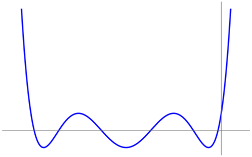
Completely log-concave polynomials and matroids



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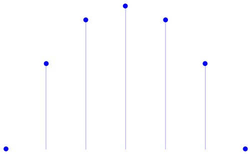

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joint work with

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Matroids

A **matroid** on ground set $[n] = \{1, \ldots, n\}$ is a nonempty collection $\mathcal{I}$ of *independent* subsets of $[n]$ satisfying:

- If $S \subseteq T$ and $T \in \mathcal{I}$, then $S \in \mathcal{I}$.
- If $S, T \in \mathcal{I}$ and $|T| > |S|$, then $\exists i \in T \backslash S$ with $S \cup \{i\} \in \mathcal{I}$.

Examples:

- linear independence of vectors $v_1, \ldots, v_n \in \mathbb{R}^d$
- cyclic independence of n edges in a graph

Independence poly. $g_M(y, z_1, \ldots, z_n) = \sum_{I \in \mathcal{I}} y^{n-|I|} \prod_{i \in I} z_i$

Results and other work

Mason's conjecture: Let $\mathcal{I}_k = \#$ indep. sets of matroid M of size k.

(i) $\mathcal{I}_k^2 \geq \mathcal{I}_{k-1} \cdot \mathcal{I}_{k+1}$ (log-concavity)

(ii) $\mathcal{I}_k^2 \geq \left(\frac{k+1}{k}\right) \cdot \mathcal{I}_{k-1} \cdot \mathcal{I}_{k+1}$

(iii) $$\left(\frac{\mathcal{I}_k}{\binom{n}{k}}\right)^2 \geq \frac{\mathcal{I}_{k-1}}{\binom{n}{k-1}} \cdot \frac{\mathcal{I}_{k+1}}{\binom{n}{k+1}}$$ (ultra log-concavity)

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Adiprasito, Huh, Katz use combinatorial Hodge theory to prove (i)

Huh, Schröter, Wang use $\uparrow$ to prove (ii)

Anari, Liu, Oveis Gharan, V. use *complete log-concavity* to prove (iii)

Brändén, Huh independently use *Lorentz polynomials* to prove (iii)

Complete log-concavity

$f \in \mathbb{R}[z_1, \ldots, z_n]$ is log-concave on $\mathbb{R}^n_{>0}$ if $f \equiv 0$ or

$$f(x) \geq 0 \text{ for all } x \in \mathbb{R}^n_{\geq 0} \quad \text{and} \quad \log(f) \text{ is concave on } \mathbb{R}^n_{>0}.$$

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For $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, let $D_v = \sum_{i=1}^n v_i \frac{\partial f}{\partial z_i}$.

$f \in \mathbb{R}[z_1, \ldots, z_n]$ is completely log-concave (CLC) on $\mathbb{R}^n_{>0}$ if for all $k \in \mathbb{N}$, $v_1, \ldots, v_k \in \mathbb{R}^n_{\geq 0}$,

$$D_{v_1} \cdots D_{v_k} f \text{ is log-concave on } \mathbb{R}^n_{\geq 0}.$$

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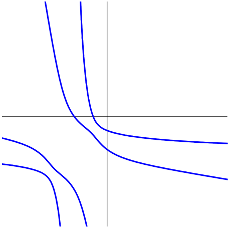
Example: $f = \prod_{i=1}^d (z + r_i) \quad \Rightarrow \quad \log(f)'' = \sum_{i=1}^d \frac{-1}{(z+r_i)^2} \leq 0$

Example: stable polynomials

$f \in \mathbb{R}[z_1, \ldots, z_n]_d$ is stable if
$f(tv + w) \in \mathbb{R}[t]$ is real rooted
for all $v \in \mathbb{R}^n_{\geq 0}, w \in \mathbb{R}^n$.

