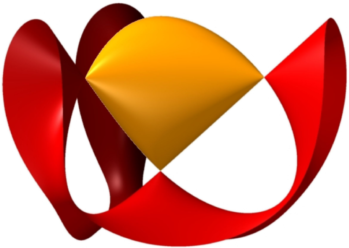
Geometry of Spectrahedra



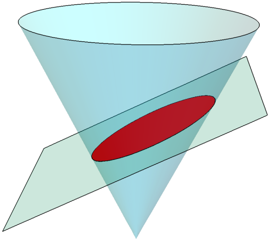
Cynthia Vinzant,
North Carolina State University

AMS Short Course on Sums of Squares
JMM 2019

Spectrahedra

Let $\mathcal{S}^n_+$ denote the convex cone of positive semidefinite matrices in $\mathcal{S}^n$.

A spectrahedron is the intersection $\mathcal{S}^n_+$ with an affine linear space L.



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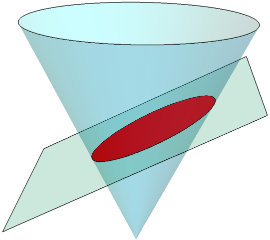

Example: for $\pi : \mathcal{S}^n \to \mathbb{R}[x_1, \ldots, x_n]_{\leq 2d}$ given by $\pi(A) = \mathrm{m}_d^T A \, \mathrm{m}_d$

$$\pi^{-1}(f) \cap \mathcal{S}_+^n$$

is the spectrahedron of sums of squares representations of f.

Spectrahedra of a sum of squares

For $n = 1$, $2d = 4$, and $f(x) = x^4 + x^2 + 1$,

$$f(x) = \begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & a \\ 0 & 1-2a & 0 \\ a & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}$$

This matrix is positive semidefinite $\Leftrightarrow a \in [-1, 1/2]$.

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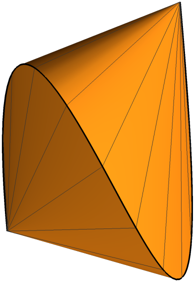
At endpoints, $a = -1, 1/2$, this matrix has rank two $\Rightarrow$

$$(x^2 - 1)^2 + (\sqrt{3}x)^2 \;\text{ and }\; (x^2 + 1/2)^2 + (\sqrt{3}/2)^2.$$

Spectrahedra

Example:

$$L = \left\{ \begin{pmatrix} 1 & x & y & z \\ x & 1 & x & y \\ y & x & 1 & x \\ z & y & x & 1 \end{pmatrix} : (x, y, z) \in \mathbb{R}^3 \right\}$$



$L \cap \mathcal{S}_+^4$

Spectrahedra

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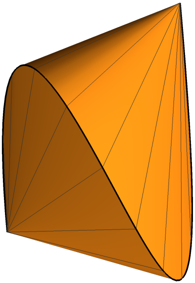

$L \cap \mathcal{S}^4_+$

Goal: understand the algebraic and convex geometry of $L \cap \mathcal{S}^n_+$

Positive semidefinite matrices

A real symmetric matrix A is positive semidefinite if the following equivalent conditions hold:

- all eigenvalues of A are ≥ 0
- all diagonal minors of A are ≥ 0
- $v^T A v \geq 0$ for all $v \in \mathbb{R}^n$
- there exists $B \in \mathbb{R}^{n \times k}$ with

$$A \;=\; BB^T \;=\; (\langle r_i, r_j \rangle)_{ij} \;=\; \sum_{i=1}^{k} c_i c_i^T$$

where $r_1, \ldots, r_n$, $c_1, \ldots, c_k$ are the rows and columns of B

Real algebraic geometry basics

Basic closed semialgebraic set = set of the form

$$\{\mathbf{p} \in \mathbb{R}^n : g_1(\mathbf{p}) \geq 0, \ldots, g_s(\mathbf{p}) \geq 0\}$$

where $g_1, \ldots, g_s \in \mathbb{R}[x_1, \ldots, x_n]$.

