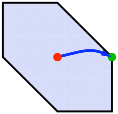
The Central Curve in Linear Programming

Cynthia Vinzant, UC Berkeley



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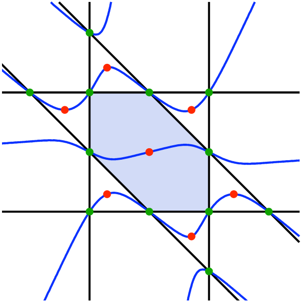

joint work with Jesús De Loera and Bernd Sturmfels

The Central Path of a Linear Program

Linear Program: $\text{Maximize}_{\mathbf{x}\in\mathbb{R}^n}$ $\mathbf{c}\cdot\mathbf{x}$ s.t. $A\cdot\mathbf{x}=\mathbf{b}$ and $\mathbf{x}\geq 0$.

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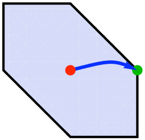
The central path is $\{\mathbf{x}^*(\lambda) \ : \ \lambda > 0\}$.
As $\lambda\to 0$, the path leads from the analytic center of the polytope, $\mathbf{x}^*(\infty)$, to the optimal vertex, $\mathbf{x}^*(0)$.



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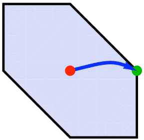

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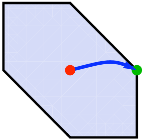

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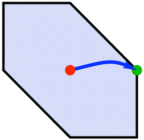

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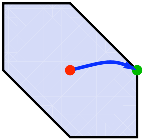

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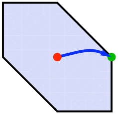
Bounds on curvature of the path $\to$ bounds on # Newton steps

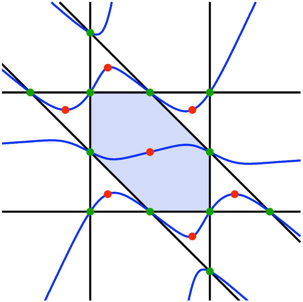
We can use concepts from algebraic geometry and matroid theory to bound the total curvature of the central path.

The Central Curve

The central curve $\mathcal{C}$ is the Zariski closure of the central path. It contains the central paths of all polyhedra in the hyperplane arrangement $\{x_i = 0\}_{i=1,\dots,n} \subset \{A \cdot \mathbf{x} = \mathbf{b}\}$.



$\longrightarrow$
Zariski closure



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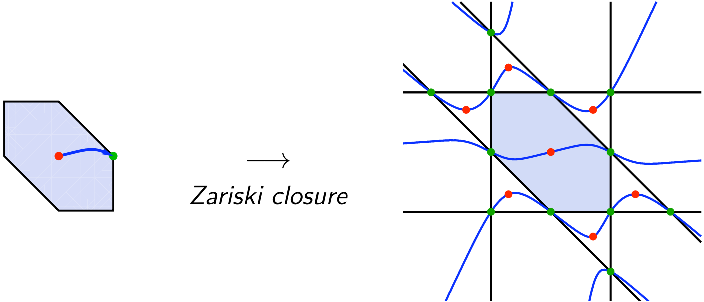

Goal: Study the nice algebraic geometry of this curve and its applications to the linear program

History and Contributions

Motivating Question: What is the maximum total curvature of the central path given the size of the matrix A?

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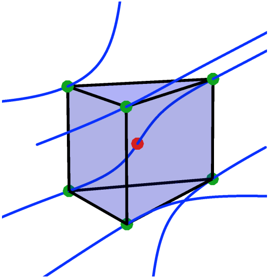
Our contribution is to use results from algebraic geometry and matroid theory to find defining equations of the central curve and refine bounds on its degree and total curvature.

Outline

- Algebraic conditions for optimality
- Degree of the curve (and other combinatorial data)
- Total curvature and the Gauss map
- Defining equations
- The primal-dual picture



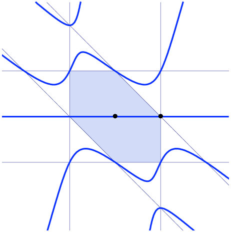
Some details

Here we assume that ...

