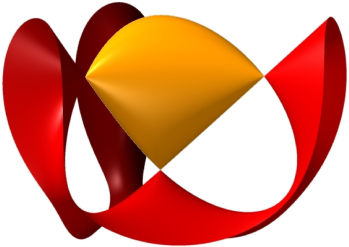
Spectrahedra

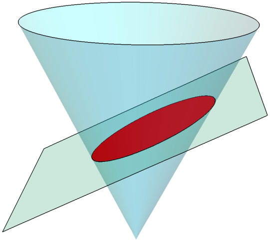


Cynthia Vinzant,
North Carolina State University
Simons Institute, Berkeley (Fall 2017)

Spectrahedra . . . What? Why?

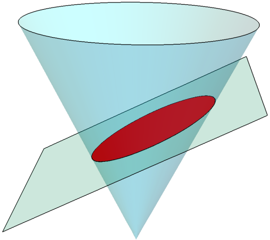
Let PSD_d denote the convex cone of positive semidefinite matrices in $\mathbb{R}^{d \times d}_{\text{sym}}$.

A spectrahedron is the intersection PSD_d with an affine linear space L.



Let PSD_d denote the convex cone of positive semidefinite matrices in $\mathbb{R}^{d\times d}_{\mathrm{sym}}$.

A spectrahedron is the intersection PSD_d with an affine linear space L.

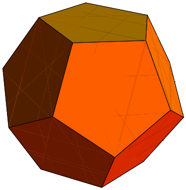


Writing $L = A_0 + \mathrm{span}_{\mathbb{R}}\{A_1, \dots, A_n\}$ identifies $L \cap PSD_d$ with

$\mathcal{S} = \{x \in \mathbb{R}^n : A(x) \succeq 0\}$ where $A(x) = A_0 + \sum_{i=1}^n x_i A_i$.

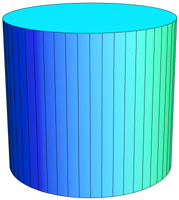
These are feasible sets of semidefinite programs (extension of linear programming with applications in combinatorial optimization, control, polynomial optimization, quantum information, . . .).

Some examples



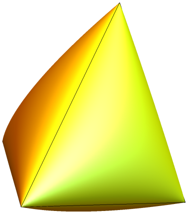
polytope

$$\begin{pmatrix} \ell_1(\underline{x}) & & 0 \\ & \ddots & \\ 0 & & \ell_{12}(\underline{x}) \end{pmatrix}$$



cylinder

$$\begin{pmatrix} 1-x & y & 0 & 0 \\ y & 1+x & 0 & 0 \\ 0 & 0 & 1-z & 0 \\ 0 & 0 & 0 & 1+z \end{pmatrix}$$



elliptope

$$\begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix}$$

Some differences with polyhedra:

- $\mathcal{S}$ can have infinitely-many faces
- dim(face) + dim(normal cone) not always equal to n.

Positive semidefinite matrices

A real symmetric matrix A is positive semidefinite if the following equivalent conditions hold:

- all eigenvalues of A are ≥ 0
- all diagonal minors of A are ≥ 0
- $v^T A v \geq 0$ for all $v \in \mathbb{R}^d$
- there exists $B \in \mathbb{R}^{d \times k}$ with

$$A = BB^T = (\langle r_i, r_j \rangle)_{ij} = \sum_{i=1}^{k} c_i c_i^T$$

where $r_1, \ldots, r_d$, $c_1, \ldots, c_k$ are the rows and cols of B

The convex cone of PSD matrices

The cone of PSD matrices $PSD_d = \mathrm{conv}(\{xx^T : x \in \mathbb{R}^d\})$.

PSD_d is self-dual under the inner product $\langle A, B\rangle = \mathrm{trace}(A \cdot B)$:

$$\begin{aligned} \langle A, B\rangle \geq 0 \text{ for all } B \in PSD_d &\Leftrightarrow \langle A, bb^T\rangle \geq 0 \text{ for all } b \in \mathbb{R}^d \\ &\Leftrightarrow b^T A b \geq 0 \text{ for all } b \in \mathbb{R}^d \\ &\Leftrightarrow A \in PSD_d \end{aligned}$$

The convex cone of PSD matrices

The cone of PSD matrices $PSD_d = \text{conv}(\{xx^T : x \in \mathbb{R}^d\})$.

