The Convex Hull of a Parametrized Curve

Cynthia Vinzant

Department of Mathematics
University of California, Berkeley

SIAM - Convex Algebraic Geometry

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Faces and Vertices of Convex Hulls

Let $\bar{\mathbf{f}}(t) = (f_1(t), \ldots, f_n(t))$ where $f_j \in \mathbb{R}[t]$ and let $\mathcal{D} \subseteq \mathbb{R}$.

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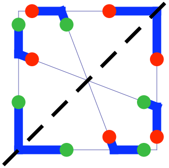
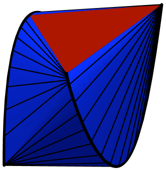
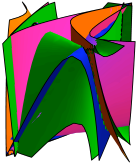




$(a_1, a_2) \quad \leftrightarrow \quad$ edge $[\overline{\mathbf{f}}(a_1), \overline{\mathbf{f}}(a_2)]$

$(a_1, a_2) \quad \leftrightarrow \quad$ facet $[\overline{\mathbf{f}}(a_1), \overline{\mathbf{f}}(a_2), \overline{\mathbf{f}}(a_3)]$

Many ways to represent the convex hull of a curve



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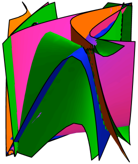

Projection of a Spectrahedron (Henrion, 2010):
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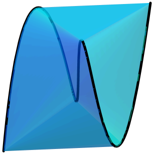

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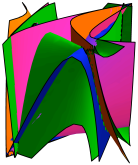

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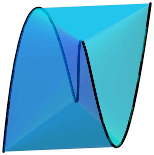
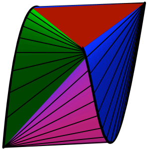


Face-vertex set (this talk):
- complete facial information of $\mathrm{conv}(C)$
- invariant under change of coordinates
- medium-hard to compute



Affine Functions $\longleftrightarrow$ Polynomials

Affine functions on $\mathbb{R}^n \longleftrightarrow$ Polynomials in $\mathrm{span}\{1, f_1, \ldots, f_n\}$

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Nonnegativity: The halfspace $\{w^T x \geq w_0\}$ contains the curve $C = \{\bar{\mathbf{f}}(t) : t \in \mathcal{D}\}$ if and only if the polynomial $g(t) \geq 0$ on $\mathcal{D}$.

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Faces: The points $\{\bar{\mathbf{f}}(a_1), \ldots, \bar{\mathbf{f}}(a_r)\}$ are the vertices of a face
$\Leftrightarrow$ there exists $g \in \mathrm{span}\{1, f_1, \ldots, f_n\}$ with $g \geq 0$ on $\mathcal{D}$
and $\{t \in \mathcal{D} \ : \ g(t) = 0\} = \{a_1, \ldots, a_r\}$

Side note: Dual Bodies

The dual cone of $\mathrm{conv}(C)$ is $\{g \in \mathrm{span}\{1, f_j\} \ : \ g \geq 0 \text{ on } \mathcal{D}\}$.

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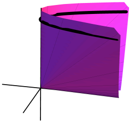

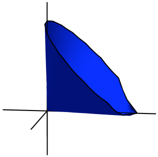

Example 1: $\mathit{conv}(1, t, t^2)$ $\{b^2 - 4ac \leq 0, a \geq 0\}$

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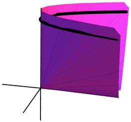

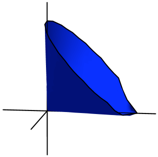

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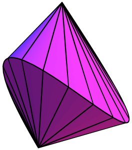

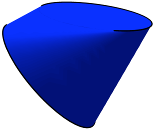

Example 2: $conv(t, 2t^2 - 1, 4t^3 - 3t)$ $\{g \in \mathbb{R}[t]_{\leq 3} : g \geq 0 \text{ on } [-1, 1]\}$

Necessary Algebraic Conditions

Prop. For $\{a_1, \ldots, a_s\} \subset \text{int}(\mathcal{D})$ and $\{a_{s+1}, \ldots, a_r\} \in \partial\mathcal{D}$, TFAE:

Let $V_r \subset \mathcal{D}^r$ be the set of $(a_1, \ldots, a_r)$ satisfying these conditions.