1) A is a $d \times n$ matrix of rank-d (possibly very special), and

2) $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^d$ are generic.

(This ensures that the central curve is irreducible and nonsingular.)



Algebraic Conditions for Optimality

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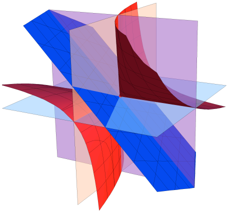

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where $\mathcal{L}_{A,\mathbf{c}}^{-1}$ denotes the coordinate-wise reciprocal $\mathcal{L}_{A,\mathbf{c}}$:

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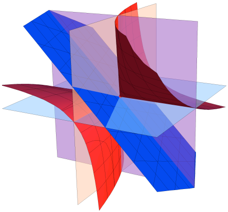

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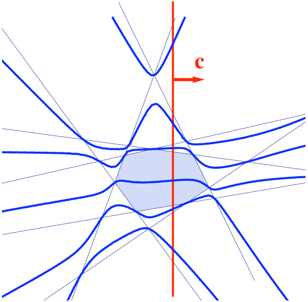
Proposition. The central curve equals the intersection of the central sheet $\mathcal{L}_{A,\mathbf{c}}^{-1}$ with the affine space $\{A\cdot\mathbf{x} = \mathbf{b}\}$.

Level sets of the cost function

Consider intersecting the central curve $\mathcal{C}$ with the level set $\{\mathbf{c} \cdot \mathbf{x} = c_0\}$.



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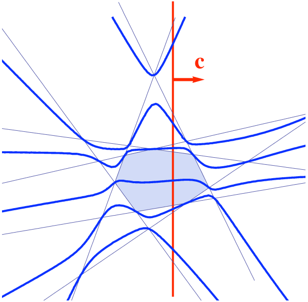

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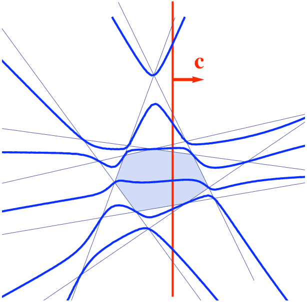

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Claim: The points $\mathcal{C} \cap \{\mathbf{c} \cdot \mathbf{x} = c_0\}$ are the analytic centers of the hyperplane arrangement $\{x_i = 0\}_{i \in [n]}$ in $\{A \cdot \mathbf{x} = \mathbf{b},\ \mathbf{c} \cdot \mathbf{x} = c_0\}$.

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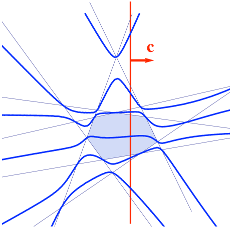

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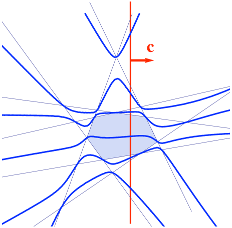

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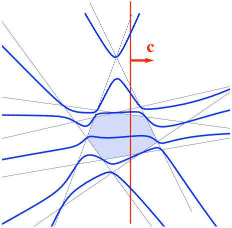

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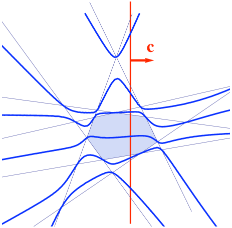
For matroid enthusiasts,
this number is the absolute value
of the Möbius invariant of $\binom{A}{\mathbf{c}}$.



