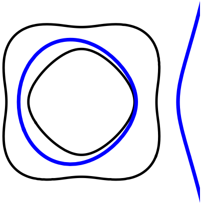
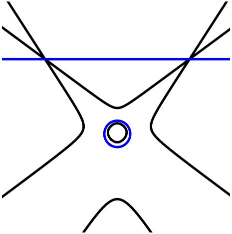
Computing Hermitian Determinantal Representations of Plane Curves

Cynthia Vinzant

University of Michigan





joint with Daniel Plaumann, Rainer Sinn, and David Speyer.

Determinantal Representations

A **determinantal representation** of $f \in \mathbb{R}[x_1, \ldots, x_n]_d$ is

$$f = \det\left(\sum_{i=1}^{n} x_i M_i\right) \quad \text{where } M_i \text{ are } d \times d \text{ matrices.}$$

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Ex: $$x^2 - y^2 - z^2 = \det\begin{pmatrix} x-y & z \\ z & x+y \end{pmatrix} = \det\left(x\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + y\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} + z\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right)$$

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A representation $\sum_i x_i M_i$ is **definite** if the M_i are real symmetric or Hermitian and there is a positive definite matrix in their span,

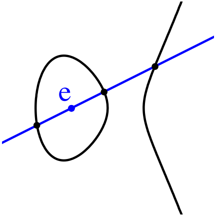
$$M(e) = \sum_i e_i M_i \succ 0 \quad \text{for some} \quad e \in \mathbb{R}^n.$$

Hyperbolic Polynomials

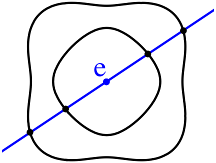
$f = \det(\sum_i x_i M_i)$ with $\sum_i e_i M_i \succ 0$

$\Rightarrow$ f is **hyperbolic** with respect to e.

(roots of $f(e + tx)$ are real for every $x \in \mathbb{R}^n$)



a hyperbolic cubic



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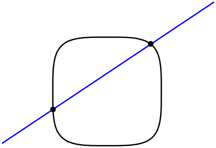

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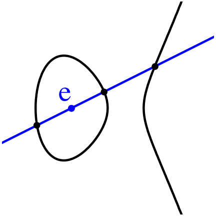

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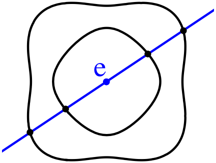

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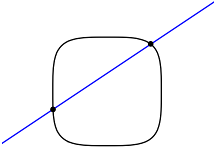

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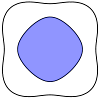
Hyperbolic plane curves consist of degree/2 nested ovals in $\mathbb{P}^2(\mathbb{R})$.

Definite Determinantal Representations

Question: What hyperbolic polynomials have definite determinantal representations $f = \det(\sum_i x_i M_i)$?



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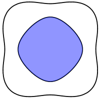

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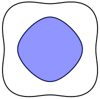

Related Question: What convex semialgebraic sets can be written as a slice of the cone of positive semidefinite matrices?

Theorem (Helton-Vinnikov 2007)

If a polynomial $f \in \mathbb{R}[x,y,z]_d$ is hyperbolic with respect to $e \in \mathbb{R}^3$ then there exist real symmetric matrices $A, B, C \in \mathbb{R}^{d\times d}_{sym}$ with

$$f = \det(xA + yB + zC) \quad \textit{and} \quad e_1 A + e_2 B + e_3 C \succ 0.$$

Constructions

Computing real symmetric determinantal representations is **hard**.

One can use . . .

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Computing *Hermitian* determinantal representations is **easier**.

Interlacing and Distinguishing Definiteness

Theorem (Plaumann-V. 2013)

For a Hermitian matrix of linear forms $M(x) = \sum_i x_i M_i$, the matrix $M(e)$ is (positive or negative) definite if and only if the top left $(d-1) \times (d-1)$ minor of M interlaces $\det(M)$ with respect to e.

