The Chow form of a reciprocal linear space

Cynthia Vinzant

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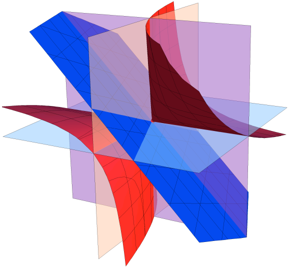

joint work with Mario Kummer, Universität Konstanz

Hyperbolicity and determinantal representations

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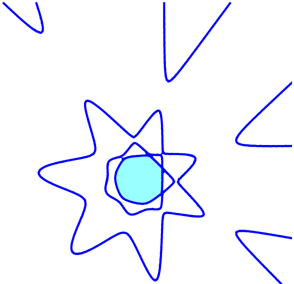

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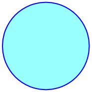

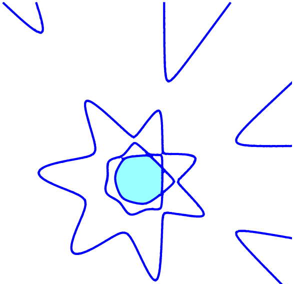

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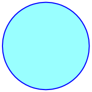
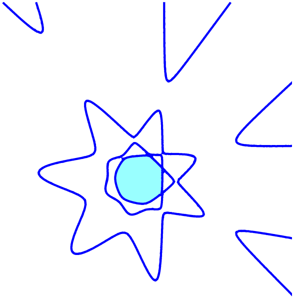


Example: $f = \det(\sum_i x_i A_i)$
where $A_1, \ldots, A_n \in \mathbb{R}^{d \times d}_{sym}$ and the matrix $\sum_i v_i A_i$ is positive definite



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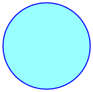

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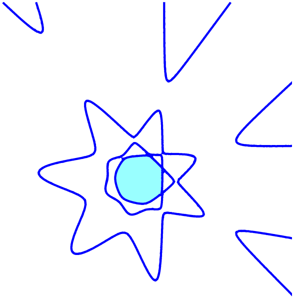

e.g. $x^2 - y^2 - z^2 = \det \begin{pmatrix} x+y & z \\ z & x-y \end{pmatrix}$

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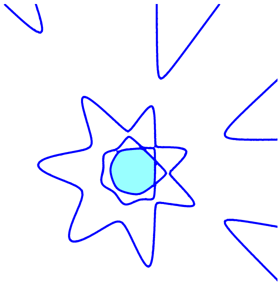

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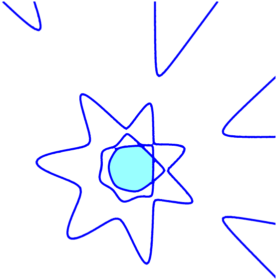

Theorem (Helton-Vinnikov 2007). A polynomial $f \in \mathbb{R}[x_1, x_2, x_3]_d$ is hyperbolic if and only if there exist $A_1, A_2, A_3 \in \mathbb{R}^{d\times d}_{sym}$ with

$$f = \det\left(\sum_i x_i A_i\right) \quad \text{and} \quad \sum_i v_i A_i \succ 0.$$

Let $X \subset \mathbb{P}^{n-1}$ be an irreducible variety of dimension $d-1$. Then

$$\{L : L^{\perp} \text{ intersects } X\} \text{ is a hypersurface in } \mathbb{G}(d-1, n-1)$$

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$L = \mathrm{span}\{a, b\} \subset \mathbb{P}^3$,

$L^\perp \cap X \neq 0 \Leftrightarrow a_0 + a_1 t + a_2 t^2 + a_3 t^3,\ b_0 + b_1 t + b_2 t^2 + b_3 t^3$ have a common root

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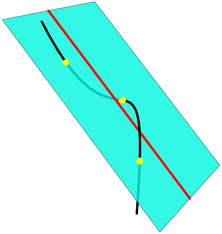
The Chow form of X is the resultant of these polynomials.

Hyperbolicity and Chow forms

A real variety $X \subset \mathbb{P}^{n-1}(\mathbb{C})$ of $codim(X) = c$ is **hyperbolic** with respect to a linear space L of dim $c - 1$ if $X \cap L = \emptyset$ and for all real linear spaces $L' \supset L$ of $\dim(L') = c$, all points $X \cap L'$ are real.



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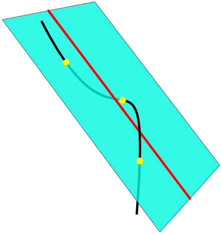

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Theorem (Shamovich-Vinnikov 2015). If a curve $X \subset \mathbb{P}^{n-1}$ is hyperbolic with respect to L, then its Chow form is a determinant

$$\det\left(\sum_{I\in\binom{[n]}{2}} p_I(M)A_I\right) \quad \text{with} \quad \sum_{I\in\binom{[n]}{2}} p_I(L^\perp)A_I \succ 0$$

for some matrices $A_I \in \mathbb{C}^{D\times D}_{Herm}$ with $D = \deg(X)$.