PSD_d is self-dual under the inner product $\langle A, B\rangle = \text{trace}(A \cdot B)$:

$$\begin{aligned} \langle A, B\rangle \geq 0 \text{ for all } B \in PSD_d \;&\Leftrightarrow\; \langle A, bb^T\rangle \geq 0 \text{ for all } b \in \mathbb{R}^d \\ &\Leftrightarrow\; b^T A b \geq 0 \text{ for all } b \in \mathbb{R}^d \\ &\Leftrightarrow\; A \in PSD_d \end{aligned}$$

Faces of PSD_d have dim $\binom{r+1}{2}$ for $r = 0, 1, \ldots, d$ and look like

$$F_L = \{A \in PSD_d \;:\; L \subseteq \ker(A)\}.$$

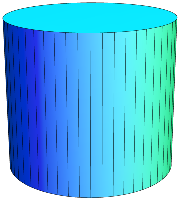
Ex: for $L = \mathrm{span}\{e_{r+1}, \ldots, e_d\}$,

$$F_L = \left\{ \begin{pmatrix} A & \underline{0} \\ \underline{0} & \underline{0} \end{pmatrix} : A \in PSD_r \right\} \cong PSD_r$$

Structure of spectrahedra

A spectrahedron $\mathcal{S} = \{x \in \mathbb{R}^n : A(x) \succeq 0\}$
is a convex, basic-closed semi-algebraic set.

$$A(x,y,z) = \begin{pmatrix} 1-x & y & 0 & 0 \\ y & 1+x & 0 & 0 \\ 0 & 0 & 1-z & 0 \\ 0 & 0 & 0 & 1+z \end{pmatrix} \quad \leftrightarrow$$



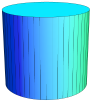
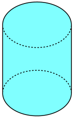
Its faces are intersections of faces of PSD_d with $\{A(x) : x \in \mathbb{R}^n\}$

$\rightarrow$ characterized by kernels of $A(x)$.

Spectrahedral Shadows: an interlude

Caution:

The projection of spectrahedron may not be a spectrahedron!

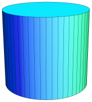
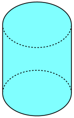
$\mathcal{S} =$  $proj(\mathcal{S}) =$  not *basic* closed

$\Rightarrow$ not a spectrahedron

Spectrahedral Shadows: an interlude

Caution:

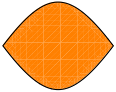
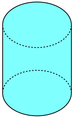
The projection of spectrahedron may not be a spectrahedron!

$\mathcal{S} =$  $\mathit{proj}(\mathcal{S}) =$  not *basic* closed

$\Rightarrow$ not a spectrahedron

Caution:

The convex dual of spectrahedron may not be a spectrahedron!

$\mathcal{S} =$  $\mathcal{S}^\circ =$  still not a spectrahedron

Spectrahedral shadows: an interlude

A spectrahedral shadow is the image of a spectrahedron under linear projection. These are convex semialgebraic sets.

Unlike spectrahedra, the class of spectrahedral shadows is closed under projection, duality, convex hull of unions, . . .

For more, come to
"An Afternoon of Real Algebraic Geometry," MSRI, Friday Sept. 15, 2-6pm.

Spectrahedral shadows: an interlude

A spectrahedral shadow is the image of a spectrahedron under linear projection. These are convex semialgebraic sets.

Unlike spectrahedra, the class of spectrahedral shadows is closed under projection, duality, convex hull of unions, . . .

Helton-Nie Conjecture (2009):
Every convex semialgebraic set is a spectrahedral shadow.

For more, come to
"An Afternoon of Real Algebraic Geometry," MSRI, Friday Sept. 15, 2-6pm.

