Numerical methods for computing real and complex tropical curves

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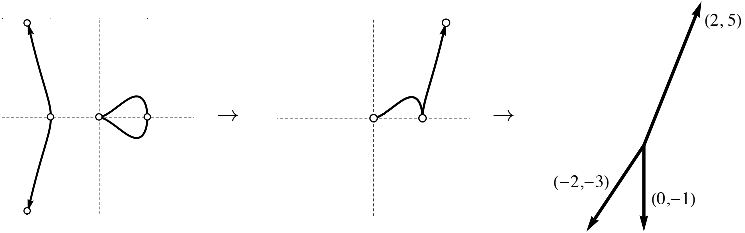

joint with Daniel Brake and Jonathan Hauenstein

Puiseux series, valuations, and tropical varieties

For $\mathbb{k} = \mathbb{R}, \mathbb{C}$, take the Puiseux series over $\mathbb{k}$:

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This is an algebraically closed ($\mathbb{k} = \mathbb{C}$) or real closed ($\mathbb{k} = \mathbb{R}$) field with valuation $\mathrm{val} : \mathbb{k}\{\{t\}\}^* \to \mathbb{Q}$:

$$\mathrm{val}\left(\sum c_q t^q\right) = \min\{q \mid c_q \neq 0\}.$$

This extends coordinate-wise to $\mathrm{val} : \mathbb{k}\{\{t\}\}^n \to \mathbb{Q}^n$.

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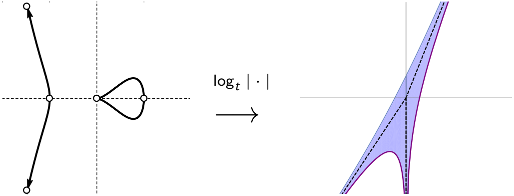
We can take the variety of $I \subset \Bbbk[x_1, \ldots, x_n]$ over $\Bbbk\{\{t\}\}$.

The $\Bbbk$-**tropical variety** of I is $\mathrm{Trop}_\Bbbk(I) = -\overline{\mathrm{val}(\mathcal{V}_{\Bbbk\{\{t\}\}} I)} \subset \mathbb{R}^n$.

Logarithmic limit sets

For $t \in \mathbb{R}_+$ and $V \subset \mathbb{k}^n$, consider the image under $\log_t(|\cdot|)$:

$$\mathcal{A}_t(I) \;= \log_t(|V|) \qquad\qquad \text{(taken coordinate-wise).}$$



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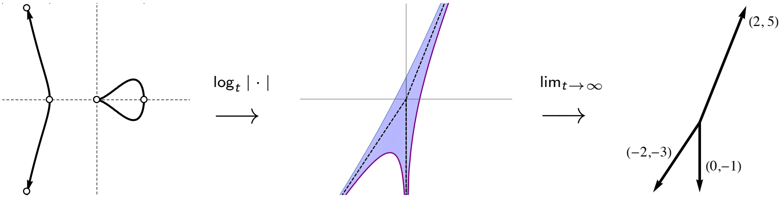

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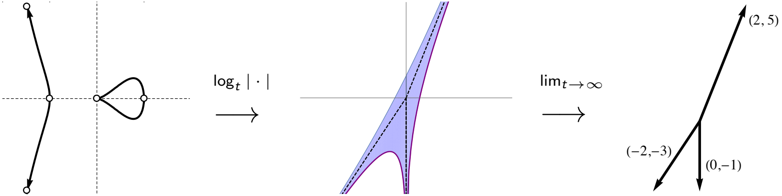
For both $\Bbbk = \mathbb{R}, \mathbb{C}$, $\mathcal{A}_\infty(\mathcal{V}_\Bbbk(I))$ equals $\mathrm{Trop}_\Bbbk(I)$.



Connections with initial ideals

For $w \in \mathbb{R}^n$, $f = \sum_\alpha c_\alpha x^\alpha$, define $\text{in}_w(f) = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha$, where $\mathcal{A}$ is the set of α maximizing $w \cdot \alpha$. Then $\text{in}_w(I) = \langle \text{in}_w(f) : f \in I \rangle$.

