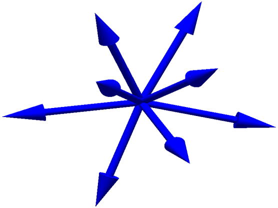
Injectivity of Hermitian frame measurements

Cynthia Vinzant

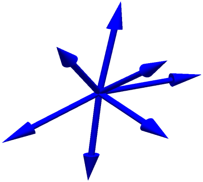
North Carolina State University



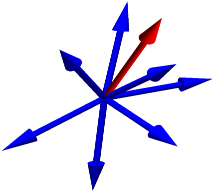
Frames and Algebraic & Combinatorial Geometry
July 31, 2015

Frames and intensity measurements

A **frame** is a collection of vectors $\Phi = \{\phi_1, \ldots, \phi_n\}$ spanning $\mathbb{C}^d$.



Frames and intensity measurements



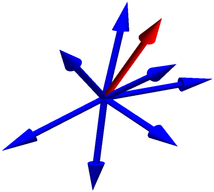
A **frame** is a collection of vectors $\Phi = \{\phi_1, \ldots, \phi_n\}$ spanning $\mathbb{C}^d$.

A frame defines **intensity measurements** of a signal $x \in \mathbb{C}^d$:

$$|\langle \phi_k, x \rangle|^2 \quad = \quad \mathrm{tr}(\phi_k \phi_k^* x x^*) \quad \text{for} \quad k = 1, \ldots, n.$$

Frames and intensity measurements

A **frame** is a collection of vectors $\Phi = \{\phi_1, \ldots, \phi_n\}$ spanning $\mathbb{C}^d$.



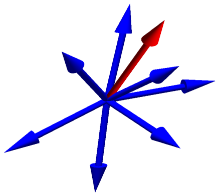
A frame defines **intensity measurements** of a signal $x \in \mathbb{C}^d$:

$$|\langle \phi_k, x \rangle|^2 \quad = \quad \mathrm{tr}(\phi_k \phi_k^* x x^*) \quad \text{for} \quad k = 1, \ldots, n.$$

Phase Retrieval: Recover xx^* from $\mathrm{tr}(\phi_k \phi_k^* x x^*)$.

Frames and intensity measurements

A **frame** is a collection of vectors $\Phi = \{\phi_1, \ldots, \phi_n\}$ spanning $\mathbb{C}^d$.



A frame defines **intensity measurements** of a signal $x \in \mathbb{C}^d$:

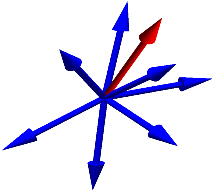
$$|\langle \phi_k, x \rangle|^2 \;=\; \mathrm{tr}(\phi_k \phi_k^* x x^*) \quad \text{for} \quad k = 1, \ldots, n.$$

Phase Retrieval: Recover xx^* from $\mathrm{tr}(\phi_k \phi_k^* x x^*)$.

Some Questions: How do we recover the signal x?

Frames and intensity measurements

A **frame** is a collection of vectors $\Phi = \{\phi_1, \ldots, \phi_n\}$ spanning $\mathbb{C}^d$.



A frame defines **intensity measurements** of a signal $x \in \mathbb{C}^d$:

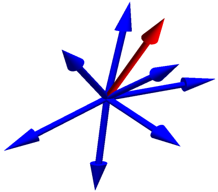
$$|\langle \phi_k, x \rangle|^2 \quad = \quad \mathrm{tr}(\phi_k \phi_k^* x x^*) \quad \text{for} \quad k = 1, \ldots, n.$$

Phase Retrieval: Recover xx^* from $\mathrm{tr}(\phi_k \phi_k^* x x^*)$.

Some Questions: How do we recover the signal x?
When is recovery of signals in $\mathbb{C}^d$ possible?

Frames and intensity measurements

A **frame** is a collection of vectors $\Phi = \{\phi_1, \ldots, \phi_n\}$ spanning $\mathbb{C}^d$.



A frame defines **intensity measurements** of a signal $x \in \mathbb{C}^d$:

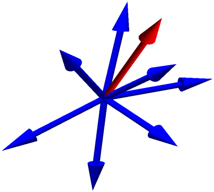
$$|\langle \phi_k, x \rangle|^2 \;=\; \mathrm{tr}(\phi_k \phi_k^* x x^*) \quad \text{for} \quad k = 1, \ldots, n.$$

Phase Retrieval: Recover xx^* from $\mathrm{tr}(\phi_k \phi_k^* x x^*)$.

Some Questions: How do we recover the signal x?
When is recovery of signals in $\mathbb{C}^d$ possible?
When is recovery of signals in $\mathbb{C}^d$ stable?

Frames and intensity measurements

A **frame** is a collection of vectors $\Phi = \{\phi_1, \ldots, \phi_n\}$ spanning $\mathbb{C}^d$.



A frame defines **intensity measurements** of a signal $x \in \mathbb{C}^d$:

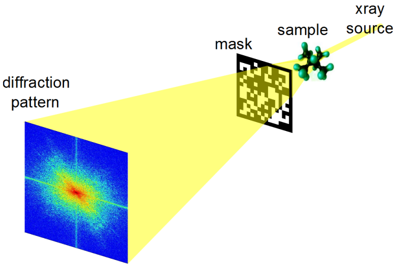
$$|\langle \phi_k, x\rangle|^2 \quad = \quad \mathrm{tr}(\phi_k \phi_k^* x x^*) \quad \text{for} \quad k = 1, \ldots, n.$$

Phase Retrieval: Recover xx^* from $\mathrm{tr}(\phi_k \phi_k^* x x^*)$.

Some Questions: How do we recover the signal x?
When is recovery of signals in $\mathbb{C}^d$ possible?
When is recovery of signals in $\mathbb{C}^d$ stable?

Motivation and Applications

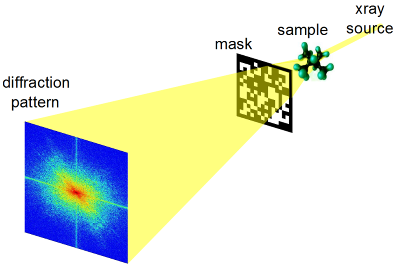
In practice the signal is some structure that is too small (DNA, crystals) or far away (astronomical phenomena) or obscured (medical images) to observe directly.



