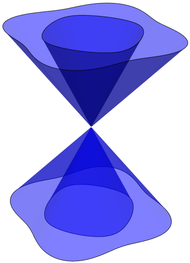
An SOS Relaxation for Hyperbolicity Cones

Cynthia Vinzant

University of Michigan



joint work with Daniel Plaumann

Hyperbolic Polynomials

A homogeneous polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]_d$ is *hyperbolic* with respect to a point $e \in \mathbb{R}^n$ if $f(e) \neq 0$ and for every $x \in \mathbb{R}^n$, all roots of $f(te + x) \in \mathbb{R}[t]$ are real.

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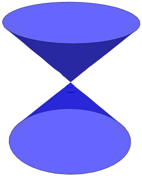

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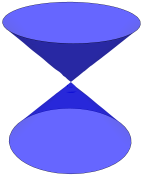

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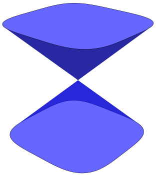

$x_1^4 - x_2^4 - x_3^4$

not hyperbolic

Hyperbolicity Cones

Its *hyperbolicity cone*, denoted $C(f, e)$,
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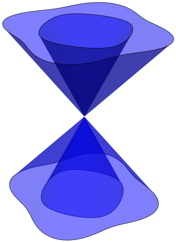

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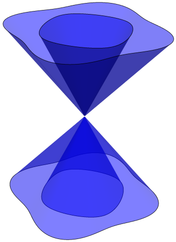

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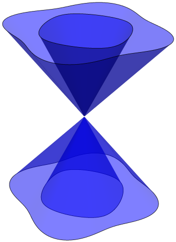

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- f is hyperbolic with respect to any point $a \in C(f, e)$.

One can use interior point methods to optimize a linear function over an affine section of a hyperbolicity cone, Güler (1997), Renegar (2006). This solves a *hyperbolic program*.

Two Important Examples of Hyperbolic Programming

f		
e		
$C(f, e)$		

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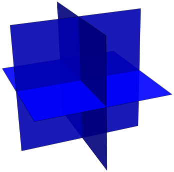
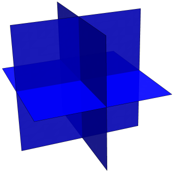
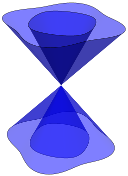
	Linear Programming	
f	$\prod_i x_i$	
e	$(1, \ldots, 1)$	
$C(f, e)$	$(\mathbb{R}_+)^n$	
		

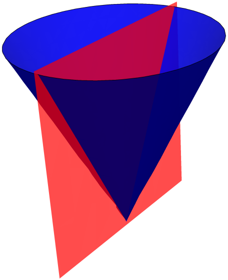
Two Important Examples of Hyperbolic Programming

	Linear Programming	Semidefinite Programming
f	$\prod_i x_i$	$\det \begin{pmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{1n} & \dots & x_{nn} \end{pmatrix}$
e	$(1, \dots, 1)$	Id_n
$C(f, e)$	$(\mathbb{R}_+)^n$	positive definite matrices
		

Some convex cones are slices of other convex cones.

Theorem

Every hyperbolicity cone is a linear slice of the cone of nonnegative polynomials.



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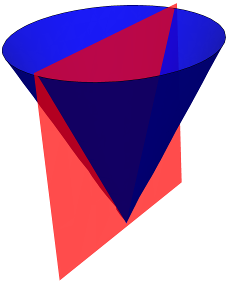
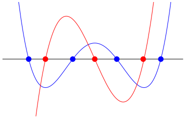
Every hyperbolicity cone is a linear slice of the cone of nonnegative polynomials.

If $f \in \mathbb{R}[x_1, \ldots, x_n]_d$ *is hyperbolic with respect to* $e \in \mathbb{R}^n$*, then its hyperbolicity cone* $C(f, e)$ *is a slice of the cone of nonnegative polynomials in* $\mathbb{R}[x_1, \ldots, x_n]_{2d-2}$.



Interlacing Derivatives

If all roots of $p(t)$ are real, then the roots of $p'(t)$ are real and interlace the roots of $p(t)$.



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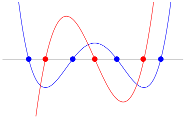
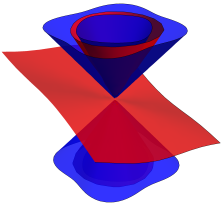
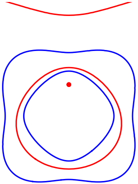


For any direction $a \in C(f, e)$ the polynomial

$$D_a(f) \;=\; \sum_i a_i \frac{\partial f}{\partial x_i} \;=\; \left(\frac{\partial}{\partial t} f(ta + x)\right)\Big|_{t=0}$$

is hyperbolic and *interlaces* f.