$\Rightarrow$ f is completely log-concave



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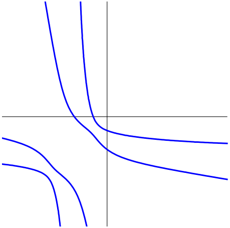

Example: $\det(\sum_{i=1}^n z_i v_i v_i^T) = \sum_{I \in \binom{[n]}{d}} \det(v_i : i \in I)^2 \prod_{i \in I} z_i$

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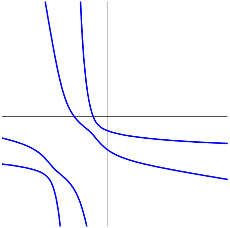

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Choe, Oxley, Sokal, Wagner: If $f = \sum_{I \in \binom{[n]}{d}} c_I \prod_{i \in I} z_i$ is stable, then $\mathrm{supp}(f) = \{I : c_I \neq 0\}$ are the bases of a matroid on $[n]$.

Brändén: Fano matroid $\neq$ support of a stable polynomial f

Equivalent conditions and univariate characterization

Gurvits: f is strongly log-concave (SLC) if

$$\partial^\alpha f = \left(\tfrac{\partial}{\partial z_1}\right)^{\alpha_1} \cdots \left(\tfrac{\partial}{\partial z_n}\right)^{\alpha_n} f \qquad \text{is log-concave on } \mathbb{R}^n_{\geq 0}.$$

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Theorem (ALOV): For $f \in \mathbb{R}[z_1, \ldots, z_n]_d$,

$$f \text{ CLC} \Leftrightarrow f \text{ SLC} \Leftrightarrow \begin{cases} \partial^\alpha f \text{ is indecomposable for all } |\alpha| \leq d-2 \\ \text{and } \partial^\alpha f \text{ is CLC for all } |\alpha| = d-2 \end{cases}$$

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Cor. (Gurvits/ALOV)

$$\sum_{k=0}^{n} a_k y^{n-k} z^k \text{ is CLC} \;\Leftrightarrow\; \left(\frac{a_k}{\binom{n}{k}}\right)^2 \geq \frac{a_{k-1}}{\binom{n}{k-1}} \cdot \frac{a_{k+1}}{\binom{n}{k+1}}$$

Complete log-concavity for matroids

Theorem. $g_M(y, z_1, \ldots, z_n) = \sum_{I \in \mathcal{I}} y^{n-|I|} \prod_{i \in \mathcal{I}} z_i$ is CLC.

(just check rank-two matroids M)

Cor: $g_M(y, z, \ldots, z) = \sum_{k=0}^{n} \mathcal{I}_k y^{n-k} z^k$ is CLC.

Cor: $\{\mathcal{I}_k\}_k$ is ultra log-concave (Mason's conjecture)

Other results

Theorem: For any matroid M, the solution to the concave program

$$\tau = \max_{p \in \mathcal{P}_M} \sum_{i=1}^{n} p_i \log \tfrac{1}{p_i} + (1 - p_i) \log \tfrac{1}{1-p_i}$$

can be computed in polynomial time and $\beta = e^{\tau}$ satisfies

$$2^{O(-r)}\beta \leq \# \text{ bases of } M \leq \beta.$$

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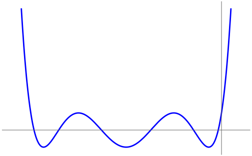
Theorem: The natural Markov Chain $P(B, B')$ on the bases of any rank-r matroid on $[n]$ mixes quickly:

$$\min\{t \in \mathbb{N} : ||P^t(B, \cdot) - \pi||_1 \leq \epsilon\} \leq r^2 \log(n/\epsilon).$$

Sum up: completely log-concave polynomials

- log-concavity of polynomial as functions
  $\Rightarrow$ log-concavity of coefficients
- many matroid polynomials are completely log-concave
- the theory of stable polynomials extends to CLC polynomials



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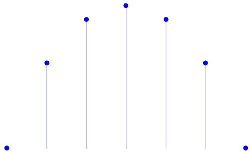

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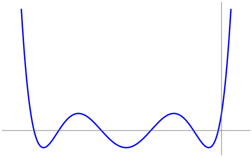

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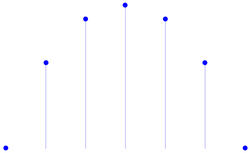

Thanks!

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