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One can define $\text{Trop}_{\mathbb{C}}(I)$ in terms of initial ideals:

$$\text{Trop}_{\mathbb{C}}(I) \;=\; \{w \in \mathbb{R}^n \;:\; \mathcal{V}(\text{in}_w(I)) \cap (\mathbb{C}^*)^n \neq \emptyset\}\,.$$

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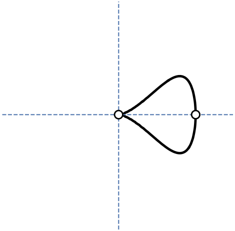
E.g. For $I = \langle (x-y)^2 + 1 \rangle$, $\mathcal{V}_\mathbb{R}(I) = \emptyset$ and $\text{Trop}_\mathbb{R}(I) = \emptyset$, but $\mathcal{V}_{\mathbb{R}^*}(\text{in}_{(1,1)}(I)) = \mathcal{V}_{\mathbb{R}^*}((x-y)^2) \neq \emptyset$.

Example: sextic plane curve

Consider $I = \langle x^6 - x^3 + y^2 \rangle$.



$\mathcal{V}_{\mathbb{R}^*}(I)$

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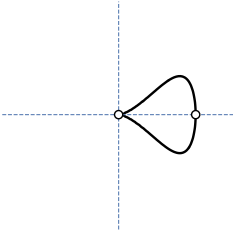
$$(x, y) = \left(t^2,\ t^3 - \frac{t^9}{2} - \frac{t^{15}}{8} + \dots\right) \quad \rightarrow \quad -\operatorname{val}(x, y) = (-2, -3)$$



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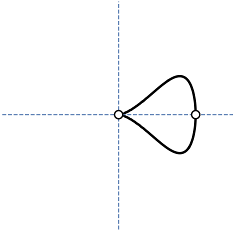

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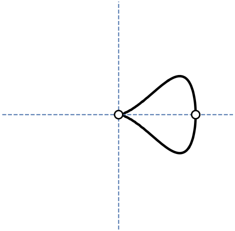

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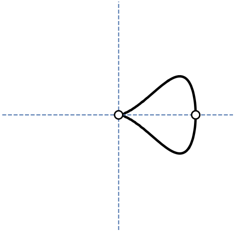

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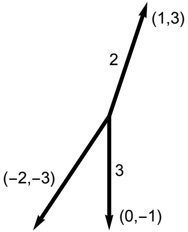
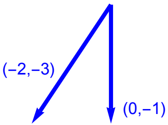


$\mathrm{Trop}_{\mathbb{C}}(I)$

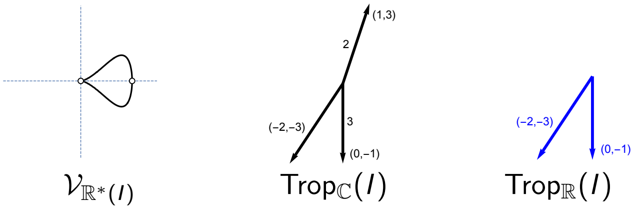


$\mathrm{Trop}_{\mathbb{R}}(I)$

Tropical varieties are polyhedral fans

Theorem (Fundamental Theorem of Tropical Geometry)

For an irreducible ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$, $\text{Trop}_{\mathbb{C}}(I)$ is a rational pure-dimensional polyhedral fan of dimensional $d = \dim(\mathcal{V}_{\mathbb{C}^*}(I))$. Counting cones with appropriate multiplicities, it is balanced.



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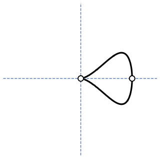
Theorem (Alessandrini, 2013)

For an ideal $I \subset \mathbb{R}[x_1, \ldots, x_n]$, $\mathrm{Trop}_{\mathbb{R}}(I)$ *is a rational polyhedral fan of dimensional* $d \leq \dim(\mathcal{V}_{\mathbb{R}^*}(I))$.



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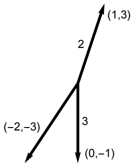

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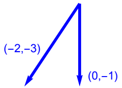

$\mathrm{Trop}_{\mathbb{R}}(I)$

Computing Tropical Varieties

Symbolic Algorithm: `Gfan` – developed by Jensen

uses Gröbner bases to compute $\mathrm{Trop}_{\mathbb{C}}(I)$ for any $I \subset \mathbb{Q}[x_1, \ldots, x_n]$.

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Numerical Algorithms:

- $\text{Trop}_{\mathbb{C}}$ of hypersurfaces (Hauenstein, Sottile, 2014)
- $\text{Trop}_{\mathbb{C}}$ of curves (Jensen, Leykin, Yu, 2015)
- $\text{Trop}_{\mathbb{C}}$ and $\text{Trop}_{\mathbb{R}}$ of curves (Brake, Hauenstein, V-)

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Numerical algorithms do not require defining equations.