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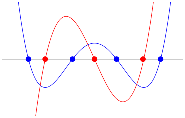

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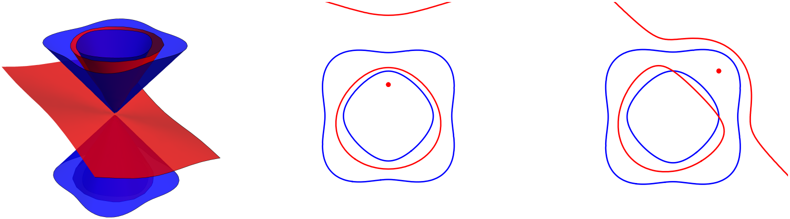
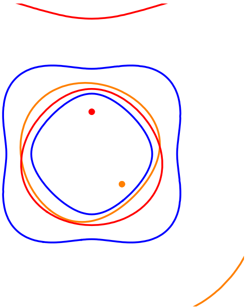


Interlacing and Nonnegativity

For any $a \in C(f, e)$, the product $D_e f \cdot D_a f$ is nonnegative on $\mathcal{V}_{\mathbb{R}}(f)$.



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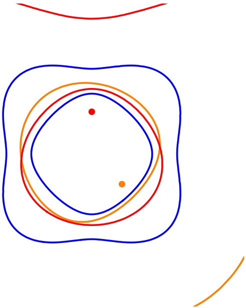

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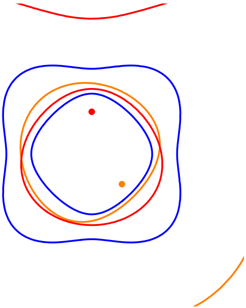


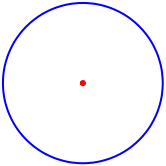
Theorem

If $f \in \mathbb{R}[x_1, \ldots, x_n]_d$ is square-free and hyperbolic with respect to the point $e \in \mathbb{R}^n$ and $f(e) > 0$, then the hyperbolicity cone $\overline{C(f,e)}$ is the following linear section of nonnegative polynomials:

$$\{\ a \in \mathbb{R}^n \ :\ D_e f \cdot D_a f - f \cdot D_e D_a f \ \geq\ 0 \quad \textit{on} \quad \mathbb{R}^n\ \}.$$

Example: the Lorentz cone

$f(x) = x_1^2 - x_2^2 - \ldots - x_n^2 \qquad e = (1, 0, \ldots, 0)$



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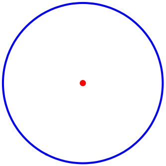

$D_e f \cdot D_a f - f \cdot D_e D_a f$

$$= \; (2x_1)(2a_1x_1 - \textstyle\sum_{j\neq 1} 2a_jx_j) - (x_1^2 - \textstyle\sum_{j\neq 1} x_j^2)(2a_1)$$

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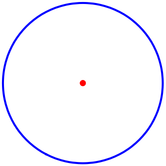

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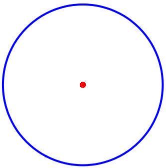

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$$\Rightarrow \quad \overline{C(f,e)} = \left\{ a \in \mathbb{R}^n \;:\; \begin{pmatrix} a_1 & -a_2 & \ldots & -a_n \\ -a_2 & a_1 & & 0 \\ \vdots & & \ddots & \vdots \\ -a_n & 0 & \ldots & a_1 \end{pmatrix} \succeq 0 \right\}$$

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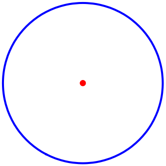

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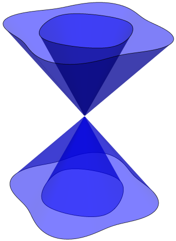
(determinant $= a_1^{n-2} f(a)$)

Sums of Squares Relaxation

Corollary

$\{a \in \mathbb{R}^n \;:\; D_e f \cdot D_a f - f \cdot D_e D_a f \text{ is a } \textit{sum of squares}\ \} \subseteq C(f, e).$



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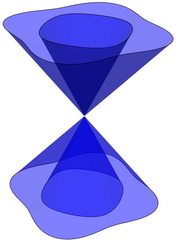
the projection of a spectrahedron!



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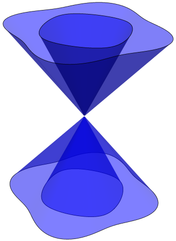
Theorem

If $f = \det(\sum_i x_i M_i)$ *where* $M_1, \ldots, M_n$ *are real symmetric matrices and* $\sum_i e_i M_i \succ 0$*, then this relaxation is exact.*



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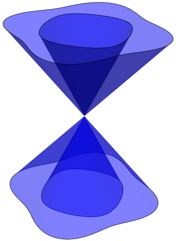

Conjecture

This relaxation is always exact and every hyperbolicity cone is the projection of a spectrahedron.

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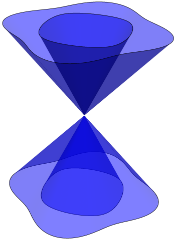

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