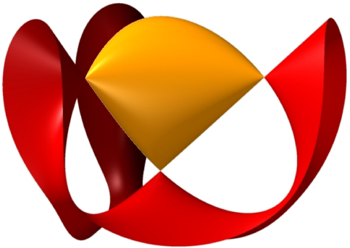
# Convex Algebraic Geometry



Cynthia Vinzant,
North Carolina State University

# What is convex algebraic geometry?

Convex algebraic geometry is the study of convex semialgebraic objects, especially those arising in optimization and statistics.

# What is convex algebraic geometry?

Convex algebraic geometry is the study of convex semialgebraic objects, especially those arising in optimization and statistics.

Many convex concepts have algebraic analogues.

| | | |
|---|---|---|
| convex duality | $\leftrightarrow$ | algebraic duality |
| convex combinations | $\leftrightarrow$ | secant varieties |
| boundary of a projection | $\leftrightarrow$ | branch locus |

# What is convex algebraic geometry?

Convex algebraic geometry is the study of convex semialgebraic objects, especially those arising in optimization and statistics.

Many convex concepts have algebraic analogues.

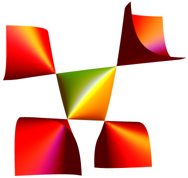
| | | |
|---|---|---|
| convex duality | $\leftrightarrow$ | algebraic duality |
| convex combinations | $\leftrightarrow$ | secant varieties |
| boundary of a projection | $\leftrightarrow$ | branch locus |

Algebraic techniques can help answer questions about these convex sets. Convexity provides additions tools and challenges.

# Motivational Example: The elliptope

The 3-elliptope is

$$\left\{ (x,y,z) \in \mathbb{R}^3 : \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0 \right\}$$

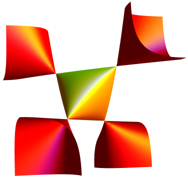


# Motivational Example: The elliptope

The 3-elliptope is

$$\left\{ (x,y,z) \in \mathbb{R}^3 : \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0 \right\}$$

- convex, semialgebraic (defined by polynomial $\leq$'s)

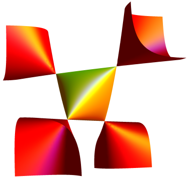


# Motivational Example: The elliptope

The 3-elliptope is

$$\left\{ (x,y,z) \in \mathbb{R}^3 : \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0 \right\}$$

- convex, semialgebraic (defined by polynomial $\leq$'s)
- a spectrahedron (feasible set of a semidefinite program)



# Motivational Example: The elliptope

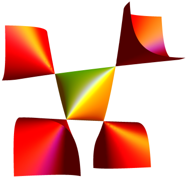
The 3-elliptope is

$$\left\{ (x,y,z) \in \mathbb{R}^3 : \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0 \right\}$$

- convex, semialgebraic (defined by polynomial $\leq$'s)
- a spectrahedron (feasible set of a semidefinite program)

appears in ...

- statistics as set of correlation matrices
- combinatorial optimization



# The elliptope: convex algebraic istructure

Convex structure

# The elliptope: convex algebraic istructure

## Convex structure

- $\infty$-many zero-dim'l faces
- 6 one-dim'l faces
- 4 vertices

# The elliptope: convex algebraic istructure

t

Convex structure

- $\infty$-many zero-dim'l faces
- 6 one-dim'l faces
- 4 vertices

Algebraic structure

Bounded by a cubic hypersurface,

$$\{(x, y, z) \in \mathbb{R}^3 : f = 2xyz - x^2 - y^2 - z^2 + 1 = 0\}$$

that has 4 nodes and contains 6 lines.

# Low-rank matrices

The elliptope exhibits *general* behavior for its size and dimension.