Reciprocal linear spaces

Given a linear space $\mathcal{L} \in \mathrm{Gr}(d, n)$, its reciprocal linear space is

$$\mathcal{L}^{-1} = \mathbb{P}\left(\overline{\left\{\left(x_1^{-1}, \ldots, x_n^{-1}\right) : x \in \mathcal{L} \cap (\mathbb{C}^*)^n\right\}}\right).$$

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De Loera-Sturmfels-V. (2012): $\mathcal{L}^{-1} \cap (\mathcal{L}^{\perp} + v)$ are analytic centers of the bounded regions in a hyperplane arrangement.

Example: $(d, n) = (2, 4)$

Take $\ell_0, \ell_1, \ell_2, \ell_3 \in \mathbb{R}[s, t]$.

Then $\mathcal{L} = \{[\ell_0 : \ell_1 : \ell_2 : \ell_3] : [s : t] \in \mathbb{P}^1\} \in \mathbb{G}(1, 3)$.

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$\mathcal{L}$ intersects the coordinate hyperplanes $\{x_i = 0\}$ in 4 points.
Remove them and take inverses to get

$$\mathcal{L}^{-1} = \overline{\{[\tfrac{1}{\ell_0} : \tfrac{1}{\ell_1} : \tfrac{1}{\ell_2} : \tfrac{1}{\ell_3}]\}} = \overline{\{[\ell_1\ell_2\ell_3 : \ell_0\ell_2\ell_3 : \ell_0\ell_1\ell_3 : \ell_0\ell_1\ell_2]\}}.$$

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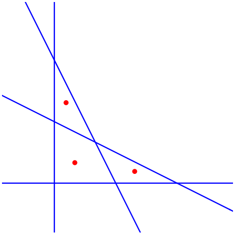

$\mathcal{L}^{-1}$ is a rational cubic curve.

Any plane L' containing $\mathcal{L}^{\perp}$ intersects $\mathcal{L}^{-1}$ in $3 = \deg(\mathcal{L}^{-1})$ real points.

Determinantal representation for $\mathcal{L}^{-1}$

Let $\mathcal{L} \in \mathbb{G}(d-1, n-1)$ not contained in a hyperplane $\{x_i = 0\}$.

Define $p(\mathcal{L}) \in \mathbb{P}(\bigwedge^d \mathbb{R}^n)$ and $\mathcal{B} = \{I \in \binom{[n]}{d} : p_I(\mathcal{L}) \neq 0\}$.

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Theorem (Kummer-V. 2016). The Chow form of $\mathcal{L}^{-1}$ can be written as a determinant

$$\det\left(\sum_{I \in \mathcal{B}} \frac{p_I(\mathcal{M})}{p_I(\mathcal{L})} A_I \right)$$

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The rowspan of the $\deg(\mathcal{L}^{-1}) \times |\mathcal{B}|$ matrix $(v_I : I \in \mathcal{B})$ is

$$\mathrm{span}\{p(\mathcal{L}) : (1, \ldots, 1) \in \mathcal{L}\} \ \cap \ (\mathbb{C}^*)^{\mathcal{B}}.$$

Generic case: the uniform matroid

If $\mathcal{B} = \binom{[n]}{d}$ the vectors $\{v_I : I \in \mathcal{B}\}$ can be taken to be

$$v_I = e_{I\setminus\{n\}} \quad \text{for} \quad I \ni n \qquad \text{and} \qquad \sum_{k=1}^{d}(-1)^k e_{I\setminus\{i_k\}} \quad \text{for} \quad I \not\ni n.$$

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Theorem (Kummer-V.). If $\mathcal{L} \in \mathrm{Gr}(2, n)$ has no zero Plücker coordinates, then the Chow form of $\mathcal{L}^{-1}$ is

$$\sum_{T \in \mathcal{T}_n} \prod_{\{i,j\} \in T} p_{ij}(\mathcal{M}) \cdot \prod_{\{k,\ell\} \in T^c} p_{k\ell}(\mathcal{L}),$$

where $\mathcal{T}_n$ denotes the set of spanning trees on n vertices.

Uniform Example: $d = 2, n = 4$

For $d = 2, n = 4$, $\mathcal{L}^{-1}$ generically has degree 3 and we can take

$$\begin{pmatrix} v_{14} & v_{24} & v_{34} & v_{12} & v_{13} & v_{23} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{pmatrix}.$$

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If $p = p(\mathcal{M})$ and $q = p(\mathcal{L})$ then the Chow form of $\mathcal{L}^{-1}$ is

$\det(\sum_I \frac{p_I}{q_I} \cdot v_I v_I^T) =$

$$\det \begin{pmatrix} \frac{p_{14}}{q_{14}} + \frac{p_{12}}{q_{12}} + \frac{p_{13}}{q_{13}} & -p_{12}/q_{12} & -p_{13}/q_{13} \\ -p_{12}/q_{12} & \frac{p_{24}}{q_{24}} + \frac{p_{12}}{q_{12}} + \frac{p_{23}}{q_{23}} & -p_{23}/q_{23} \\ -p_{13}/q_{13} & -p_{23}/q_{23} & \frac{p_{34}}{q_{34}} + \frac{p_{13}}{q_{13}} + \frac{p_{23}}{q_{23}} \end{pmatrix}.$$

Closing thoughts

We need more examples of hyperbolic varieties!

The varieties $\mathcal{L}^{-1}$ are hyperbolic with respect to an orthant in the Grassmannian, and the Chow forms we found only involve square-free monomials in $\mathbb{C}[p_I(\mathcal{M})]$.

There will be nice combinatorics and geometry in the following:

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Thanks!