(picture from Candés-Eldar-Strohmer-Voroninski 2013)

Motivation and Applications

In practice the signal is some structure that is too small (DNA, crystals) or far away (astronomical phenomena) or obscured (medical images) to observe directly.

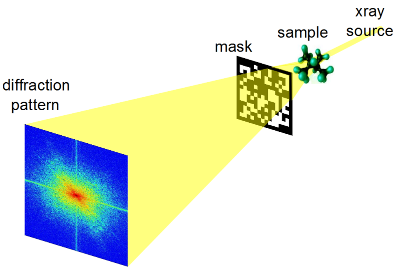


(picture from Candés-Eldar-Strohmer-Voroninski 2013)

If some measurements are possible, then one hopes to reconstruct this structure.

Motivation and Applications

In practice the signal is some structure that is too small (DNA, crystals) or far away (astronomical phenomena) or obscured (medical images) to observe directly.



(picture from Candés-Eldar-Strohmer-Voroninski 2013)

If some measurements are possible, then one hopes to reconstruct this structure.

Here our signal x lies in a finite-dimensional space ($\mathbb{C}^d$), and its measurements are modeled by $|\langle \phi_k, x\rangle|^2$ for $\phi_k \in \mathbb{C}^d$.

Phase Retrieval: recovering a vector from its measurements

When do the measurements $\mathrm{tr}(\phi_k \phi_k^* x x^*)$ **determine** $xx^* \in \mathbb{C}^{d\times d}_{Herm}$?

Phase Retrieval: recovering a vector from its measurements

When do the measurements $\mathrm{tr}(\phi_k\phi_k^* xx^*)$ **determine** $xx^* \in \mathbb{C}^{d\times d}_{Herm}$?

That is, for what collections of vectors $\Phi = (\phi_1 \ldots \phi_n)$ is the map

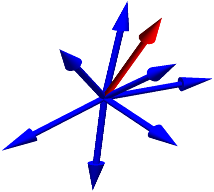
$$\mathcal{M}_\Phi : \left\{ \begin{matrix} \text{rank-1 Hermitian} \\ d \times d \text{ matrices} \end{matrix} \right\} \to \mathbb{R}^n \quad \text{given by} \quad X \mapsto (\mathrm{tr}(\phi_k\phi_k^* \cdot X))_k$$

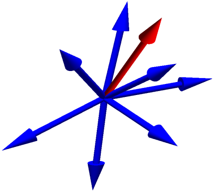
injective?

(Heinosaari–Mazzarella–Wolf, 2011):

For $n < 4d - 2\alpha - 4$, $\mathcal{M}_\Phi$ is not injective,

where $\alpha = \#$ of 1's in binary expansion of $d-1$.





(Heinosaari–Mazzarella–Wolf, 2011):

For $n < 4d - 2\alpha - 4$, $\mathcal{M}_\Phi$ is not injective,

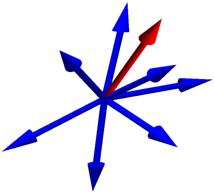
where $\alpha = \#$ of 1's in binary expansion of $d-1$.

Conjecture (Bandeira-Cahill-Mixon-Nelson, 2013)

(a) If $n < 4d - 4$, then $\mathcal{M}_\Phi$ is not injective.

(b) If $n \geq 4d - 4$, then $\mathcal{M}_\Phi$ is injective for generic Φ.

How many measurements for injectivity? About $4d$.



(Heinosaari–Mazzarella–Wolf, 2011):
For $n < 4d - 2\alpha - 4$, $\mathcal{M}_\Phi$ is not injective,
where $\alpha = \#$ of 1's in binary expansion of $d-1$.

Conjecture (Bandeira-Cahill-Mixon-Nelson, 2013)
(a) If $n < 4d - 4$, then $\mathcal{M}_\Phi$ is not injective.
(b) If $n \geq 4d - 4$, then $\mathcal{M}_\Phi$ is injective for generic Φ.

(Conca–Edidin–Hering–V., 2014)
For $n \geq 4d - 4$, $\mathcal{M}_\Phi$ is injective for generic $\Phi \in \mathbb{C}^{d\times n}$.
If $d = 2^k + 1$ and $n < 4d - 4$, $\mathcal{M}_\Phi$ is not injective.

A nice reformulation of non-injectivity

Observation (Bandeira-Cahill-Mixon-Nelson):

$\mathcal{M}_\Phi$ is non-injective $\Leftrightarrow$ $\exists$ a nonzero matrix $Q \in \mathbb{C}^{d\times d}_{Herm}$ with

$$\text{rank}(Q) \leq 2 \quad \text{and} \quad \phi_k^* Q \phi_k = 0 \quad \text{for each } 1 \leq k \leq n. \qquad (*)$$

A nice reformulation of non-injectivity

Observation (Bandeira-Cahill-Mixon-Nelson):

$\mathcal{M}_\Phi$ is non-injective $\Leftrightarrow$ $\exists$ a nonzero matrix $Q \in \mathbb{C}^{d\times d}_{Herm}$ with

$$\operatorname{rank}(Q) \leq 2 \quad \text{and} \quad \phi_k^* Q \phi_k = 0 \quad \text{for each } 1 \leq k \leq n. \qquad (*)$$

Why?