Curves are tractable and are used in internal computations for $\text{Trop}_{\mathbb{C}}$ of larger dimensional varieties.

Strategy for computing $\text{Trop}_{\mathbb{k}}(I)$

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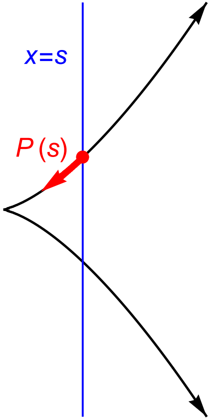
Cauchy Integrals: If $f(z)$ is analytic on $\{z \in \mathbb{C} : |z| \leq \tau\}$, then

$$f^{(k)}(0) = \frac{k!}{2\pi} \int_0^{2\pi} \frac{f(\tau e^{i\theta})}{(\tau e^{i\theta})^{k+1}} d\theta,$$

and $f(z) = \sum_{k=0}^{\infty} f^{(k)}(0) \cdot \frac{1}{k!} \cdot z^k$ for $|z| \leq \tau$.

Finding an analytic parametrization: monodromy

Suppose $\{P(s) : s \in [0, \tau]\} \subset C$ and $P_j(s) = s$.



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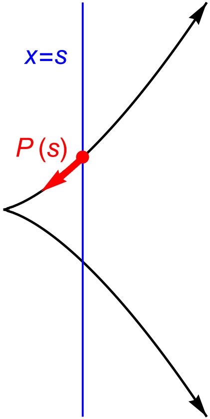

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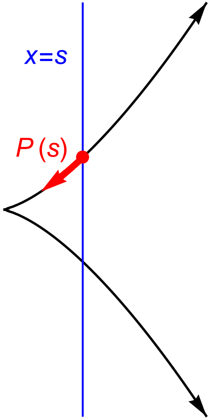

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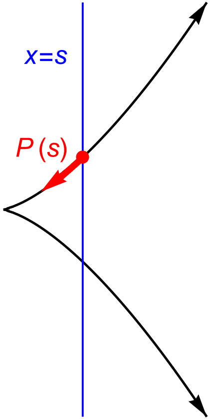

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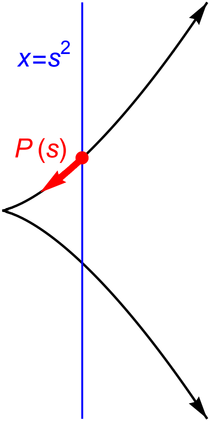

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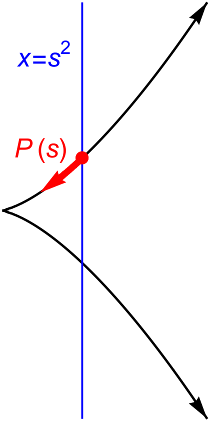
Cycle number $= 2$

Finding an analytic parametrization: monodromy



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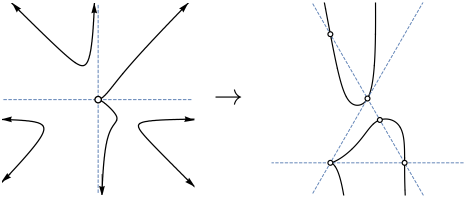
Re-parametrize: $s \mapsto s^2$, $P(s) = (s^2, s^3)$

Let $C \subset \mathbb{C}^n$ be an irreducible curve.

- Take $\overline{C} \subset \mathbb{P}^n(\mathbb{C})$ and an affine slice $\widehat{C} = \{\ell = 1\} \cap \overline{C}$ containing all the points $\overline{C} \cap \mathcal{V}(x_0 x_1 \cdot x_n)$.



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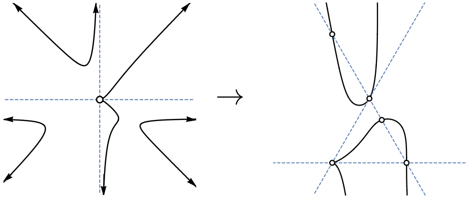
- Compute $T^* = \min(|T|)$ such that $\{x_j - T\}$ is tangent to $\widehat{C}$.



