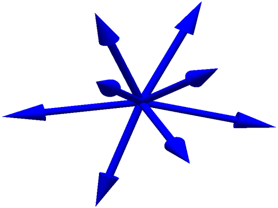
An algebraic approach to phase retrieval

Cynthia Vinzant

University of Michigan



joint with Aldo Conca, Dan Edidin, and Milena Hering.

I learned about frame theory from . . .

American Institute of Mathematics

Frame theory intersects geometry

July 29 to August 2, 2013

at the

American Institute of Mathematics, Palo Alto, California

organized by

Bernhard Bodmann, Gitta Kutyniok, and Tim Roemer

Frames and intensity measurements

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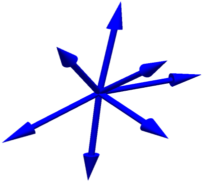

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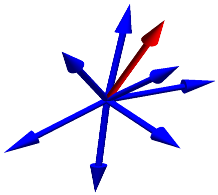

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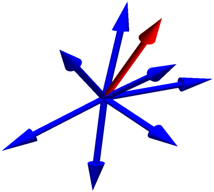

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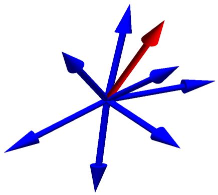

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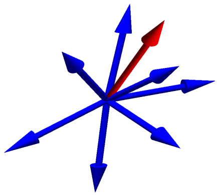

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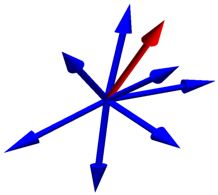

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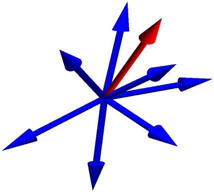

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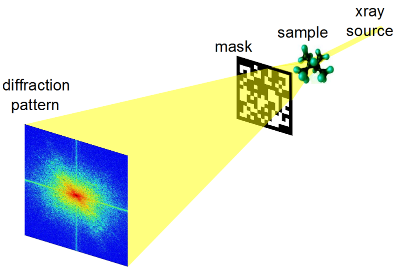
In practice the signal is some structure that is too small (DNA, crystals) or far away (astronomical phenomena) or obscured (medical images) to observe directly.



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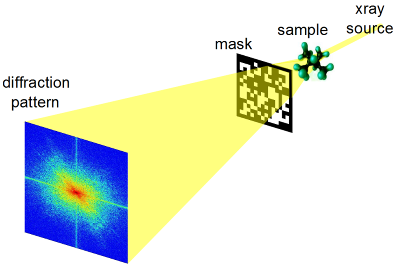

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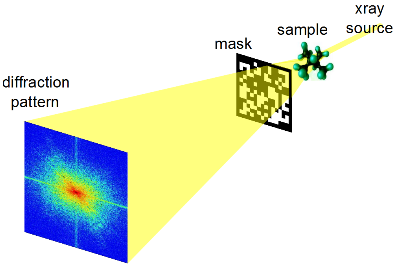

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Here our signal x lies in a finite-dimensional space ($\mathbb{C}^d$), and its measurements are modeled by $|\langle \phi_k, x\rangle|^2$ for $\phi_k \in \mathbb{C}^d$.

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Better question: When is the map $\mathcal{M}_\Phi$ injective?

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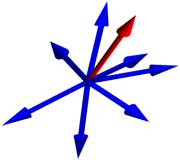

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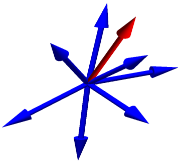

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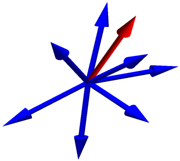

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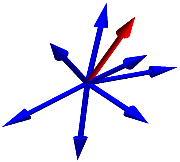
Conjecture (Bandeira-Cahill-Mixon-Nelson, 2013)

(a) If $n < 4d-4$, then $\mathcal{M}_\Phi$ is not injective.