A nice reformulation of non-injectivity

Observation (Bandeira-Cahill-Mixon-Nelson):

$\mathcal{M}_\Phi$ is non-injective $\Leftrightarrow$ $\exists$ a nonzero matrix $Q \in \mathbb{C}^{d\times d}_{Herm}$ with

$$\text{rank}(Q) \leq 2 \quad \text{and} \quad \phi_k^* Q \phi_k = 0 \quad \text{for each } 1 \leq k \leq n. \qquad (*)$$

Why? $\quad \mathcal{M}_\Phi(x) = \mathcal{M}_\Phi(y) \Leftrightarrow \phi_k^* x x^* \phi_k = \phi_k^* y y^* \phi_k \quad \forall k$

A nice reformulation of non-injectivity

Observation (Bandeira-Cahill-Mixon-Nelson):

$\mathcal{M}_\Phi$ is non-injective $\Leftrightarrow$ $\exists$ a nonzero matrix $Q \in \mathbb{C}^{d\times d}_{Herm}$ with

$$\text{rank}(Q) \leq 2 \quad \text{and} \quad \phi_k^* Q \phi_k = 0 \quad \text{for each } 1 \leq k \leq n. \qquad (*)$$

Why?
$$\begin{aligned} \mathcal{M}_\Phi(x) = \mathcal{M}_\Phi(y) &\Leftrightarrow \phi_k^* x x^* \phi_k = \phi_k^* y y^* \phi_k \quad \forall k \\ &\Leftrightarrow \phi_k^*(xx^* - yy^*)\phi_k = 0 \quad \forall k \end{aligned}$$

A nice reformulation of non-injectivity

Observation (Bandeira-Cahill-Mixon-Nelson):

$\mathcal{M}_\Phi$ is non-injective $\Leftrightarrow$ $\exists$ a nonzero matrix $Q \in \mathbb{C}^{d\times d}_{Herm}$ with

$$\text{rank}(Q) \leq 2 \quad \text{and} \quad \phi_k^* Q \phi_k = 0 \quad \text{for each } 1 \leq k \leq n. \qquad (*)$$

Why?

$$\begin{aligned} \mathcal{M}_\Phi(x) = \mathcal{M}_\Phi(y) &\Leftrightarrow \phi_k^* x x^* \phi_k = \phi_k^* y y^* \phi_k \quad \forall k \\ &\Leftrightarrow \phi_k^* \underbrace{(xx^* - yy^*)}_{\text{rank } 2} \phi_k = 0 \quad \forall k \end{aligned}$$

A nice reformulation of non-injectivity

Observation (Bandeira-Cahill-Mixon-Nelson):

$\mathcal{M}_\Phi$ is non-injective $\Leftrightarrow$ $\exists$ a nonzero matrix $Q \in \mathbb{C}^{d\times d}_{Herm}$ with

$$\mathrm{rank}(Q) \leq 2 \quad \text{and} \quad \phi_k^* Q \phi_k = 0 \quad \text{for each } 1 \leq k \leq n. \qquad (*)$$

Why? $$\begin{aligned} \mathcal{M}_\Phi(x) = \mathcal{M}_\Phi(y) &\Leftrightarrow \phi_k^* x x^* \phi_k = \phi_k^* y y^* \phi_k \quad \forall k \\ &\Leftrightarrow \phi_k^* \underbrace{(xx^* - yy^*)}_{\text{rank } 2} \phi_k = 0 \quad \forall k \end{aligned}$$

More algebraic question: When does $(\mathrm{span}_{\mathbb{R}}\{\phi_1\phi_1^*, \ldots, \phi_n\phi_n^*\})^\perp$ intersect the rank-2 locus of $\mathbb{C}^{d\times d}_{Herm}$?

Getting (Real) Algebraic

Consider the incidence set

$$\left\{(\Phi, Q) \in \mathbb{P}(\mathbb{C}^{d\times n}) \times \mathbb{P}(\mathbb{C}^{d\times d}_{Herm}) \ : \ \text{rank}(Q) \le 2 \text{ and } \phi_k^* Q \phi_k = 0 \ \ \forall k\right\}.$$

Getting (Real) Algebraic

Consider the incidence set

$$\left\{(\Phi, Q) \in \mathbb{P}(\mathbb{C}^{d\times n}) \times \mathbb{P}(\mathbb{C}^{d\times d}_{Herm}) \ : \ \text{rank}(Q) \leq 2 \text{ and } \phi_k^* Q \phi_k = 0 \ \ \forall k\right\}.$$

$$\Phi \in \mathbb{C}^{d\times n} \qquad \longrightarrow \qquad A + \mathrm{i}B \text{ where } A, B \in \mathbb{R}^{d\times n}$$

Getting (Real) Algebraic

Consider the incidence set

$$\left\{(\Phi, Q) \in \mathbb{P}(\mathbb{C}^{d\times n}) \times \mathbb{P}(\mathbb{C}^{d\times d}_{Herm}) \;:\; \text{rank}(Q) \leq 2 \text{ and } \phi_k^* Q \phi_k = 0 \;\; \forall k\right\}.$$

$$\begin{array}{lcl} \Phi \in \mathbb{C}^{d\times n} & \longrightarrow & A + \mathrm{i}B \text{ where } A, B \in \mathbb{R}^{d\times n} \\ Q \in \mathbb{C}^{d\times d}_{Herm} & \longrightarrow & X + \mathrm{i}Y \text{ where } X \in \mathbb{R}^{d\times d}_{sym},\; Y \in \mathbb{R}^{d\times d}_{skew} \end{array}$$

Getting (Real) Algebraic

Consider the incidence set

$$\left\{(\Phi, Q) \in \mathbb{P}(\mathbb{C}^{d\times n}) \times \mathbb{P}(\mathbb{C}^{d\times d}_{Herm}) \;:\; \text{rank}(Q) \leq 2 \text{ and } \phi_k^* Q \phi_k = 0 \;\; \forall k\right\}.$$

$\Phi \in \mathbb{C}^{d\times n} \longrightarrow A + \mathrm{i}B$ where $A, B \in \mathbb{R}^{d\times n}$

$Q \in \mathbb{C}^{d\times d}_{Herm} \longrightarrow X + \mathrm{i}Y$ where $X \in \mathbb{R}^{d\times d}_{sym}$, $Y \in \mathbb{R}^{d\times d}_{skew}$

incidence set $\longrightarrow$ *real projective variety* in $\mathbb{P}((\mathbb{R}^{d\times n})^2) \times \mathbb{P}(\mathbb{R}^{d\times d}_{sym} \times \mathbb{R}^{d\times d}_{skew})$

Getting (Real) Algebraic

Consider the incidence set

$$\left\{(\Phi, Q) \in \mathbb{P}(\mathbb{C}^{d\times n}) \times \mathbb{P}(\mathbb{C}^{d\times d}_{Herm}) \ : \ \text{rank}(Q) \leq 2 \text{ and } \phi_k^* Q \phi_k = 0 \ \ \forall k\right\}.$$

$\Phi \in \mathbb{C}^{d\times n} \longrightarrow A + \mathrm{i}B$ where $A, B \in \mathbb{R}^{d\times n}$

$Q \in \mathbb{C}^{d\times d}_{Herm} \longrightarrow X + \mathrm{i}Y$ where $X \in \mathbb{R}^{d\times d}_{sym}$, $Y \in \mathbb{R}^{d\times d}_{skew}$

incidence set $\longrightarrow$ *real projective variety* in $\mathbb{P}((\mathbb{R}^{d\times n})^2) \times \mathbb{P}(\mathbb{R}^{d\times d}_{sym} \times \mathbb{R}^{d\times d}_{skew})$

Consequence: The bad frames, $\{\Phi : \mathcal{M}_\Phi$ is non-injective$\}$, are the projection of a real (projective) variety.