Spectrahedral shadows: an interlude

A spectrahedral shadow is the image of a spectrahedron under linear projection. These are convex semialgebraic sets.

Unlike spectrahedra, the class of spectrahedral shadows is closed under projection, duality, convex hull of unions, ...

Helton-Nie Conjecture (2009):
Every convex semialgebraic set is a spectrahedral shadow.

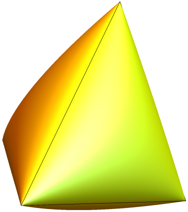
Counterexample by Scheiderer in 2016.

Open: What is the smallest dimension of a counterexample?

For more, come to
"An Afternoon of Real Algebraic Geometry," MSRI, Friday Sept. 15, 2-6pm.

Example: Elliptopes

The $d \times d$ elliptope is



$$\mathcal{E}_d = \{A \in PSD_d : A_{ii} = 1 \text{ for all } i\}$$

$= \{d \times d$ correlation matrices$\}$ in stats

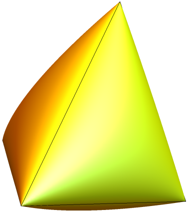
$\mathcal{E}_d$ has 2^{d-1} matrices of rank-one: $\{xx^T : x \in \{-1,1\}^d\}$, corresponding to cuts in the complete graph K_d.

Example: Elliptopes

The $d \times d$ elliptope is

$$\mathcal{E}_d = \{A \in PSD_d : A_{ii} = 1 \text{ for all } i\}$$

$= \{d \times d$ correlation matrices$\}$ in stats



$\mathcal{E}_d$ has 2^{d-1} matrices of rank-one: $\{xx^T : x \in \{-1,1\}^d\}$, corresponding to cuts in the complete graph K_d.

$$\begin{aligned} MAXCUT &= \max_{S \subset [d]} \sum_{i \in S, j \in S^c} w_{ij} = \max_{x \in \{-1,1\}^d} \sum_{i,j} w_{ij} \frac{(1 - x_i x_j)}{2} \\ &= \max_{A \in \mathcal{E}_d, \mathrm{rk}(A)=1} \sum_{i,j} w_{ij} \frac{(1 - A_{ij})}{2} \leq \max_{A \in \mathcal{E}_d} \sum_{i,j} w_{ij} \frac{(1 - A_{ij})}{2}. \end{aligned}$$

Goemans-Williamson use this to give $\approx .87$ optimal cuts of graphs.

Example: Univariate Moments

$\mathcal{S} = \mathrm{conv}\{(t, t^2, \ldots, t^{2d}) : t \in \mathbb{R}\}$ is a spectrahedron in $\mathbb{R}^{2d}$

$\mathcal{S} = \{x \in \mathbb{R}^{2d} : M(x) \succeq 0\}$ where $M(x) = (x_{i+j-2})_{1 \leq i,j \leq d+1}$

Example: Univariate Moments

$\mathcal{S} = \text{conv}\{(t, t^2, \ldots, t^{2d}) : t \in \mathbb{R}\}$ is a spectrahedron in $\mathbb{R}^{2d}$

$\mathcal{S} = \{x \in \mathbb{R}^{2d} : M(x) \succeq 0\}$ where $M(x) = (x_{i+j-2})_{1 \le i,j \le d+1}$

Ex. (d=1): $\text{conv}\{(t, t^2) : t \in \mathbb{R}\} = \left\{(x_1, x_2) : \begin{pmatrix} 1 & x_1 \\ x_1 & x_2 \end{pmatrix} \succeq 0\right\}$

Example: Univariate Moments

$\mathcal{S} = \text{conv}\{(t, t^2, \ldots, t^{2d}) : t \in \mathbb{R}\}$ is a spectrahedron in $\mathbb{R}^{2d}$

$\mathcal{S} = \{x \in \mathbb{R}^{2d} : M(x) \succeq 0\}$ where $M(x) = (x_{i+j-2})_{1 \leq i,j \leq d+1}$