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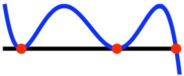

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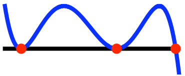
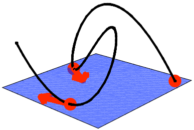


2. $\{\bar{\mathbf{f}}(a_j)\} \cup \{\bar{\mathbf{f}}(a_j) + \bar{\mathbf{f}}'(a_j) \ : \ a_j \in \text{int}\,\mathcal{D}\}$ lie in a common hyperplane



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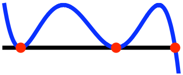

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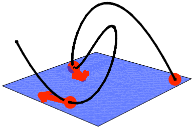

3. $$\text{rank}\left(\begin{bmatrix} 1 & \ldots & 1 & 1 & \ldots & 1 \\ \bar{\mathbf{f}}(a_1) & \ldots & \bar{\mathbf{f}}(a_r) & \bar{\mathbf{f}}'(a_1) & \ldots & \bar{\mathbf{f}}'(a_s) \end{bmatrix}\right) \leq n$$

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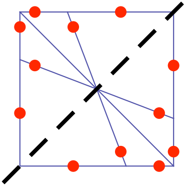
Exposing the real faces

"Discriminant": $\mathcal{S} := \pi(V_{r+1}) \cup \text{sing}(V_r) \cup \ldots$ has codim-1 in V_r.

To test which points in $V_r \backslash \mathcal{S}$ are the vertices of a face on conv(C) it suffices to test one point in each connected component of $V_r \backslash \mathcal{S}$.

Example: $C = \{(t,\ 4t^3 - 3t,\ 16t^5 - 20t^3 + 5t)\ :\ t \in [-1, 1]\}$

V_2 (potential edges)
$\pi(V_3)$ (potential edges of 2-faces)



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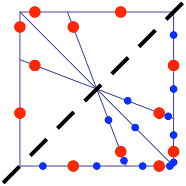

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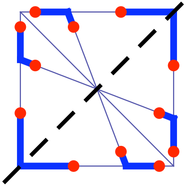

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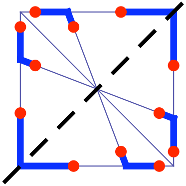

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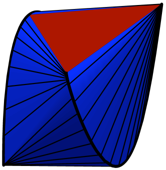
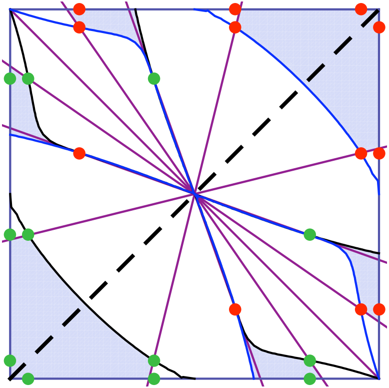


Audience Challenge: Visualizing 4-dim'l convex bodies

Example: $C = \{(t, t^3, t^5, t^7) \ : \ t \in [-1, 1]\} \subset \mathbb{R}^4$



edges of 4-faces

edges of 3-faces

edges

Stratification of the Grassmannian

For fixed $\mathcal{D}$, the face-vertex sets of conv(C) depend only on $\mathrm{span}\{1, f_1(t), \ldots, f_n(t)\} \subset \mathbb{R}[t]$.

If deg$(f_j) \leq d$ then $\mathrm{span}\{f_1, \ldots, f_n\}$ is a point in Gr$(n, \mathbb{R}[t]_{\leq d})$.

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Example: $C = \{(f_1(t), f_2(t)) \ : \ t \in \mathbb{R}\}$ with $\deg(f_j) \leq 4$.

The set of planes in $\mathbb{R}[t]_{\leq 4}$ containing a polynomial $(t-a)^4$ form a hypersurface $\mathcal{H}$ of in $\mathrm{Gr}(2,4)$, given by

$$2p_{2,3}^3 - 12p_{1,3}p_{2,3}p_{2,4} + 27p_{1,2}p_{2,4}^2 + 32p_{1,3}^2p_{3,4} - 72p_{1,2}p_{1,4}p_{3,4} - 24p_{1,2}p_{2,3}p_{3,4}$$

The **number of edges** of conv(C) is determined by what region of $\mathrm{Gr}(2,4)\backslash\mathcal{H}$ contains the plane $\mathrm{span}\{f_1, f_2\}$.

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Thanks!