($\Rightarrow$ a closed semialgebraic subset of $\mathbb{P}((\mathbb{R}^{d\times n})^2)$)

Dimensions

The rank ≤ 2 matrices in $\mathbb{C}^{d\times d}$ are a variety of

$$\text{dimension } 4d-4 \quad \text{and} \quad \text{degree } \binom{2d-3}{d-2}^2 / (2d-3).$$

Dimensions

The rank ≤ 2 matrices in $\mathbb{C}^{d\times d}$ are a variety of

$$\text{dimension } 4d-4 \quad \text{and} \quad \text{degree } \binom{2d-3}{d-2}^2 / (2d-3).$$

Theorem. For $n \geq 4d-4$ and generic $\Phi \in \mathbb{C}^{d\times n} \cong (\mathbb{R}^{d\times n})^2$, there is no non-zero matrix of rank ≤ 2 in $\{\phi_1\phi_1^*, \ldots, \phi_n\phi_n^*\}^\perp$.

Dimensions

The rank ≤ 2 matrices in $\mathbb{C}^{d\times d}$ are a variety of

$$\text{dimension } 4d-4 \quad \text{and} \quad \text{degree } \binom{2d-3}{d-2}^2/(2d-3).$$

Theorem. For $n \geq 4d-4$ and generic $\Phi \in \mathbb{C}^{d\times n} \cong (\mathbb{R}^{d\times n})^2$, there is no non-zero matrix of rank ≤ 2 in $\{\phi_1\phi_1^*, \ldots, \phi_n\phi_n^*\}^\perp$.

This means there is a polynomial $f(A,B)$ in $2dn$ variables that vanishes on $\{(A,B) \in (\mathbb{R}^{d\times n})^2 : \mathcal{M}_{A+\mathbb{i}B} \text{ is non-injective}\}$.

Example: $d = 2, \;\; n = 4d - 4 = 4$

A 2×2 Hertmitian matrix Q defines the real quadratic polynomial

$$q(a,b,c,d) = \begin{pmatrix} a - \mathrm{i}c & b - \mathrm{i}d \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} + \mathrm{i}y_{12} \\ x_{12} - \mathrm{i}y_{12} & x_{22} \end{pmatrix} \begin{pmatrix} a + \mathrm{i}c \\ b + \mathrm{i}d \end{pmatrix}$$

Example: $d = 2, \;\; n = 4d - 4 = 4$

A 2×2 Hertmitian matrix Q defines the real quadratic polynomial

$$q(a,b,c,d) = \begin{pmatrix} a - \mathrm{i}c & b - \mathrm{i}d \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} + \mathrm{i}y_{12} \\ x_{12} - \mathrm{i}y_{12} & x_{22} \end{pmatrix} \begin{pmatrix} a + \mathrm{i}c \\ b + \mathrm{i}d \end{pmatrix}$$
$$= x_{11}(a^2 + c^2) + x_{22}(b^2 + d^2) + 2x_{12}(ab + cd) + 2y_{12}(bc - ad).$$

Example: $d = 2, \;\; n = 4d - 4 = 4$

A 2×2 Hertmitian matrix Q defines the real quadratic polynomial

$$q(a,b,c,d) = \begin{pmatrix} a - \mathrm{i}c & b - \mathrm{i}d \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} + \mathrm{i}y_{12} \\ x_{12} - \mathrm{i}y_{12} & x_{22} \end{pmatrix} \begin{pmatrix} a + \mathrm{i}c \\ b + \mathrm{i}d \end{pmatrix}$$
$$= x_{11}(a^2 + c^2) + x_{22}(b^2 + d^2) + 2x_{12}(ab + cd) + 2y_{12}(bc - ad).$$

Since any Q has rank ≤ 2, the frame

$$\Phi = \begin{pmatrix} a_1 + \mathrm{i}c_1 & a_2 + \mathrm{i}c_2 & a_3 + \mathrm{i}c_3 & a_4 + \mathrm{i}c_4 \\ b_1 + \mathrm{i}d_1 & b_2 + \mathrm{i}d_2 & b_3 + \mathrm{i}d_3 & b_4 + \mathrm{i}d_4 \end{pmatrix}$$

defines injective measurements $\mathcal{M}_\Phi$ whenever

Example: $d = 2, \;\; n = 4d - 4 = 4$

A 2×2 Hertmitian matrix Q defines the real quadratic polynomial

$$q(a,b,c,d) = \begin{pmatrix} a - \mathrm{i}c & b - \mathrm{i}d \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} + \mathrm{i}y_{12} \\ x_{12} - \mathrm{i}y_{12} & x_{22} \end{pmatrix} \begin{pmatrix} a + \mathrm{i}c \\ b + \mathrm{i}d \end{pmatrix}$$
$$= x_{11}(a^2 + c^2) + x_{22}(b^2 + d^2) + 2x_{12}(ab + cd) + 2y_{12}(bc - ad).$$