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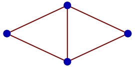

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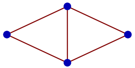

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$$\Rightarrow \quad \deg(\mathcal{C}) = \sum_{i=0}^{d} h_i \quad \text{and} \quad \text{genus}(\mathcal{C}) = 1 - \sum_{j=0}^{d} (1-j) h_j \ .$$

Total Curvature

Classic differential geometry: The total curvature of any real algebraic curve $\mathcal{C}$ in $\mathbb{R}^m$ is the arc length of its image under the Gauss map $\gamma : \mathcal{C} \to \mathbb{S}^{m-1}$. This quantity is bounded above by π times the degree of the projective Gauss curve in $\mathbb{P}^{m-1}$. That is,

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Theorem: The degree of the projective Gauss curve of the central curve $\mathcal{C}$ satisfies a bound in terms of matroid invariants:

$$\deg(\gamma(\mathcal{C})) \quad \leq \quad 2 \cdot \sum_{i=1}^{d} i \cdot h_i \quad \leq \quad 2 \cdot (n-d-1) \cdot \binom{n-1}{d-1}.$$

Equations

Proudfoot and Speyer (2006) also prove that the equations defining $\mathcal{L}_{A,\mathbf{c}}^{-1}$ are the homogeneous polynomials

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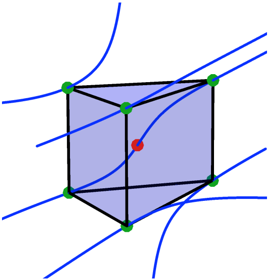
Cocircuit $v = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \end{pmatrix}$ produces $-2x_2x_3 + 1x_1x_3 + 1x_1x_2$.

Example

$(n = 5,\ d = 2)$

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 1 & 2 & 0 & 4 & 0 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$



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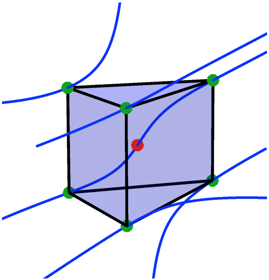
$x_1 + x_2 + x_3 - 3$
$x_4 + x_5 - 2$



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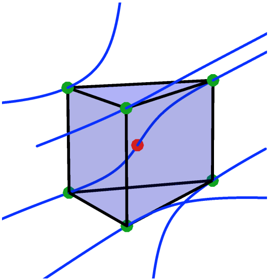

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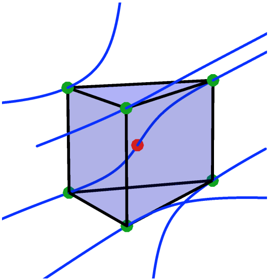

$$h = (1, 2, 2) \ \Rightarrow\ \deg(\mathcal{C}) = 5, \quad \text{total curvature}(\mathcal{C}) \leq 12\pi\ (\leq 16\pi)$$

Duality

Dual LP: Minimize$_{\mathbf{y}\in\mathbb{R}^d}$ $\mathbf{b}^T\mathbf{y}$: $A^T\mathbf{y}-\mathbf{s}=\mathbf{c}$, $\mathbf{s}\geq 0$

$\longleftrightarrow$ Minimize$_{\mathbf{s}\in\mathbb{R}^n}$ $\mathbf{v}^T\mathbf{s}$: $B\cdot\mathbf{s}=B\cdot\mathbf{c}$, $\mathbf{s}\geq 0$,

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The primal-dual central path is cut out by the system of polynomial equations

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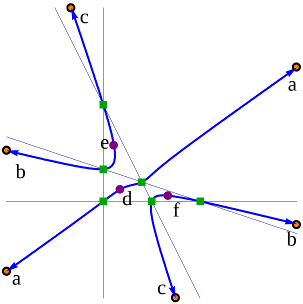
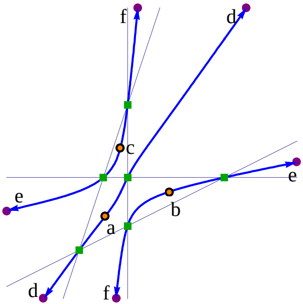
Examine $\lambda \to 0$ and $\lambda \to \infty$.