Ex. (d=1): $\text{conv}\{(t, t^2) : t \in \mathbb{R}\} = \left\{(x_1, x_2) : \begin{pmatrix} 1 & x_1 \\ x_1 & x_2 \end{pmatrix} \succeq 0\right\}$

Minimization of univariate polynomial of degree $\leq 2d$

$\rightarrow$Minimization of linear function over $\mathcal{S}$

Example: Univariate Moments

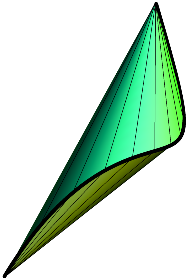
$\mathcal{S} = \mathrm{conv}\{(t, t^2, \ldots, t^{2d}) : t \in \mathbb{R}\}$ is a spectrahedron in $\mathbb{R}^{2d}$

$\mathcal{S} = \left\{x \in \mathbb{R}^{2d} : M(x) \succeq 0\right\}$ where $M(x) = (x_{i+j-2})_{1 \le i,j \le d+1}$

Ex. (d=1): $\mathrm{conv}\{(t, t^2) : t \in \mathbb{R}\} = \left\{(x_1, x_2) : \begin{pmatrix} 1 & x_1 \\ x_1 & x_2 \end{pmatrix} \succeq 0\right\}$

Minimization of univariate polynomial of degree $\le 2d$

$\rightarrow$Minimization of linear function over $\mathcal{S}$



Ex: $\mathrm{conv}\{(t, t^2, t^3) : t \in [-1, 1]\}$

$$= \left\{x \in \mathbb{R}^3 : \begin{pmatrix} 1 \pm x_1 & x_1 \pm x_2 \\ x_1 \pm x_2 & x_2 \pm x_3 \end{pmatrix} \succeq 0\right\}$$

Extreme Points: Pataki range

$\mathcal{S} = \{x \in \mathbb{R}^n : A(x) \succeq 0\}, \quad \dim(\mathcal{S}) = n,\ A_i \in \mathbb{R}^{d\times d}_{\mathrm{sym}}.$

If x is an extreme point of $\mathcal{S}$ and r is the rank of $A(x)$ then

$$\binom{r+1}{2} + n \ \leq \ \binom{d+1}{2}$$

Furthermore if $A_0, \ldots, A_n$ are generic, then $n \geq \binom{d-r+1}{2}$.

The interval of $r \in \mathbb{Z}_+$ satisfying both $\leq$'s is the Pataki range.

Extreme Points: Pataki range

$\mathcal{S} = \{x \in \mathbb{R}^n : A(x) \succeq 0\}$, $\dim(\mathcal{S}) = n$, $A_i \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$.

If x is an extreme point of $\mathcal{S}$ and r is the rank of $A(x)$ then

$$\binom{r+1}{2} + n \ \leq \ \binom{d+1}{2}$$

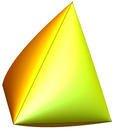
Furthermore if $A_0, \ldots, A_n$ are generic, then $n \geq \binom{d-r+1}{2}$.

The interval of $r \in \mathbb{Z}_+$ satisfying both $\leq$'s is the Pataki range.

Open: For each d, n, is there a spectrahedron with an extreme point of each rank in the Pataki range?

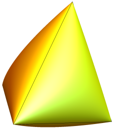
Facial structure

Example: $d = 3, n = 3$ Pataki range: $r = 1, 2$



Facial structure

Example: $d = 3, n = 3$ Pataki range: $r = 1, 2$



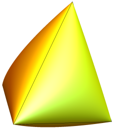
Counting rank-1 matrices:

$\{X : \text{rank}(X) \leq 1\}$ is variety of codim 3 and degree 4 in $\mathbb{R}^{3\times 3}_{sym}$.

$\Rightarrow 0, 1, 2, 3, 4$ or ∞ rank-1 matrices in $\mathcal{S}$ (generically 0,2, or 4)

Facial structure

Example: $d = 3, n = 3$ Pataki range: $r = 1, 2$



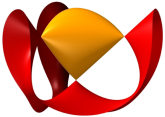
Counting rank-1 matrices:

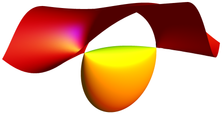
$\{X : \text{rank}(X) \leq 1\}$ is variety of codim 3 and degree 4 in $\mathbb{R}^{3\times 3}_{sym}$.