Since any Q has rank ≤ 2, the frame

$$\Phi = \begin{pmatrix} a_1 + \mathrm{i}c_1 & a_2 + \mathrm{i}c_2 & a_3 + \mathrm{i}c_3 & a_4 + \mathrm{i}c_4 \\ b_1 + \mathrm{i}d_1 & b_2 + \mathrm{i}d_2 & b_3 + \mathrm{i}d_3 & b_4 + \mathrm{i}d_4 \end{pmatrix}$$

defines injective measurements $\mathcal{M}_\Phi$ whenever

$$\det \begin{pmatrix} a_1^2 + c_1^2 & b_1^2 + d_1^2 & a_1 b_1 + c_1 d_1 & b_1 c_1 - a_1 d_1 \\ a_2^2 + c_2^2 & b_2^2 + d_2^2 & a_2 b_2 + c_2 d_2 & b_2 c_2 - a_2 d_2 \\ a_3^2 + c_3^2 & b_3^2 + d_3^2 & a_3 b_3 + c_3 d_3 & b_3 c_3 - a_3 d_3 \\ a_4^2 + c_4^2 & b_4^2 + d_4^2 & a_4 b_4 + c_4 d_4 & b_4 c_4 - a_4 d_4 \end{pmatrix} \neq 0.$$

Fewer measurements: $d = 2^a + 1$

When $n < 4d - 4$, the linear space $\{Q \in \mathbb{C}^{d\times d} \;:\; \phi_k^* Q \phi_k = 0 \;\forall k\}$ will contain rank-2 matrices. Must one be *Hermitian*?

Fewer measurements: $d = 2^a + 1$

When $n < 4d - 4$, the linear space $\{Q \in \mathbb{C}^{d\times d} \ : \ \phi_k^* Q \phi_k = 0 \ \forall k\}$ will contain rank-2 matrices. Must one be *Hermitian*?

When $d = 2^a + 1$, then the degree of $\{rk - 2 \text{ in } \mathbb{C}^{d\times d}\}$ is odd.
$\Rightarrow$ For any $\Phi \in \mathbb{C}^{d\times n}$ with $n < 4d - 4$, M_Φ is not injective.

Fewer measurements: $d = 2^a + 1$

When $n < 4d - 4$, the linear space $\{Q \in \mathbb{C}^{d\times d} \;:\; \phi_k^* Q \phi_k = 0 \;\forall k\}$ will contain rank-2 matrices. Must one be *Hermitian*?

When $d = 2^a + 1$, then the degree of $\{rk - 2 \text{ in } \mathbb{C}^{d\times d}\}$ is odd.
$\Rightarrow$ For any $\Phi \in \mathbb{C}^{d\times n}$ with $n < 4d - 4$, M_Φ is not injective.

Example: $d = 3$, $4d - 5 = 7$

For $\phi_1, \ldots, \phi_7 \in \mathbb{C}^3$, we expect $\{Q : \phi_k^* Q \phi_k = 0\}$ = a line in $\mathbb{P}(\mathbb{C}^{3\times 3})$.

Fewer measurements: $d = 2^a + 1$

When $n < 4d - 4$, the linear space $\{Q \in \mathbb{C}^{d\times d} \ : \ \phi_k^* Q \phi_k = 0 \ \forall k\}$ will contain rank-2 matrices. Must one be *Hermitian*?

When $d = 2^a + 1$, then the degree of $\{rk - 2 \text{ in } \mathbb{C}^{d\times d}\}$ is odd.
$\Rightarrow$ For any $\Phi \in \mathbb{C}^{d\times n}$ with $n < 4d - 4$, M_Φ is not injective.

Example: $d = 3$, $4d - 5 = 7$

For $\phi_1, \ldots, \phi_7 \in \mathbb{C}^3$, we expect $\{Q : \phi_k^* Q \phi_k = 0\} =$ a line in $\mathbb{P}(\mathbb{C}^{3\times 3})$.
$\Rightarrow \{Q : \phi_k^* Q \phi_k = 0\} \cap V(\det(Q)) = 3$ points in $\mathbb{P}(\mathbb{C}^{3\times 3})$.

Fewer measurements: $d = 2^a + 1$

When $n < 4d - 4$, the linear space $\{Q \in \mathbb{C}^{d\times d} \ : \ \phi_k^* Q \phi_k = 0 \ \forall k\}$ will contain rank-2 matrices. Must one be *Hermitian*?

When $d = 2^a + 1$, then the degree of $\{rk - 2 \text{ in } \mathbb{C}^{d\times d}\}$ is odd.
$\Rightarrow$ For any $\Phi \in \mathbb{C}^{d\times n}$ with $n < 4d - 4$, M_Φ is not injective.

Example: $d = 3$, $4d - 5 = 7$

For $\phi_1, \ldots, \phi_7 \in \mathbb{C}^3$, we expect $\{Q : \phi_k^* Q \phi_k = 0\}$ = a line in $\mathbb{P}(\mathbb{C}^{3\times 3})$.
$\Rightarrow \{Q : \phi_k^* Q \phi_k = 0\} \cap V(\det(Q)) = 3$ points in $\mathbb{P}(\mathbb{C}^{3\times 3})$.

Since 3 is odd and the linear space $\{Q : \phi_k^* Q \phi_k = 0\}$ is invariant under $Q \mapsto Q^*$, at least one rank-2 matrix must be Hermitian.

Fewer measurements: $d = 2^a + 1$

When $n < 4d - 4$, the linear space $\{Q \in \mathbb{C}^{d\times d} \ : \ \phi_k^* Q \phi_k = 0 \ \forall k\}$ will contain rank-2 matrices. Must one be *Hermitian*?

When $d = 2^a + 1$, then the degree of $\{rk - 2 \text{ in } \mathbb{C}^{d\times d}\}$ is odd.
$\Rightarrow$ For any $\Phi \in \mathbb{C}^{d\times n}$ with $n < 4d - 4$, M_Φ is not injective.

Example: $d = 3$, $4d - 5 = 7$

For $\phi_1, \ldots, \phi_7 \in \mathbb{C}^3$, we expect $\{Q : \phi_k^* Q \phi_k = 0\} =$ a line in $\mathbb{P}(\mathbb{C}^{3\times 3})$.
$\Rightarrow \{Q : \phi_k^* Q \phi_k = 0\} \cap V(\det(Q)) = 3$ points in $\mathbb{P}(\mathbb{C}^{3\times 3})$.

Since 3 is odd and the linear space $\{Q : \phi_k^* Q \phi_k = 0\}$ is invariant under $Q \mapsto Q^*$, at least one rank-2 matrix must be Hermitian.