Example: $\mathcal{S}^n_+$ and $\mathcal{S}^n_+ \cap L$ (given by diagonal minors ≥ 0)

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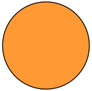
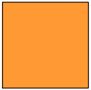
Semialgebraic set = finite boolean combination (complements, intersections, and unions) of basic closed semialgebraic sets



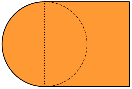
basic closed

$1 - x^2 - y^2 \geq 0$



basic closed

$x(2-x) \geq 0, 1 - y^2 \geq 0$



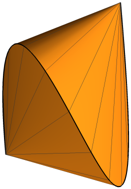
not basic closed

(union)

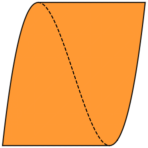
Real algebraic geometry basics

Tarski-Seidenberg Theorem
The projection of a semialgebraic set is semialgebraic.

$$S = \{(x, y, z) \,:\, (y+x)^2 \leq (z+1)(x+1),\ (y-x)^2 \leq (z-1)(x-1),\ x^2 \leq 1\}$$



S



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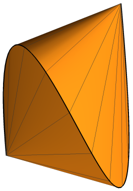

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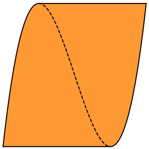

$\pi_{xz}(S)$

$$\pi_{xz}(S) = \{(x,z) \;:\; -1 \leq z \leq 4x^3 - 3x,\; x \leq 1/2\} \cup \{(x,z) \;:\; 4x^3 - 3x \leq z \leq 1,\; -1/2 \leq x\}$$

Computation: Cylindrical Algebraic Decomposition

Convexity basics

A subset $C \subseteq \mathbb{R}^d$ is ...

... *convex* if for $\mathbf{x}, \mathbf{y} \in C$, $\lambda \in [0,1]$, $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in C$

... a *convex cone* if it is convex and $\mu C \subseteq C$ for $\mu \in \mathbb{R}_{\geq 0}$

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The convex hull of $S \subseteq \mathbb{R}^d$ is

$$\mathrm{conv}(S) = \left\{ \sum_{i=1}^{k} \lambda_i \mathbf{p_i} \;:\; \mathbf{p}_i \in S,\ \lambda_i \geq 0,\ \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

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The conical hull of $S \subseteq \mathbb{R}^d$ is

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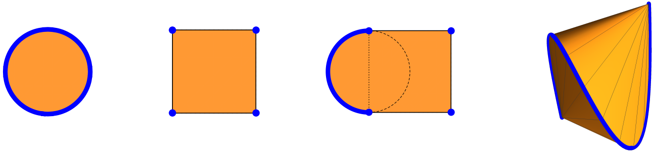
Convexity basics : extreme points/rays

An extreme point of a convex set C is a point $\mathbf{p} \in C$ such that

$$\mathbf{p} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y} \quad \text{for} \quad \mathbf{x}, \mathbf{y} \in C, \lambda \in (0,1) \quad \Rightarrow \quad \mathbf{x} = \mathbf{y} = \mathbf{p}.$$



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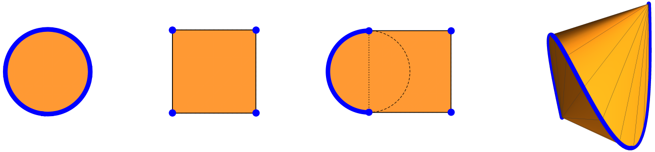

Krein-Milman Theorem

A convex compact set is the convex hull of its extreme points.

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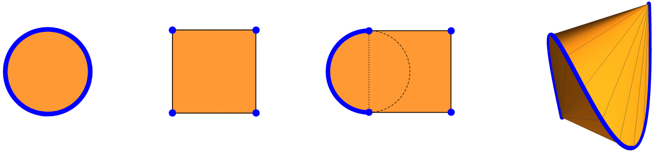

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An extreme ray of a convex cone C is a ray $\mathbb{R}_+\mathbf{r} \subseteq C$ such that

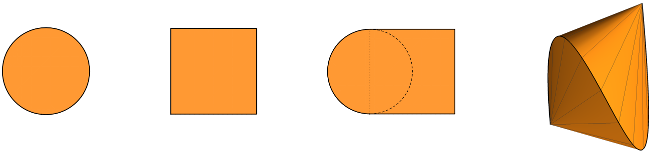
$$\mathbf{r} = \lambda \mathbf{x} + \mu \mathbf{y} \text{ for } \mathbf{x}, \mathbf{y} \in C, \lambda, \mu \in \mathbb{R}_+ \quad \Rightarrow \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}_+\mathbf{r}.$$

Convexity basics : faces

Extreme points and rays are examples of faces.