(b) If $n \geq 4d-4$, then $\mathcal{M}_\Phi$ is injective for generic Φ.



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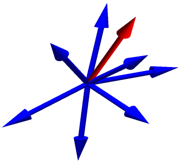

We prove (b) by writing injectivity as an *algebraic condition*.

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$\mathcal{M}_\Phi$ is non-injective $\Leftrightarrow$ $\exists$ a nonzero matrix $Q \in \mathbb{C}^{d\times d}_{Herm}$ with

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More algebraic question: When does $(\mathrm{span}_{\mathbb{R}}\{\phi_1\phi_1^*, \ldots, \phi_n\phi_n^*\})^\perp$ intersect the rank-2 locus of $\mathbb{C}^{d\times d}_{Herm}$?

Getting (Real) Algebraic

Consider the incidence set

$$\left\{ (\Phi, Q) \in \mathbb{P}(\mathbb{C}^{d\times n}) \times \mathbb{P}(\mathbb{C}^{d\times d}_{Herm}) \ : \ \text{rank}(Q) \leq 2 \text{ and } \phi_k^* Q \phi_k = 0 \ \ \forall k \right\}.$$

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Consequence: The bad frames, $\{\Phi : \mathcal{M}_\Phi$ is non-injective$\}$, are the projection of a real (projective) variety.

($\Rightarrow$ a closed semialgebraic subset of $\mathbb{P}((\mathbb{R}^{d\times n})^2)$)

Getting (Complex) Algebraic

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Sketch of proof: The preimage $\pi_2^{-1}(Q)$ of any matrix Q is the product of n quadratic hypersurfaces.

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As a consequence, for $n \geq 4d - 4$,

$$\dim\big(\pi_1(\mathcal{B}_{d,n})\big) \leq 2dn - 2 \quad \text{and} \quad \operatorname{codim}\big(\pi_1(\mathcal{B}_{d,n})\big) \geq 1.$$

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The set of bad frames is small.

Theorem (CEH-)

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Corollary

For $n \geq 4d - 4$, $\mathcal{M}_\Phi$ is injective for generic $\Phi \in \mathbb{C}^{d\times n} \cong (\mathbb{R}^{d\times n})^2$.
There is a Zariski-open set of frames Φ for which $\mathcal{M}_\Phi$ is injective.

Example: $d = 2, \;\; n = 4d - 4 = 4$

A 2×2 Hertmitian matrix Q defines the real quadratic polynomial

$$q(a, b, c, d) = \begin{pmatrix} a - \mathrm{i}c & b - \mathrm{i}d \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} + \mathrm{i}y_{12} \\ x_{12} - \mathrm{i}y_{12} & x_{22} \end{pmatrix} \begin{pmatrix} a + \mathrm{i}c \\ b + \mathrm{i}d \end{pmatrix}$$

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Conjecture

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Smallest open question: $(d = 4,\ n = 4d - 5 = 11)$

Given vectors $\phi_1, \ldots, \phi_{11} \in \mathbb{C}^4$, does there always exist a Hermitian rank-two matrix $Q \in \mathbb{C}^{4\times 4}_{Herm}$ for which $\phi_k^* Q \phi_k = 0$?

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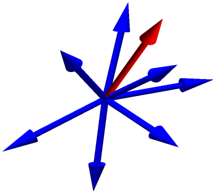
We expect 20 rank-two matrices $Q \in \mathbb{P}(\mathbb{C}^{d\times d})$ with $\phi_k^* Q \phi_k = 0$.
Must there be a Hermitian one?

Final Thoughts

Frame theory and phase retrieval bring together many areas of mathematics and produce interesting algebraic questions.

Tools from algebraic geometry can be used to tackle problems that look very non-algebraic.



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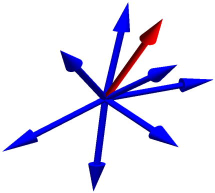

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Thanks!