$\rightarrow \mathcal{M}_\Phi$ is *not injective*

Fewer measurements: $d = 4$

We parametrize $\mathbb{C}^{4\times 4}_{Herm}$ with $\mathbb{R}^{16}$:

$$Q = \begin{pmatrix} x_{11} & x_{12} + \mathbb{i} y_{12} & x_{13} + \mathbb{i} y_{13} & x_{14} + \mathbb{i} y_{14} \\ x_{12} - \mathbb{i} y_{12} & x_{22} & x_{23} + \mathbb{i} y_{23} & x_{24} + \mathbb{i} y_{24} \\ x_{13} - \mathbb{i} y_{13} & x_{23} - \mathbb{i} y_{23} & x_{33} & x_{34} + \mathbb{i} y_{34} \\ x_{14} - \mathbb{i} y_{14} & x_{24} - \mathbb{i} y_{24} & x_{34} - \mathbb{i} y_{34} & x_{44} \end{pmatrix}.$$

Fewer measurements: $d = 4$

We parametrize $\mathbb{C}^{4\times 4}_{Herm}$ with $\mathbb{R}^{16}$:

$$Q = \begin{pmatrix} x_{11} & x_{12} + \mathbb{i}y_{12} & x_{13} + \mathbb{i}y_{13} & x_{14} + \mathbb{i}y_{14} \\ x_{12} - \mathbb{i}y_{12} & x_{22} & x_{23} + \mathbb{i}y_{23} & x_{24} + \mathbb{i}y_{24} \\ x_{13} - \mathbb{i}y_{13} & x_{23} - \mathbb{i}y_{23} & x_{33} & x_{34} + \mathbb{i}y_{34} \\ x_{14} - \mathbb{i}y_{14} & x_{24} - \mathbb{i}y_{24} & x_{34} - \mathbb{i}y_{34} & x_{44} \end{pmatrix}.$$

Let m_{jk} be the 3×3 minor $\det(Q_{[4]\setminus j,\ [4]\setminus k}) \in \mathbb{Q}[i][x_{jk}, y_{jk}]$.

The matrix Q has rank $\leq 2 \quad \Leftrightarrow \quad m_{jk} = 0$ for all $1 \leq j, k \leq 4$.

Fewer measurements: $d = 4$

We parametrize $\mathbb{C}^{4\times 4}_{Herm}$ with $\mathbb{R}^{16}$:

$$Q = \begin{pmatrix} x_{11} & x_{12} + \mathbb{i}y_{12} & x_{13} + \mathbb{i}y_{13} & x_{14} + \mathbb{i}y_{14} \\ x_{12} - \mathbb{i}y_{12} & x_{22} & x_{23} + \mathbb{i}y_{23} & x_{24} + \mathbb{i}y_{24} \\ x_{13} - \mathbb{i}y_{13} & x_{23} - \mathbb{i}y_{23} & x_{33} & x_{34} + \mathbb{i}y_{34} \\ x_{14} - \mathbb{i}y_{14} & x_{24} - \mathbb{i}y_{24} & x_{34} - \mathbb{i}y_{34} & x_{44} \end{pmatrix}.$$

Let m_{jk} be the 3×3 minor $\det(Q_{[4]\backslash j,\ [4]\backslash k}) \in \mathbb{Q}[i][x_{jk}, y_{jk}]$.

The matrix Q has rank $\leq 2 \quad \Leftrightarrow \quad m_{jk} = 0$ for all $1 \leq j, k \leq 4$.

The map $\mathcal{M}_\Phi$ is injective if and only if there is **no real non-zero solution** $(x_{11}, \ldots, y_{34}) \in \mathbb{R}^{16}$ to the equations

$$m_{11} = m_{12} = \ldots = m_{44} = 0 \ \text{ and } \ \phi_k^* Q \phi_k = 0 \ \ \forall\ k.$$

An injective frame with $d = 4,\ n = 11$

$$\Phi =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 9\mathbb{i} & 1-\mathbb{i} & -2+4\mathbb{i} & -3+\mathbb{i} & 3-3\mathbb{i} & -3+5\mathbb{i} & -3+8\mathbb{i} \\ 0 & 0 & 1 & 0 & -5-7\mathbb{i} & -5-2\mathbb{i} & -4-2\mathbb{i} & 1-8\mathbb{i} & -8+7\mathbb{i} & 5+6\mathbb{i} & 5-5\mathbb{i} \\ 0 & 0 & 0 & 1 & -6-7\mathbb{i} & -1-8\mathbb{i} & 3+8\mathbb{i} & 7-6\mathbb{i} & -6-2\mathbb{i} & 2\mathbb{i} & -6-4\mathbb{i} \end{pmatrix}$$

For $k = 1, \ldots, 11$, let $\ell_k \;=\; \phi_k^* Q \phi_k \in \mathbb{R}[x_{jk}, y_{jk} : 1 \le j \le k \le 4]$.

An injective frame with $d = 4,\ n = 11$

$$\Phi =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 9i & 1-i & -2+4i & -3+i & 3-3i & -3+5i & -3+8i \\ 0 & 0 & 1 & 0 & -5-7i & -5-2i & -4-2i & 1-8i & -8+7i & 5+6i & 5-5i \\ 0 & 0 & 0 & 1 & -6-7i & -1-8i & 3+8i & 7-6i & -6-2i & 2i & -6-4i \end{pmatrix}$$

For $k = 1, \ldots, 11$, let $\ell_k = \phi_k^* Q \phi_k \in \mathbb{R}[x_{jk}, y_{jk} : 1 \leq j \leq k \leq 4]$.

For example, $\ell_1 = x_{11}$ and

$$\ell_5 = x_{11} - 10x_{13} - 12x_{14} + 81x_{22} - 126x_{23} - 126x_{24} + 74x_{33} + 158x_{34} + 85x_{44} - 18y_{12} + 14y_{13} + 14y_{14} - 90y_{23} - 108y_{24} + 14y_{34}.$$

An injective frame with $d = 4$, $n = 11$

$$\Phi =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 9i & 1-i & -2+4i & -3+i & 3-3i & -3+5i & -3+8i \\ 0 & 0 & 1 & 0 & -5-7i & -5-2i & -4-2i & 1-8i & -8+7i & 5+6i & 5-5i \\ 0 & 0 & 0 & 1 & -6-7i & -1-8i & 3+8i & 7-6i & -6-2i & 2i & -6-4i \end{pmatrix}$$

For $k = 1, \ldots, 11$, let $\ell_k = \phi_k^* Q \phi_k \in \mathbb{R}[x_{jk}, y_{jk} : 1 \leq j \leq k \leq 4]$.