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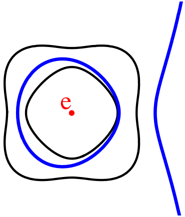
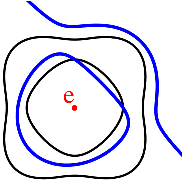


interlacer



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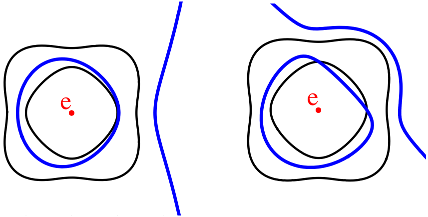

interlacer non-interlacer

Example of an interlacer: the directional derivative

$$\sum_{i=1}^{n} e_i \frac{\partial f}{\partial x_i}$$

Interlude: adjugate matrices

For a matrix M, let M^{adj} denote its adjugate (or classical adjoint).

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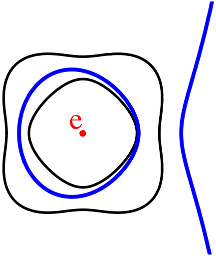
- $(M^{adj})_{11}$ interlaces $\det(M) \quad \Rightarrow \quad M(e) \succ 0.$

Theorem (Plaumann-V. 2013)

Suppose $g_1 \in \mathbb{R}[x, y, z]_{d-1}$ *interlaces* f *with respect to* $e \in \mathbb{R}^3$ *and split the points* $\mathcal{V}(f, g_1)$ *into disjoint sets* $S \cup \overline{S}$.



Interlacers → Definite Determinantal Representations

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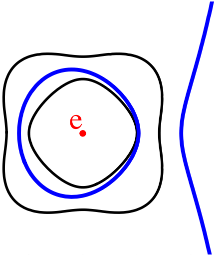
If $g = (g_1, \ldots, g_d)$ *is a basis* $\mathbb{C}[x, y, z]_{d-1} \cap \mathcal{I}(S)$,



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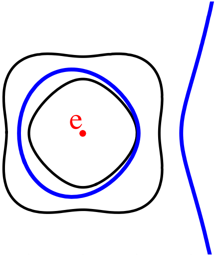
If $g = (g_1, \ldots, g_d)$ is a basis $\mathbb{C}[x,y,z]_{d-1} \cap \mathcal{I}(S)$, then there is a Hermitian matrix $M = xA + yB + zC$ with



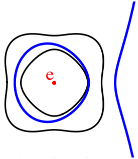
$(M^{adj})_{(1,\cdot)} = g$,

$M(e) \succ 0$, *and*

$f = \det(M)$.

Algorithm (PSSV)

Input: $f \in \mathbb{R}[x, y, z]_d$ and $e \in \mathbb{R}^3$ with f hyperbolic w.resp. to e.



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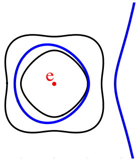

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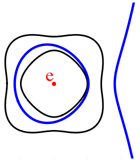

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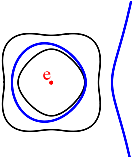

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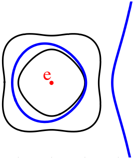

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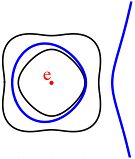

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- In the $3d^2$ variables $A_{i,j}, B_{i,j}, C_{i,j}$, solve the $2d\binom{d+2}{2}$ affine equations coming from the polynomial vector equations

$$\begin{aligned} g \cdot (xA + yB + zC) &= (f, 0 \ldots 0) \\ (xA + yB + zC) \cdot \overline{g}^T &= (f, 0 \ldots 0)^T. \end{aligned}$$

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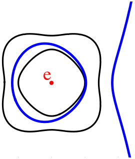

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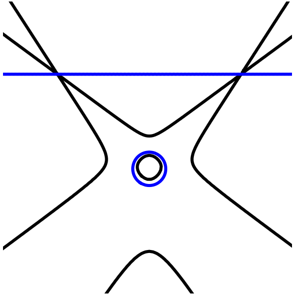
Output: Hermitian matrices $A, B, C \in \mathbb{C}^{d\times d}$ with
$f = \det(xA + yB + zC)$ and $e_1A + e_2B + e_3C \succ 0$.