Global Geometry: a nice curve in $\mathbb{P}^n \times \mathbb{P}^n$

Primal (**x**-space)

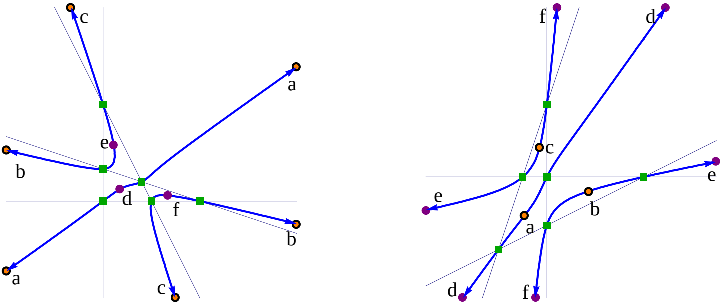


Dual (**s**-space)



Global Geometry: a nice curve in $\mathbb{P}^n \times \mathbb{P}^n$

Primal (**x**-space) | Dual (**s**-space)



vertices $\longleftrightarrow$ vertices

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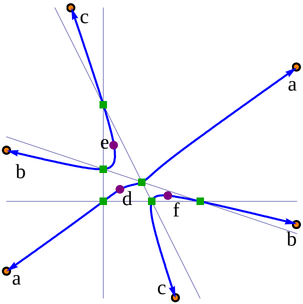

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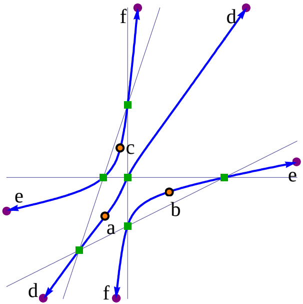

vertices	$\longleftrightarrow$	vertices
analytic centers	$\longleftrightarrow$	points at ∞

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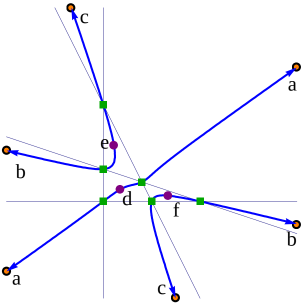
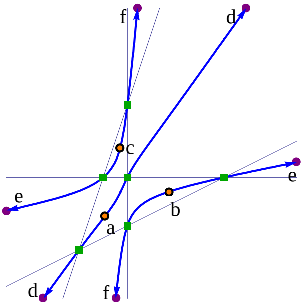
Primal (**x**-space) Dual (**s**-space)





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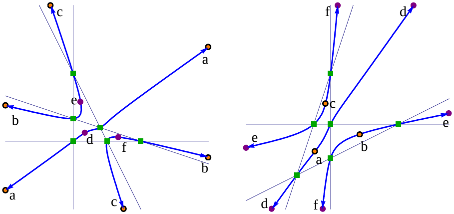

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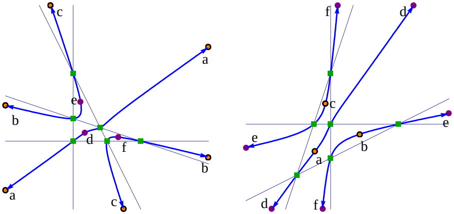

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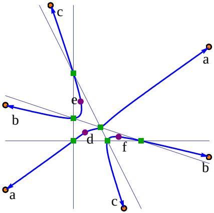

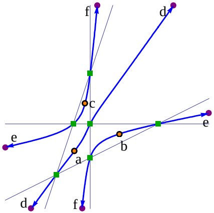

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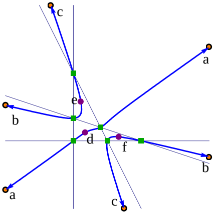

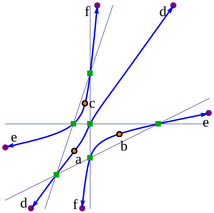

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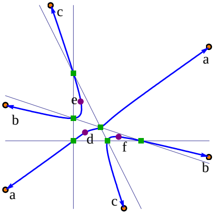

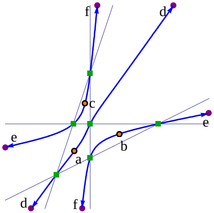

Thanks!