For example, $\ell_1 = x_{11}$ and

$$\ell_5 = x_{11} - 10x_{13} - 12x_{14} + 81x_{22} - 126x_{23} - 126x_{24} + 74x_{33} + 158x_{34} + 85x_{44} - 18y_{12} + 14y_{13} + 14y_{14} - 90y_{23} - 108y_{24} + 14y_{34}.$$

We can use symbolic methods to verify that there is **no non-zero solution** $(x_{jk}, y_{jk}) \in \mathbb{R}^{16}$ to $m_{11} = \ldots = m_{44} = \ell_1 = \ldots \ell_{11} = 0$.

An injective frame with $d = 4,\ n = 11$

$$\Phi =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 9i & 1-i & -2+4i & -3+i & 3-3i & -3+5i & -3+8i \\ 0 & 0 & 1 & 0 & -5-7i & -5-2i & -4-2i & 1-8i & -8+7i & 5+6i & 5-5i \\ 0 & 0 & 0 & 1 & -6-7i & -1-8i & 3+8i & 7-6i & -6-2i & 2i & -6-4i \end{pmatrix}$$

For $k = 1, \ldots, 11$, let $\ell_k = \phi_k^* Q \phi_k \in \mathbb{R}[x_{jk}, y_{jk} : 1 \leq j \leq k \leq 4]$.

For example, $\ell_1 = x_{11}$ and

$$\ell_5 = x_{11} - 10x_{13} - 12x_{14} + 81x_{22} - 126x_{23} - 126x_{24} + 74x_{33} + 158x_{34} + 85x_{44} - 18y_{12} + 14y_{13} + 14y_{14} - 90y_{23} - 108y_{24} + 14y_{34}.$$

We can use symbolic methods to verify that there is **no non-zero solution** $(x_{jk}, y_{jk}) \in \mathbb{R}^{16}$ to $m_{11} = \ldots = m_{44} = \ell_1 = \ldots \ell_{11} = 0$.

$\Rightarrow \mathcal{M}_\Phi$ **is injective**

Certifying injectivity of Φ

We want to show that there are no non-zero solutions to

$$m_{11} = m_{12} = \ldots = m_{44} = \ell_1 = \ldots \ell_{11} = 0. \qquad (*)$$

(Actually, the solution set is 10 pairs of complex conjugate lines in $\mathbb{C}^{16}$.)

Certifying injectivity of Φ

We want to show that there are no non-zero solutions to

$$m_{11} = m_{12} = \ldots = m_{44} = \ell_1 = \ldots \ell_{11} = 0. \qquad (*)$$

(Actually, the solution set is 10 pairs of complex conjugate lines in $\mathbb{C}^{16}$.)

Strategy:

1. Find $f \in \mathbb{Q}[x_{34}, y_{34}]$ satisfying: $f(x_{34}, y_{34}) = 0$ if and only if the point $(x_{34}, y_{34}) \in \mathbb{C}^2$ can be extended to a solution $(x_{jk}, y_{jk}) \in \mathbb{C}^{16}$ of (*). GB
2. Check that $f(x_{34}, 1)$ has no real roots. Sturm Sequences
3. Check that there are no non-zero solutions $(x_{jk}, y_{jk}) \in \mathbb{C}^{16}$ to (*) with $y_{34} = 0$. GB

The degree-20 polynomial f

$$
\begin{aligned}
f(x,y) = {} & \\
& 4759968569745446624632941235848317972215004335443712508202580090260692859720627225484588720209848521523 2 \cdot x^{20} \\
& -9408757898677587698385207544032012686757747191942419403886561777856441943421668927931239678701185110917 12 \cdot x^{19}y \\
& +8079760677210192071804090111142610477024725441627364213141746522285905327070793719538623768982021441867 008 \cdot x^{18}y^{2} \\
& -4039076119385512227738119861674476347949768089560889759338652081079474904180163379696825629934525056798 9120 \cdot x^{17}y^{3} \\
& +1316163699161712083349773390645033718595763912689290644681189359000173652951856270420783825929203590239 63120 \cdot x^{16}y^{4} \\
& -2930143953295830258772603727896289422633385156858345889638963396132176909535601120631345912044691669037 30584 \cdot x^{15}y^{5} \\
& +4580697380327306959961441352487913380075697108775299383780927457830775495589761570255507459619722253400 79644 \cdot x^{14}y^{6} \\
& -5173690715936272198475209439244544585611474515244956750989070213709762812172996403114894657046923686152 64514 \cdot x^{13}y^{7} \\
& +4525989792302552884426716279347073780027478930147173884948180216545288751973456241545086261140379729015 00688 \cdot x^{12}y^{8} \\
& -3726489629089989125062840863318293346597041580385723887626070813975403978918752880203278418002758078963 31363 \cdot x^{11}y^{9} \\
& +3682328648215806636083625072247318422248169481663757922519581898984133499430591999918507459208575873464 22247 \cdot x^{10}y^{10} \\
& -4036357117318856838318622860038798713682858360905769539302388231747011112630825131743283190918248458784 08842 \cdot x^{9}y^{11} \\
& +3909211915449450601064540973487640801752188774101560792079769947965884448045745835258520461161334060634 92232 \cdot x^{8}y^{12} \\
& -3032822467435356773800177458896813711365404193801126904332399474919797642268623791827771429742112422014 36038 \cdot x^{7}y^{13} \\
& +1844793803200490451976865054438239601533846094289877804325730051093976579264406885582986834930923436853 87706 \cdot x^{6}y^{14} \\
& -8748531134946098282444899249804604349842739617932165019824281993965335236316505756427803378950027337397 3662 \cdot x^{5}y^{15} \\
& +3201652076372467643713417459481895553698485776946191554627380432236585669329009090385178872977727504041 1744 \cdot x^{4}y^{16} \\
& -8843043103455739360596137302837349740785483274132912552686735695145524362028265118639059872092039716064 999 \cdot x^{3}y^{17} \\
& +1775125426181341100587099980276312627299716879819457817398603067248151810981307579223879621865024794510 283 \cdot x^{2}y^{18} \\
& -2415271186523114884330387721689130740259912144531886286475890570332460720769964895775316661853363323084 62 \cdot xy^{19} \\
& +1789221783272048344039984590283109020243476322910421222065808511084122010609114807044576623410638172200 0 \cdot y^{20}
\end{aligned}
$$

The set of injective frames

Both $\{\Phi' : \mathcal{M}_{\Phi'} \text{ is injective}\}$ and $\{\Phi' : \mathcal{M}_{\Phi'} \text{ is non-injective}\}$ are full-dimensional semialgebraic sets in $\mathbb{C}^{4\times 11} \cong (\mathbb{R}^{4\times 11})^2$.

