\ 2 & 5 & 1 & 7 \end{bmatrix}$	$\begin{bmatrix} -1 & -2 & 1 & -2 \\ -2 & -3 & 2 & -6 \\ 1 & 2 & -1 & 2 \\ -2 & -6 & 2 & 0 \end{bmatrix}$	$\begin{bmatrix} 4 & 4 & 2 & -2 \\ 4 & 0 & 4 & -2 \\ 2 & 4 & 0 & -1 \\ -2 & -2 & -1 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 1 & 2 & 1 \\ -1 & -1 & 2 & -1 \\ 2 & -2 & -3 & -1 \\ 1 & -1 & -1 & 0 \end{bmatrix}$
(8, 4) :	$\begin{bmatrix} 16 & -4 & -16 & 10 \\ -4 & 18 & 0 & -13 \\ -16 & 0 & 20 & -9 \\ 10 & -13 & -9 & 19 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & -5 & 6 & 1 \\ -1 & 6 & -7 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -16 & 0 & -8 \\ -16 & 0 & 16 & -16 \\ 0 & 16 & 0 & 8 \\ -8 & -16 & 8 & -16 \end{bmatrix}$	$\begin{bmatrix} 7 & 9 & 16 & 3 \\ 9 & -9 & -12 & 9 \\ 16 & -12 & -15 & 15 \\ 3 & 9 & 15 & 0 \end{bmatrix}$
(10, 6) :	$\begin{bmatrix} 18 & -13 & 15 & 1 \\ -13 & 22 & 2 & -16 \\ 15 & 2 & 30 & -20 \\ 1 & -16 & -20 & 30 \end{bmatrix}$	$\begin{bmatrix} -15 & 7 & 8 & 5 \\ 7 & -3 & -4 & -3 \\ 8 & -4 & -4 & -2 \\ 5 & -3 & -2 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -8 & -15 \\ -3 & 0 & -15 & -7 \end{bmatrix}$	$\begin{bmatrix} -15 & 0 & -6 & 2 \\ 0 & 15 & 6 & 8 \\ -6 & 6 & 0 & 4 \\ 2 & 8 & 4 & 4 \end{bmatrix}$
(6, 0) :	$\begin{bmatrix} 3 & 6 & -4 & -4 \\ 6 & 13 & -5 & -5 \\ -4 & -5 & 19 & 20 \\ -4 & -5 & 20 & 23 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & -3 & 0 \\ -1 & 3 & 6 & 0 \\ -3 & 6 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 8 & 2 & -2 & 2 \\ 2 & -4 & -2 & 2 \\ -2 & -2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & 1 & 3 \\ -2 & -5 & -11 & -15 \\ 1 & -11 & -8 & -6 \\ 3 & -15 & -6 & 0 \end{bmatrix}$
(8, 2) :	$\begin{bmatrix} 3 & -3 & 3 & -1 \\ -3 & 4 & -3 & 2 \\ 3 & -3 & 5 & 0 \\ -1 & 2 & 0 & 2 \end{bmatrix}$	$\begin{bmatrix} -1 & 1 & -1 & -2 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -2 & 0 & 0 & -4 \end{bmatrix}$	$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & 3 & -1 & 2 \\ 0 & -1 & -1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$
(10, 4) :	$\begin{bmatrix} 5 & -1 & -3 & 1 \\ -1 & 2 & 2 & 0 \\ -3 & 2 & 4 & -1 \\ 1 & 0 & -1 & 3 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -4 & -4 & -2 \\ 0 & -4 & -4 & -2 \\ 0 & -2 & -2 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 4 & -4 & -6 \\ 4 & 0 & 2 & 1 \\ -4 & 2 & -4 & -4 \\ -6 & 1 & -4 & -3 \end{bmatrix}$	$\begin{bmatrix} -3 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ -2 & 0 & -1 & -1 \end{bmatrix}$
(8, 0) :	$\begin{bmatrix} 9 & 0 & -7 & -10 \\ 0 & 5 & 0 & 2 \\ -7 & 0 & 15 & 5 \\ -10 & 2 & 5 & 13 \end{bmatrix}$	$\begin{bmatrix} 8 & 6 & 5 & 8 \\ 6 & -8 & -5 & -4 \\ 5 & -5 & -3 & -2 \\ 8 & -4 & -2 & 0 \end{bmatrix}$	$\begin{bmatrix} 8 & 4 & 11 & 4 \\ 4 & 0 & 10 & 0 \\ 11 & 10 & 5 & 10 \\ 4 & 0 & 10 & 0 \end{bmatrix}$	$\begin{bmatrix} -4 & -4 & 2 & 4 \\ -4 & -4 & 2 & 4 \\ 2 & 2 & 0 & 0 \\ 4 & 4 & 0 & 0 \end{bmatrix}$
(10, 2) :	$\begin{bmatrix} 29 & -22 & 4 & -4 \\ -22 & 26 & -7 & 5 \\ 4 & -7 & 25 & -6 \\ -4 & 5 & -6 & 5 \end{bmatrix}$	$\begin{bmatrix} -1 & -4 & -1 & -4 \\ -4 & -12 & -4 & -14 \\ -1 & -4 & -1 & -4 \\ -4 & -14 & -4 & -15 \end{bmatrix}$	$\begin{bmatrix} -5 & 9 & 6 & 7 \\ 9 & 8 & -2 & 5 \\ 6 & -2 & -4 & -2 \\ 7 & 5 & -2 & 3 \end{bmatrix}$	$\begin{bmatrix} -5 & 16 & -1 & -10 \\ 16 & -12 & 20 & 4 \\ -1 & 20 & 7 & -14 \\ -10 & 4 & -14 & 0 \end{bmatrix}$
(10, 0) :	$\begin{bmatrix} 51 & -34 & 5 & 60 \\ -34 & 147 & 30 & -37 \\ 5 & 30 & 99 & 40 \\ 60 & -37 & 40 & 135 \end{bmatrix}$	$\begin{bmatrix} 15 & 97 & 64 & 36 \\ 97 & -13 & -50 & 76 \\ 64 & -50 & -63 & 40 \\ 36 & 76 & 40 & 48 \end{bmatrix}$	$\begin{bmatrix} -27 & 45 & -27 & 51 \\ 45 & 0 & -30 & 10 \\ -27 & -30 & 48 & -44 \\ 51 & 10 & -44 & 24 \end{bmatrix}$	$\begin{bmatrix} -60 & 30 & 10 & -52 \\ 30 & 45 & -55 & -2 \\ 10 & -55 & 40 & 32 \\ -52 & -2 & 32 & -32 \end{bmatrix}$