For general $A_0, A_1, A_2, A_3 \in \mathbb{R}^{3\times 3}_{sym}$ the set of matrices

$$\{A_0 + xA_1 + yA_2 + zA_3 : (x, y, z) \in \mathbb{R}^3 \text{ or } \mathbb{C}^3\}$$

contains 4 rank-one matrices over $\mathbb{C}$ and 0,2, or 4 over $\mathbb{R}$.

# Low-rank matrices

The elliptope exhibits *general* behavior for its size and dimension.

For general $A_0, A_1, A_2, A_3 \in \mathbb{R}^{3\times 3}_{sym}$ the set of matrices

$$\{A_0 + xA_1 + yA_2 + zA_3 : (x, y, z) \in \mathbb{R}^3 \text{ or } \mathbb{C}^3\}$$

contains 4 rank-one matrices over $\mathbb{C}$ and 0,2, or 4 over $\mathbb{R}$.

Why? The set of matrices of rank $\leq 1$ is variety of codimension 3 and degree 4 in $\mathbb{R}^{3\times 3}_{sym} \cong \mathbb{R}^6$.

# Low-rank matrices

The elliptope exhibits *general* behavior for its size and dimension.

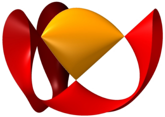
For general $A_0, A_1, A_2, A_3 \in \mathbb{R}^{3\times3}_{sym}$ the set of matrices

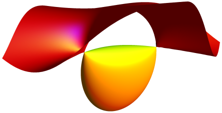
$$\{A_0 + xA_1 + yA_2 + zA_3 : (x, y, z) \in \mathbb{R}^3 \text{ or } \mathbb{C}^3\}$$

contains 4 rank-one matrices over $\mathbb{C}$ and 0,2, or 4 over $\mathbb{R}$.

Why? The set of matrices of rank $\leq 1$ is variety of codimension 3 and degree 4 in $\mathbb{R}^{3\times3}_{sym} \cong \mathbb{R}^6$.

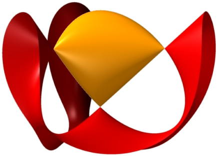
If $A_0 \succ 0$, there will always be 2 or 4 matrices of rank-one over $\mathbb{R}$.

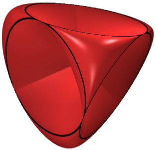




# Algebraic and convex duality

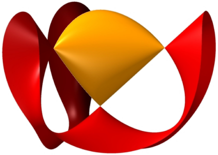
We can use duality in algebraic geometry to calculate hypersurface bounding the dual of a convex body.

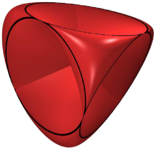




# Algebraic and convex duality

We can use duality in algebraic geometry to calculate hypersurface bounding the dual of a convex body.

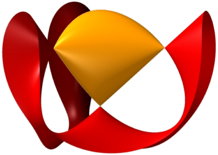


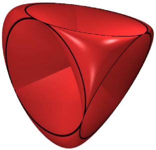


The dual of the elliptope is bounded by the union of a quartic surface and four planes.

# Algebraic and convex duality

We can use duality in algebraic geometry to calculate hypersurface bounding the dual of a convex body.





The dual of the elliptope is bounded by the union of a quartic surface and four planes. Writing down the solution of a random linear optimization problem over the elliptope requires solving a degree four polynomial.

# Moments and Sums of Squares

The cone of nonnegative polynomials

$$\mathrm{NN}_{n,2d} = \{p \in \mathbb{R}[x_1,\ldots,x_n]_{\leq 2d} : p(x) \geq 0 \text{ for all } \mathbb{R}^n\}$$

is convex, semialgebraic, and contains the cone of sums of squares

$$\mathrm{SOS}_{n,2d} = \{h_1^2 + \ldots + h_r^2 : h_j \in \mathbb{R}[x_1,\ldots,x_n]_{\leq d}\}.$$