A quartic example

$$\mathbf{f} = \mathbf{x}^4 - 4\mathbf{x}^2\mathbf{y}^2 + \mathbf{y}^4 - 4\mathbf{x}^2\mathbf{z}^2 - 2\mathbf{y}^2\mathbf{z}^2 + \mathbf{z}^4$$

(hyperbolic w.resp. to $(1, 0, 0)$)



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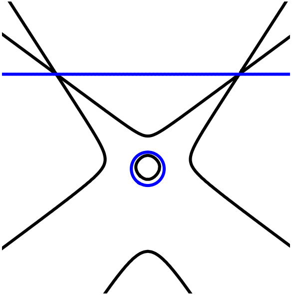

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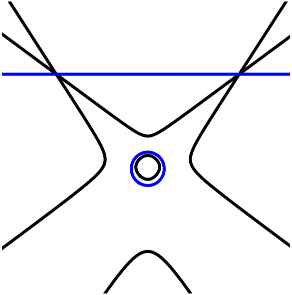
Split $\mathcal{V}(f) \cap \mathcal{V}(g_1) = S \cup \overline{S}$ where
$S = \{[0 : \pm 1 : 1],\ [2 : \pm\sqrt{3} : i],\ [2 : i : \pm\sqrt{3}]\}$.



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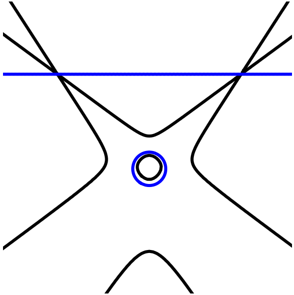

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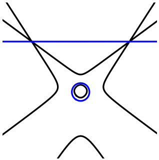
The cubics in $\mathbb{C}[x, y, z]_3$ vanishing at S are spanned by
$g = (g_1, g_2, g_3, g_4)$, where

$$g_2 = ix^3 + 4ixy^2 - 4x^2z - 4y^2z + 4z^3,$$
$$g_3 = -3ix^3 + 4x^2y + 4ixy^2 - 4y^3 + 4yz^2,$$
$$g_4 = -x^3 - 2ix^2y - 2ix^2z + 4xyz.$$

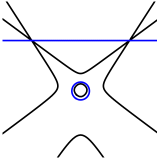
Example: $f = x^4 - 4x^2y^2 + y^4 - 4x^2z^2 - 2y^2z^2 + z^4$



From the vector of cubics $g = (g_1, \ldots, g_4)$,
we solve the polynomial vector equations

$$g \cdot (xA + yB + zC) = (f, 0) \quad \text{and} \quad (xA + yB + zC) \cdot \overline{g}^T = (f, 0)^T.$$

Example: $f = x^4 - 4x^2y^2 + y^4 - 4x^2z^2 - 2y^2z^2 + z^4$



From the vector of cubics $g = (g_1, \ldots, g_4)$,
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one entry $\rightarrow$ 15 affine equations in A_{ij}, B_{ij}, C_{ij}

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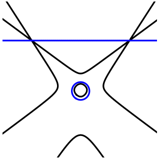

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Unique solution:

$$xA + yB + zC = \frac{1}{8}\begin{pmatrix} 14x & 2z & 2ix - 2y & 2i(y-z) \\ 2z & x & 0 & -ix + 2y \\ -2ix - 2y & 0 & x & ix - 2z \\ -2i(y-z) & ix + 2y & -ix - 2z & 4x \end{pmatrix}$$

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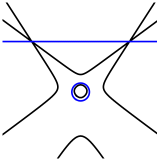

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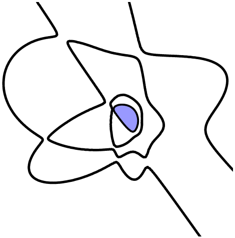
determinant $= (1/256) \cdot f$, positive definite at $(x, y, z) = (1, 0, 0)$

Numerical computations



For randomly generated hyperbolic polynomials, this method computes determinantal representations fairly quickly (in Mathematica).

Average computation times:

degree	5	6	7	8	9	10	15
time (sec)	0.4	0.8	1.7	3.2	6.1	10.7	110

References

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- J. W. Helton and V. Vinnikov. Linear matrix inequality representation of sets. *Comm. Pure Appl. Math.*, **60(5)**, (2007) 654–674.
- Anton Leykin and D. Plaumann. Determinantal representations of hyperbolic curves via polynomial homotopy continuation. arXiv:1212.3506
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Thanks!