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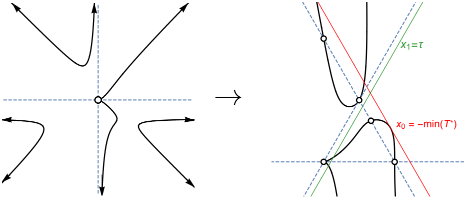
- Take $\overline{C} \subset \mathbb{P}^n(\mathbb{C})$ and an affine slice $\widehat{C} = \{\ell = 1\} \cap \overline{C}$ containing all the points $\overline{C} \cap \mathcal{V}(x_0 x_1 \cdot x_n)$.
- Compute $T^* = \min(|T|)$ such that $\{x_j - T\}$ is tangent to $\widehat{C}$.
- For $\tau < T^*$, slice $\widehat{C}$ with $\{x_j - \tau\}$.



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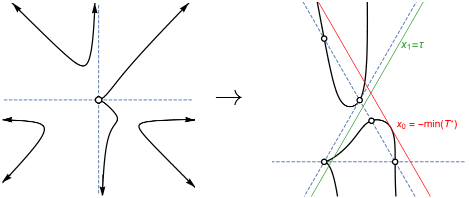

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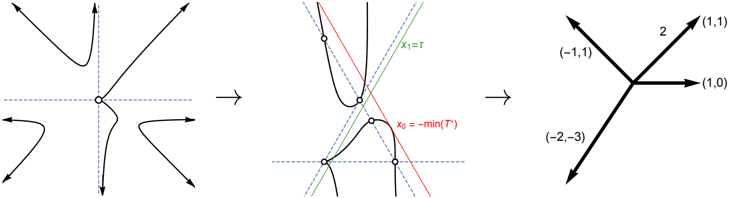
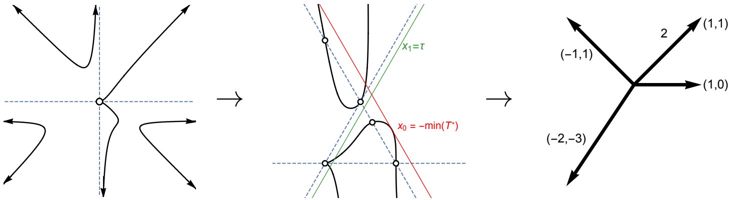


Example: quartic plane curve

Replace $C = \mathcal{V}(x_1^3 x_2 - x_1 x_2^3 + x_1^3 - x_2^2) \subset \mathbb{C}^2$ with
$\widehat{C} = \mathcal{V}(x_1^3 x_2 - x_1 x_2^3 + x_0 x_1^3 - x_0^2 x_2^2, \;\; x_0 + x_1 + 2x_2 - 1) \subset \mathbb{C}^3$.



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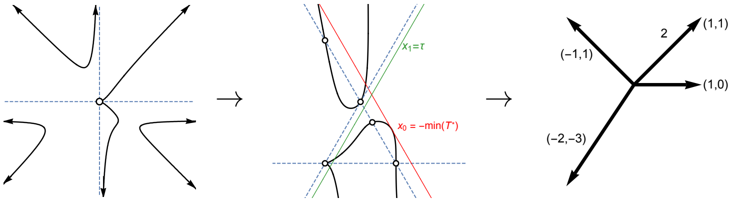
$\widehat{C} \cap \mathcal{V}(x_0x_1x_2) = \{(0,1,0), (0,1/3,1/3), (0,0,1/2), (0,-1,1), (1,0,0)\}$



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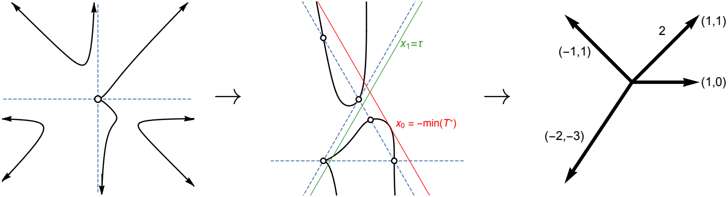
The point $p \approx (0.8293, 0.1, 0.0354) \subset \widehat{C} \cap \{x_1 = .1\}$ tracks to $(1,0,0)$.