We say $F \subseteq C$ is a *face* of C if F is convex and

$$\lambda \mathbf{x} + (1-\lambda)\mathbf{y} \in F \;\text{ for }\; \mathbf{x}, \mathbf{y} \in C, \lambda \in (0,1) \quad \Rightarrow \quad \mathbf{x}, \mathbf{y} \in F.$$



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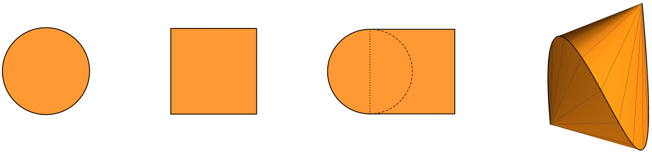

Example: $F = \{\mathbf{x} \in C : \langle \mathbf{c}, \mathbf{x}\rangle \geq \langle \mathbf{c}, \mathbf{y}\rangle \text{ for all } \mathbf{y} \in C\}$

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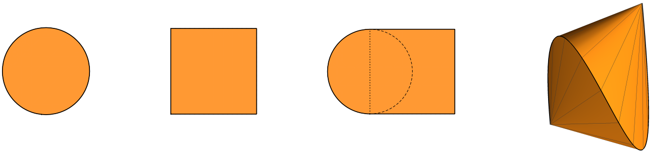

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Note: Faces of $C \cap L$ has the form $F \cap L$ where F = face of C.

Extreme points of $C \cap L$ need not be extreme points of C!

Convexity basics : faces of the PSD cone

Example: $\mathcal{S}_+^n = \text{conv}(\{xx^T : x \in \mathbb{R}^n\})$ is a convex cone.

Its extreme rays are $\{\mathbb{R}_+ xx^T : x \in \mathbb{R}^n\}$.

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Faces of $\mathcal{S}_+^n$ have dim $\binom{r+1}{2}$ for $r = 0, 1, \ldots, n$ and look like

$$F_V = \{A \in \mathcal{S}_+^n \ : \ V \subseteq \ker(A)\}.$$

Ex: for $V = \mathrm{span}\{e_{r+1}, \ldots, e_n\}$,

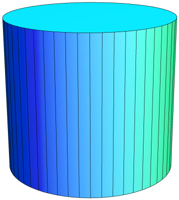
$$F_V \ = \ \left\{ \begin{pmatrix} B & \underline{0} \\ \underline{0} & \underline{0} \end{pmatrix} \ : \ B \in \mathcal{S}_+^r \right\} \ \cong \ \mathcal{S}_+^r$$

Convexity basics : faces of spectrahedra

Faces of $\mathcal{S}_+^n \cap L$ have the form $F_V = \{A \in \mathcal{S}_+^n \cap L \ : \ V \subseteq \ker(A)\}$.

Example:

$$L = \left\{ A(x,y,z) = \begin{pmatrix} 1-x & y & 0 & 0 \\ y & 1+x & 0 & 0 \\ 0 & 0 & 1-z & 0 \\ 0 & 0 & 0 & 1+z \end{pmatrix} : (x,y,z) \in \mathbb{R}^3 \right\}$$



$\mathcal{S}_+^4 \cap L$

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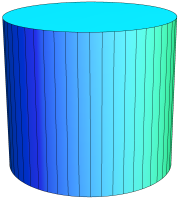


$\mathcal{S}_+^4 \cap L$

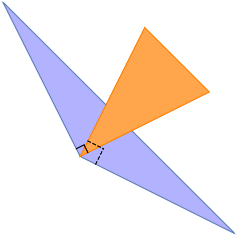
V		F_V
$\mathrm{span}_{\mathbb{R}}\{e_3\}$	$\rightarrow$	2−dim'l face $z = 1$
$\mathrm{span}_{\mathbb{R}}\{e_1\}$	$\rightarrow$	edge $x = 1, y = 0$
$\mathrm{span}_{\mathbb{R}}\{e_1, e_3\}$	$\rightarrow$	point $(x,y,z) = (1,0,1)$

Dual cones

For a convex convex cone $C \subseteq \mathbb{R}^d$,
the *dual cone* is

$$C^* = \{\mathbf{c} \in \mathbb{R}^d : \langle \mathbf{c}, \mathbf{x} \rangle \geq 0 \text{ for all } \mathbf{x} \in C\}.$$

For closed cones, $(C^*)^* = C$.



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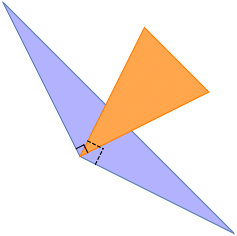

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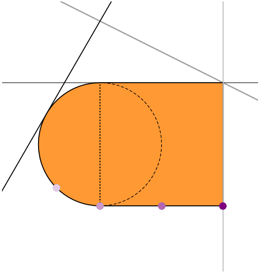
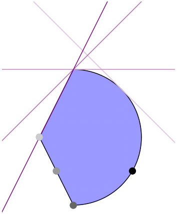
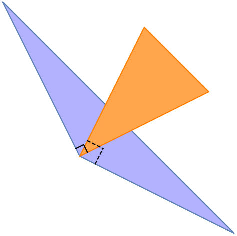
Visualizing with $c_1 = x_1 = 1$:





Dual cones: visualization challenge



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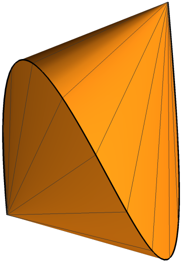
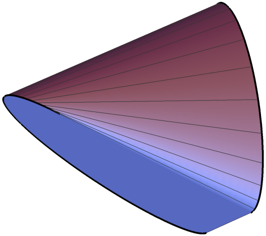
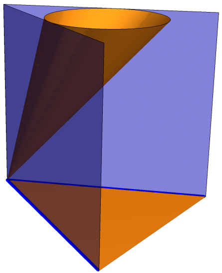


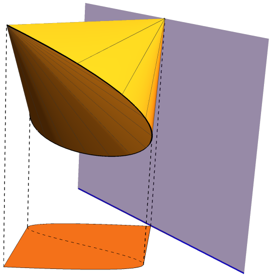


Dual cones: projection and slicing

Consider (orthogonal) projection $\pi_L : \mathbb{R}^d \to L$.

For a convex cone $C \subseteq \mathbb{R}^d$, what linear inequalities define $\pi_L(C)$?





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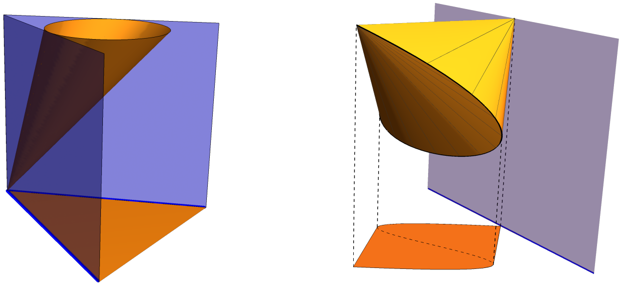
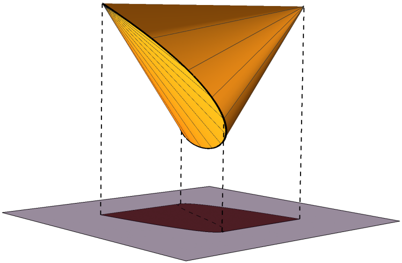


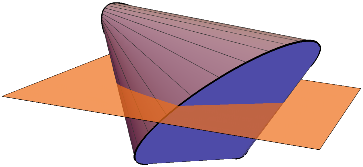
Ans: $\{\ell \in C^* : \ell \text{ is constant on preimages of } \pi_L\} \cong C^* \cap L$

Projection and slicing are dual operations.

For (orthogonal) projection $\pi_L : \mathbb{R}^d \to L$.

$$(\pi_L(C))^* \;=\; C^* \cap L \qquad \text{and} \qquad (C \cap L)^* \;=\; \overline{\pi_L(C^*)}$$





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Dual cones: PSD cone

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$\mathcal{S}_+^n$ is self-dual under the inner product $\langle A, B\rangle = \text{trace}(A \cdot B)$:

$$\begin{aligned}\langle A, B\rangle \geq 0 \text{ for all } B \in \mathcal{S}_+^n \;&\Leftrightarrow\; \langle A, bb^T\rangle \geq 0 \text{ for all } b \in \mathbb{R}^n \\ &\Leftrightarrow\; b^T A b \geq 0 \text{ for all } b \in \mathbb{R}^n \\ &\Leftrightarrow\; A \in \mathcal{S}_+^n\end{aligned}$$

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Dual cones: PSD cone

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Cor: {spectrahedral shadows} are closed under projection, duality

Dual cones: sums of squares

For any subspace $L \subset \mathcal{S}^n$,

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Recall that $\Sigma_{n,\leq 2d} = \pi_L(\mathcal{S}^N_+)$ where $\pi_L(A) = \mathrm{m}_d(\mathbf{x})^T A \, \mathrm{m}_d(\mathbf{x})$

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When $\Sigma_{n,\leq 2d} = P_{n,\leq 2d}$, this gives that

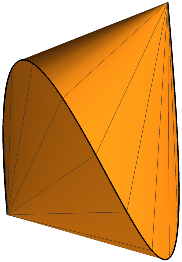
$$P^*_{n,\leq 2d} \;=\; \mathrm{conv}(\mathrm{m}_{2d}(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\}$$

is a spectrahedron.