# Moments and Sums of Squares

The cone of nonnegative polynomials

$$\mathrm{NN}_{n,2d} = \{p \in \mathbb{R}[x_1, \ldots, x_n]_{\leq 2d} : p(x) \geq 0 \text{ for all } \mathbb{R}^n\}$$

is convex, semialgebraic, and contains the cone of sums of squares

$$\mathrm{SOS}_{n,2d} = \{h_1^2 + \ldots + h_r^2 : h_j \in \mathbb{R}[x_1, \ldots, x_n]_{\leq d}\}.$$

The dual cone to $NN_{n,2d}$ is the cone of moments of degree $\leq 2d$:

$$NN_{n,2d}^{\circ} = \mathrm{conv}\{\lambda(1, x_1, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_n^{2d}) : \lambda \in \mathbb{R}, x \in \mathbb{R}^n\}.$$

# Moments and Sums of Squares

The cone of nonnegative polynomials

$$\mathrm{NN}_{n,2d} = \{p \in \mathbb{R}[x_1, \ldots, x_n]_{\leq 2d} : p(x) \geq 0 \text{ for all } \mathbb{R}^n\}$$

is convex, semialgebraic, and contains the cone of sums of squares

$$\mathrm{SOS}_{n,2d} = \{h_1^2 + \ldots + h_r^2 : h_j \in \mathbb{R}[x_1, \ldots, x_n]_{\leq d}\}.$$

The dual cone to $NN_{n,2d}$ is the cone of moments of degree $\leq 2d$:

$$NN^{\circ}_{n,2d} = \mathrm{conv}\{\lambda(1, x_1, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_n^{2d}) : \lambda \in \mathbb{R}, x \in \mathbb{R}^n\}.$$

Duality reverses inclusion, so $NN^{\circ}_{n,2d} \subseteq \mathrm{SOS}^{\circ}_{n,2d}$.

# Moments and Sums of Squares

The cone of nonnegative polynomials

$$\mathrm{NN}_{n,2d} = \{p \in \mathbb{R}[x_1, \ldots, x_n]_{\leq 2d} : p(x) \geq 0 \text{ for all } \mathbb{R}^n\}$$

is convex, semialgebraic, and contains the cone of sums of squares

$$\mathrm{SOS}_{n,2d} = \{h_1^2 + \ldots + h_r^2 : h_j \in \mathbb{R}[x_1, \ldots, x_n]_{\leq d}\}.$$

The dual cone to $NN_{n,2d}$ is the cone of moments of degree $\leq 2d$:

$$NN_{n,2d}^{\circ} = \mathrm{conv}\{\lambda(1, x_1, \ldots, x_n, x_1^2, x_1 x_2, \ldots, x_n^{2d}) : \lambda \in \mathbb{R}, x \in \mathbb{R}^n\}.$$

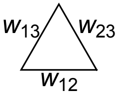
Duality reverses inclusion, so $NN_{n,2d}^{\circ} \subseteq \mathrm{SOS}_{n,2d}^{\circ}$.

Moreover $\mathrm{SOS}_{n,2d}^{\circ}$ is a spectrahedron!

# Sums of squares and the Goemans-Williamson relaxation

MAXCUT: Given weights $w_e \in \mathbb{R}$ to the edges of a graph $G = (V, E)$, find a cut $V \to \{\pm 1\}$ maximizing the summed weight of mixed edges.

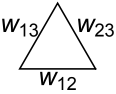
$\max \sum w_{ij}(1 - x_i x_j)$ s.t. $x_1^2 = x_2^2 = x_3^2 = 1$



# Sums of squares and the Goemans-Williamson relaxation

MAXCUT: Given weights $w_e \in \mathbb{R}$ to the edges of a graph $G = (V, E)$, find a cut $V \to \{\pm 1\}$ maximizing the summed weight of mixed edges.