$\Rightarrow 0, 1, 2, 3, 4$ or ∞ rank-1 matrices in $\mathcal{S}$ (generically 0,2, or 4)

There must be ≥ 1 rank-1 matrix. Why? Topology!

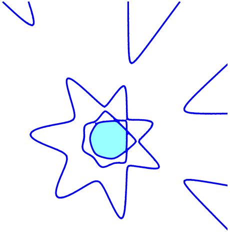
If $\partial\mathcal{S}$ has no rank-1 matrices, then the map $S^2 \cong \partial\mathcal{S} \to \mathbb{P}^2(\mathbb{R})$ given by $x \mapsto \ker(A(x))$ is an embedding. $\Rightarrow\Leftarrow$





(For more see Friedland, Robbin, Sylvester,1984)

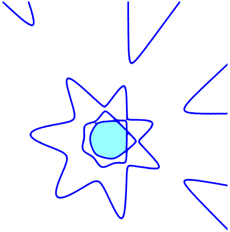
Another connection with topology



Suppose $A_0 = I$ and let $f(x) = \det(A(x))$.

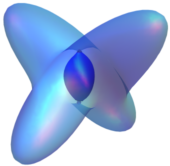
$\Rightarrow$ f is hyperbolic, i.e.
for every $x \in \mathbb{R}^n$, $f(tx) \in \mathbb{R}[t]$ is real-rooted.

Another connection with topology



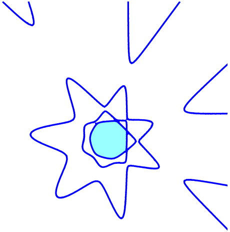
Suppose $A_0 = I$ and let $f(x) = \det(A(x))$.

$\Rightarrow$ f is hyperbolic, i.e.
for every $x \in \mathbb{R}^n$, $f(tx) \in \mathbb{R}[t]$ is real-rooted.



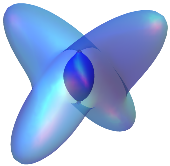
If $V_{\mathbb{R}}(f)$ is smooth, this implies that that $V_{\mathbb{R}}(f) \subset \mathbb{P}^{n-1}$ consists of $\lfloor d/2 \rfloor$ nested spheres.

Another connection with topology



Suppose $A_0 = I$ and let $f(x) = \det(A(x))$.

$\Rightarrow$ f is hyperbolic, i.e.
for every $x \in \mathbb{R}^n$, $f(tx) \in \mathbb{R}[t]$ is real-rooted.



If $V_{\mathbb{R}}(f)$ is smooth, this implies that that $V_{\mathbb{R}}(f) \subset \mathbb{P}^{n-1}$ consists of $\lfloor d/2 \rfloor$ nested spheres.

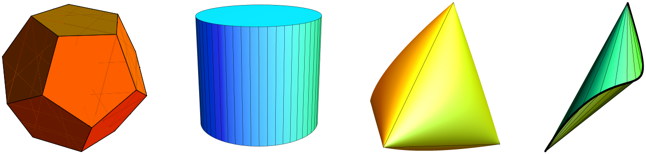
Open (Generalized Lax Conjecture):
Is every hyperbolicity region a spectrahedron?

Combinatorics of spectrahedra

What is the "f-vector" of a spectrahedron?

Extreme points and faces of $\mathcal{S}$ come with a lot of discrete data ...

dimension, matrix rank, dimension of normal cone, degree,
$\#$ number of connected components, Betti $\#$s, ...



Combinatorics of spectrahedra

What is the "f-vector" of a spectrahedron?

Extreme points and faces of $\mathcal{S}$ come with a lot of discrete data ...

dimension, matrix rank, dimension of normal cone, degree,
$\#$ number of connected components, Betti $\#$s, ...

Very open: What values are possible?