# Many flavors of quartic spectrahedra

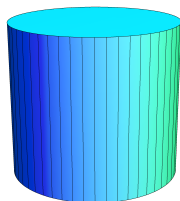


# Special Quartic Spectrahedra

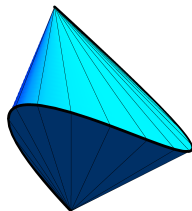
Non-generically, the span of  $A_0, A_1, A_2, A_3$  might contain a curve of rank-two matrices.



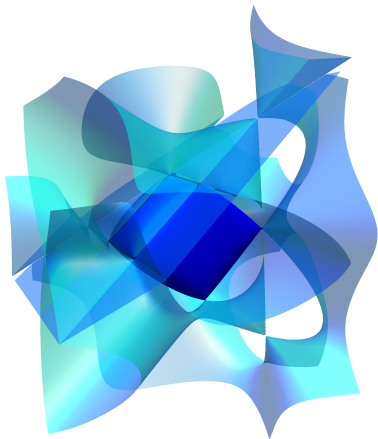
$$\begin{pmatrix} x_0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 \\ 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & x_3 \end{pmatrix}$$



$$\begin{pmatrix} x_0 + x_1 & x_2 & 0 & 0 \\ x_2 & x_0 - x_1 & 0 & 0 \\ 0 & 0 & x_0 + x_3 & 0 \\ 0 & 0 & 0 & x_0 - x_3 \end{pmatrix}$$



$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_0 & x_1 & x_2 \\ x_2 & x_1 & x_0 & x_1 \\ x_3 & x_2 & x_1 & x_0 \end{pmatrix}$$



Spectrahedra can be understood using **beautiful** and **classical** algebraic geometry.

There is still lots to understand. What are the **combinatorial types** of spectrahedra of higher dimensions and degrees?

Thanks for your attention!