$\max \sum w_{ij}(1 - x_i x_j)$ s.t. $x_1^2 = x_2^2 = x_3^2 = 1$

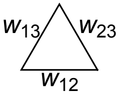



$= \max \sum w_{ij}(1 - y_{ij})$ s.t. $y$ belongs to

$$C = \operatorname{conv}\{(x_1x_2, x_1x_3, x_2x_3) : x \in \{\pm 1\}^3\}$$

# Sums of squares and the Goemans-Williamson relaxation

MAXCUT: Given weights $w_e \in \mathbb{R}$ to the edges of a graph $G = (V, E)$, find a cut $V \to \{\pm 1\}$ maximizing the summed weight of mixed edges.




$\max \sum w_{ij}(1 - x_i x_j)$ s.t. $x_1^2 = x_2^2 = x_3^2 = 1$

$= \max \sum w_{ij}(1 - y_{ij})$ s.t. $y$ belongs to

$$C = \mathrm{conv}\{(x_1x_2, x_1x_3, x_2x_3) : x \in \{\pm 1\}^3\}$$

The dual convex body is

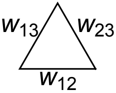
$$C^\circ = \{(a_{12}, a_{13}, a_{23}) : \sum a_{ij} x_i x_j \leq 1 \text{ for } x \in \{\pm 1\}^3\}$$

$$\subseteq \{(a_{12}, a_{13}, a_{23}) : 1 - \sum a_{ij} x_i x_j \text{ is SOS } mod \ \langle x_j^2 - 1 \rangle\} = S$$

# Sums of squares and the Goemans-Williamson relaxation

MAXCUT: Given weights $w_e \in \mathbb{R}$ to the edges of a graph $G = (V, E)$, find a cut $V \to \{\pm 1\}$ maximizing the summed weight of mixed edges.

$\max \sum w_{ij}(1 - x_i x_j)$ s.t. $x_1^2 = x_2^2 = x_3^2 = 1$



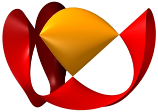

$= \max \sum w_{ij}(1 - y_{ij})$ s.t. $y$ belongs to

$$C = \mathrm{conv}\{(x_1x_2, x_1x_3, x_2x_3) : x \in \{\pm 1\}^3\}$$

The dual convex body is

$$C^\circ = \{(a_{12}, a_{13}, a_{23}) : \sum a_{ij}x_ix_j \leq 1 \text{ for } x \in \{\pm 1\}^3\}$$

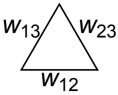
$$\subseteq \{(a_{12}, a_{13}, a_{23}) : 1 - \sum a_{ij}x_ix_j \text{ is SOS } mod \ \langle x_j^2 - 1\rangle\} = S$$

Then $C \subseteq S^\circ =$ 

# Sums of squares and the Goemans-Williamson relaxation

MAXCUT: Given weights $w_e \in \mathbb{R}$ to the edges of a graph $G = (V, E)$, find a cut $V \to \{\pm 1\}$ maximizing the summed weight of mixed edges.

$\max \sum w_{ij}(1 - x_i x_j)$ s.t. $x_1^2 = x_2^2 = x_3^2 = 1$



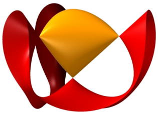

$= \max \sum w_{ij}(1 - y_{ij})$ s.t. $y$ belongs to

$$C = \mathrm{conv}\{(x_1x_2, x_1x_3, x_2x_3) : x \in \{\pm 1\}^3\}$$

The dual convex body is

$$C^\circ = \{(a_{12}, a_{13}, a_{23}) : \sum a_{ij}x_i x_j \leq 1 \text{ for } x \in \{\pm 1\}^3\}$$

$$\subseteq \{(a_{12}, a_{13}, a_{23}) : 1 - \sum a_{ij}x_i x_j \text{ is SOS } mod\ \langle x_j^2 - 1\rangle\} = S$$

Then $C \subseteq \mathrm{S}^\circ =$ 

**Thanks!**