The set of injective frames

Both $\{\Phi' : \mathcal{M}_{\Phi'} \text{ is injective}\}$ and $\{\Phi' : \mathcal{M}_{\Phi'} \text{ is non-injective}\}$ are full-dimensional semialgebraic sets in $\mathbb{C}^{4\times 11} \cong (\mathbb{R}^{4\times 11})^2$.

For example, we can vary the last entry of $\Phi =$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 9i & 1-i & -2+4i & -3+i & 3-3i & -3+5i & -3+8i \\ 0 & 0 & 1 & 0 & -5-7i & -5-2i & -4-2i & 1-8i & -8+7i & 5+6i & 5-5i \\ 0 & 0 & 0 & 1 & -6-7i & -1-8i & 3+8i & 7-6i & -6-2i & 2i & a+ib \end{pmatrix}.$$

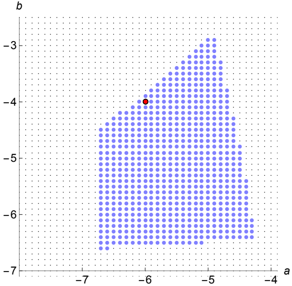
The set of injective frames

Both $\{\Phi' : \mathcal{M}_{\Phi'} \text{ is injective}\}$ and $\{\Phi' : \mathcal{M}_{\Phi'} \text{ is non-injective}\}$ are full-dimensional semialgebraic sets in $\mathbb{C}^{4\times 11} \cong (\mathbb{R}^{4\times 11})^2$.

For example, we can vary the last entry of $\Phi =$

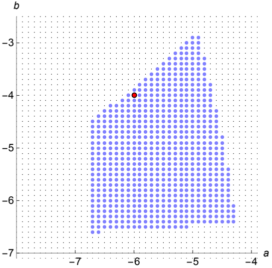
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 9i & 1-i & -2+4i & -3+i & 3-3i & -3+5i & -3+8i \\ 0 & 0 & 1 & 0 & -5-7i & -5-2i & -4-2i & 1-8i & -8+7i & 5+6i & 5-5i \\ 0 & 0 & 0 & 1 & -6-7i & -1-8i & 3+8i & 7-6i & -6-2i & 2i & a+ib \end{pmatrix}.$$

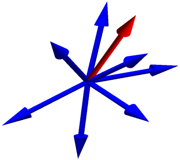
The set $\{\Phi' : \mathcal{M}_{\Phi'} \text{ is injective}\}$ contains an open ball around Φ.



Final thoughts and questions

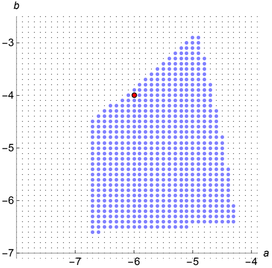
Algebraic methods are useful for some problems in frame theory, especially computing small examples.





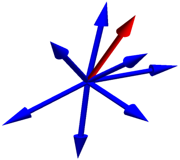
Final thoughts and questions

Algebraic methods are useful for some problems in frame theory, especially computing small examples.



Modified $4d - 4$ **Conjecture.**

For $\Phi \in \mathbb{C}^{d \times 4d-5}$, the probability p_d that $\mathcal{M}_\Phi$ is injective is less than 1 and $p_d \to 0$ as $d \to \infty$.

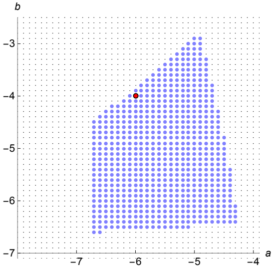


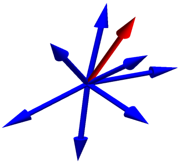
Final thoughts and questions

Algebraic methods are useful for some problems in frame theory, especially computing small examples.

Modified $4d-4$ Conjecture.

For $\Phi \in \mathbb{C}^{d\times 4d-5}$, the probability p_d that $\mathcal{M}_\Phi$ is injective is less than 1 and $p_d \to 0$ as $d \to \infty$.

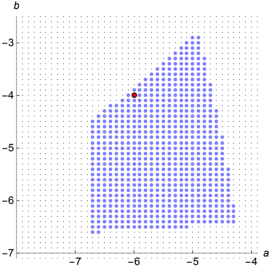




- What is the smallest n for which there exists $\Phi \in \mathbb{C}^{d\times n}$ with $\mathcal{M}_\Phi$ injective?
- For $d \neq 2^k + 1$ and $n = 4d-5$, can we *construct* $\Phi \in \mathbb{C}^{d\times n}$ with $\mathcal{M}_\Phi$ injective?
- How can we *efficiently* guarantee injectivity?

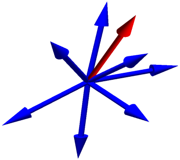
Final thoughts and questions

Algebraic methods are useful for some problems in frame theory, especially computing small examples.



Modified $4d-4$ **Conjecture.**

For $\Phi \in \mathbb{C}^{d\times 4d-5}$, the probability p_d that $\mathcal{M}_\Phi$ is injective is less than 1 and $p_d \to 0$ as $d \to \infty$.



- What is the smallest n for which there exists $\Phi \in \mathbb{C}^{d\times n}$ with $\mathcal{M}_\Phi$ injective?
- For $d \neq 2^k+1$ and $n = 4d-5$, can we *construct* $\Phi \in \mathbb{C}^{d\times n}$ with $\mathcal{M}_\Phi$ injective?
- How can we *efficiently* guarantee injectivity?

Thanks!