Low-rank sums-of-squares representations

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joint work with Greg Blekherman,
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Sums of squares and nonnegative polynomials

A representation of a element $f \in R$ as a sum of squares over a ring R (usually $\mathbb{R}[x_0, \ldots, x_n]$ or a quotient) is an expression

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Let $\Sigma_{n,2d}$ denote the sums of squares in $\mathbb{R}[x_0, \ldots, x_n]_{2d}$
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Theorem (Hilbert): $\Sigma_{n,2d} = P_{n,2d}$ if and only if

$$n = 1 \quad \text{or} \quad 2d = 2 \quad \text{or} \quad (n, 2d) = (2, 4).$$

Motzkin non-example: $x^2y^4 + x^4y^2 - 3x^2y^2z^2 + z^6 \in P_{2,6} \backslash \Sigma_{2,6}$

Number of squares

$n = 1$**:** A nonnegative bivariate form is a sum of *two* squares
Proof: Factor $f = (p + \mathbb{i} q)(p - \mathbb{i} q) = p^2 + q^2$ where $p, q \in \mathbb{R}[x_0, x_1]_d$

$2d = 2$**:** A nonnegative quadratic form in $P_{n,2}$ is a sum of $n + 1$ squares
Proof: Diagonalization of quadratic forms

$(n, 2d) = (2, 4)$**:** A nonnegative ternary quartic is a sum of *three* squares
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Our goal: Unify/generalize these results using varieties of minimal degree

Quadratic forms on varieties

Let ...

$X \subset \mathbb{P}^N(\mathbb{C})$ = a real, nondegenerate irreducible variety equal to $\overline{X(\mathbb{R})}^{Zar}$,

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$\Sigma_X = \{\ell_1^2 + \ldots + \ell_r^2 : \ell_i \in \mathbb{R}[X]_1\} \subset \mathbb{R}[X]_2$, and

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$\nu_d(\mathbb{P}^n)$ has minimal degree $\Leftrightarrow$ $n = 1$, $d = 1$, or $(n, d) = (2, 2)$
Corollary: Hilbert's result.

Varieties of minimal degree

Theorem: If X is a variety of minimal degree, then any $q \in P_X$ is a sum of $\dim(X) + 1$ squares. For generic q this bound is tight.

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Rational normal scrolls and biforms

Rational normal scroll X = closure of the image of $\mathbb{C} \times \mathbb{P}^{n-1}$ under

$$(t, x) \mapsto [x_1 : x_1 t : \ldots : x_1 t^{d_1} : \ldots : x_n : x_n t : \ldots : x_n t^{d_n}] \in \mathbb{P}^{d_1 + \ldots + d_n + n - 1}$$

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A quadratic form on X corresponds to a *biform* (Choi, Lam, Reznick), which can be written as

$$f = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}^T \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{1n} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{where} \quad a_{ij} \in \mathbb{R}[t]_{\leq d_i + d_j}.$$

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f is nonnegative $\Leftrightarrow A = (a_{ij})_{ij}$ is positive semidefinite for all $t \in \mathbb{R}$

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Cor: If A is p.s.d. for all $t \in \mathbb{R}$, then $A = BB^T$ where $B \in \mathbb{R}[t]^{n \times (n+1)}$.

Biforms: an example ($n = 2, d_1 = 1, d_2 = 2$)

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$$f = (1 + t^2)x^2 + (1 + t^2 + t^4)y^2 \;=\; \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} 1 + t^2 & 0 \\ 0 & 1 + t^2 + t^4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

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The representation of $f = y^2 + (xt + yt)^2 + (x - yt^2)^2$ gives

$$\begin{pmatrix} 1 + t^2 & 0 \\ 0 & 1 + t^2 + t^4 \end{pmatrix} = BB^T \quad \text{where} \quad B = \begin{pmatrix} 0 & t & 1 \\ 1 & t & -t^2 \end{pmatrix}.$$

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Conjecture: If $X \subset \mathbb{P}^N$ is a *variety* of minimal degree, then a generic $q \in P_X$ has $2^{\operatorname{codim}(X)}$ representations as a sum of $\dim(X) + 1$ squares.

(Possible proof by Hanselka and Sinn)

Real vs. complex representations

Remarkably, the number of real representations as sums of few squares is more stable over $\mathbb{R}$ than $\mathbb{C}$.

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Example. There are four surfaces of minimal degree in $\mathbb{P}^5$:
the cone over $\nu_4(\mathbb{P}^1)$, $\nu_2(\mathbb{P}^2)$, and the rational normal scrolls X_{d_1,d_2} with $(d_1, d_2) = (2, 2), (1, 3)$.

A general element $q \in P_X$ is a sum of 3 squares $q = h_1^2 + h_2^2 + h_3^2$.
If $h_1, h_2, h_3 \in \mathbb{F}[X]_1$, say the representation is over the field $\mathbb{F}$.

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$\mathrm{cone}(\nu_4(\mathbb{P}^1))$	8	35
$\nu_2(\mathbb{P}^2)$	8	63
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