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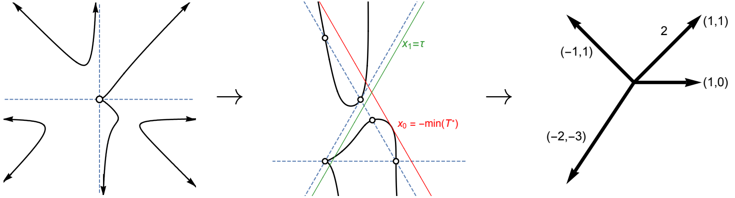
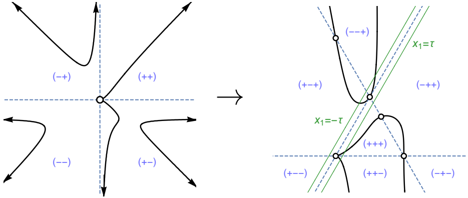
Valuation of the path is $(0,2,3) \quad \rightarrow \quad (-2,-3) \in \text{Trop}_{\mathbb{C}}(C)$.



Real Tropical Strategy

We can compute $\mathrm{Trop}_{\mathbb{R}}(C)$ similarly to $\mathrm{Trop}_{\mathbb{C}}(C)$.

This requires checking $\{x_j = \pm\tau\}$ for real points and only considering real paths converging to $\widehat{C} \cap \{x_j = 0\}$.



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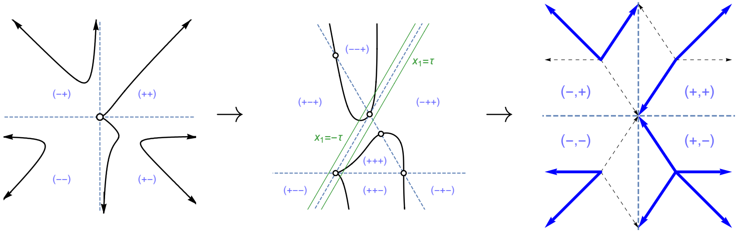
Keeping track of signs of the parametrized paths gives the signed real tropical variety.



Computing curves in large spaces

A central curve defined by . . .

$$\begin{bmatrix} x_1 s_1 - x_j s_j \quad \text{for} \quad j = 2, \ldots, 7 \\ -u_0 + t^2 - x_1 \\ -v_0 + t^4 - x_2 \\ u_1 - x_3 \\ v_1 - x_4 \\ t(u_0 + v_0) - v_1 - x_5 \\ t^2 u_0 - u_1 - x_6 \\ t^2 v_0 - u_1 - x_7) \\ t^2 x_2 + x_3 + x_7 - t^6 \\ t^2 x_1 + x_3 + x_6 - t^4 \\ tx_1 + tx_2 + x_4 + x_5 - t^3 - t^5 \\ s_1 - ts_5 - t^2 s_6 \\ s_2 - ts_5 - t^2 s_7 - 1 \\ s_3 - s_6 - s_7 \\ s_4 - s_5 \end{bmatrix}$$

Allamigeon, Benchimol, Gaubert, and Joswig use real tropical methods to construct a family of linear programs whose central curves have high total curvature.

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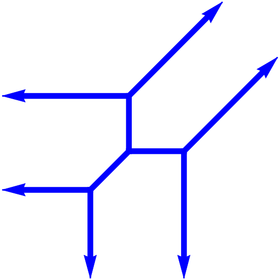
Our algorithm find that the tropical variety $\text{Trop}_{\mathbb{C}}(C) = \text{Trop}_{\mathbb{R}}(C)$ consists of 10 rays with multiplicites $6, 4, 3, 2, 2, 1, 1, 1, 1, 1$.

Looking beyond



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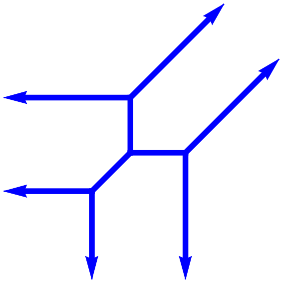

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We would like to develop these methods for varieties of dimension ≥ 1, like *surfaces*, over both $\mathbb{C}$ and $\mathbb{R}$.

Tropicalizations of varieties higher dimension can also be used to compute tropical varieties of ideals in $\Bbbk\{\{t\}\}[x_1, \ldots, x_n]$, which are used in many applications (like central curves).

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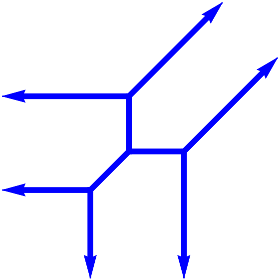

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