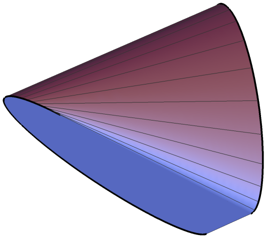
Duality and SOS

$$C = \operatorname{conv}\{\lambda(1, t, 2t^2 - 1, 4t^3 - 3t) : t \in [-1, 1], \lambda \geq 0\}$$

$$C^* = \{(a, b, c, d) : a + bt + c(2t^2 - 1) + d(4t^3 - 3t) \geq 0 \text{ for } t \in [-1, 1]\}$$



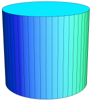
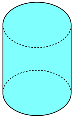
C
spectrahedron



C^*
spec. shadow

Caution:

The projection of spectrahedron may not be a spectrahedron!

$C =$  $\pi(C) =$ 

not *basic* closed

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Spectrahedral shadows: an interlude

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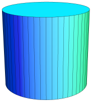
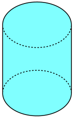
$C =$  $\pi(C) =$ 

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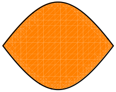
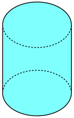
$C =$  $C^* =$ 

still not a spectrahedron

Spectrahedral shadows: an interlude

A spectrahedral shadow is the image of a spectrahedron under linear projection. These are convex semialgebraic sets.

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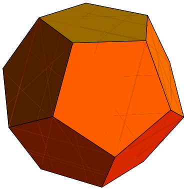
Counterexample by Scheiderer in 2016: $P_{3,\leq 6}$.

Open: What is the smallest dimension of a counterexample?

Back to spectrahedra $L \cap \mathcal{S}^n_+$

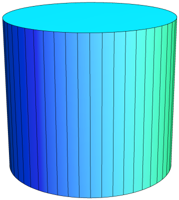
Parametrize L by $A(\mathbf{x}) = A_0 + x_1 A_1 + \ldots, x_d A_d$.

Then $L \cap \mathcal{S}^n_+ \cong \{\mathbf{x} \in \mathbb{R}^d : A(\mathbf{x}) \succeq 0\}$.



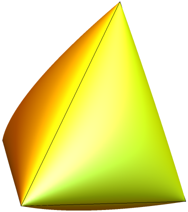
polytope

$$\begin{pmatrix} \ell_1(\underline{x}) & & 0 \\ & \ddots & \\ 0 & & \ell_{12}(\underline{x}) \end{pmatrix}$$



cylinder

$$\begin{pmatrix} 1-x & y & 0 & 0 \\ y & 1+x & 0 & 0 \\ 0 & 0 & 1-z & 0 \\ 0 & 0 & 0 & 1+z \end{pmatrix}$$

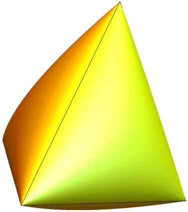


elliptope

$$\begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix}$$

Example: Elliptopes

The $n \times n$ elliptope is



$$\mathcal{E}_n = \{A \in PSD_n : A_{ii} = 1 \text{ for all } i\}$$

$= \{n \times n \text{ correlation matrices}\}$ in stats

$\mathcal{E}_n$ has 2^{n-1} matrices of rank-one: $\{\mathbf{x}\mathbf{x}^T : \mathbf{x} \in \{-1,1\}^n\}$, corresponding to cuts in the complete graph K_n.

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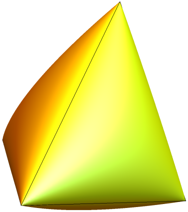

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$$\begin{aligned} MAXCUT &= \max_{S \subset [n]} \sum_{i \in S, j \in S^c} w_{ij} = \max_{x \in \{-1,1\}^d} \sum_{i,j} w_{ij} \frac{(1 - x_i x_j)}{2} \\ &= \max_{A \in \mathcal{E}_n, \mathrm{rk}(A)=1} \sum_{i,j} w_{ij} \frac{(1 - A_{ij})}{2} \leq \max_{A \in \mathcal{E}_n} \sum_{i,j} w_{ij} \frac{(1 - A_{ij})}{2}. \end{aligned}$$

Goemans-Williamson use this to give $\approx .87$ optimal cuts of graphs.

Example: Univariate Moments

$C = \text{conv}\{(t, t^2, \ldots, t^{2d}) : t \in \mathbb{R}\}$ is a spectrahedron in $\mathbb{R}^{2d}$

$C = \left\{\mathbf{x} \in \mathbb{R}^{2d} : M(\mathbf{x}) \succeq 0\right\}$ where $M(\mathbf{x}) = (x_{i+j-2})_{1 \le i,j \le d+1}$

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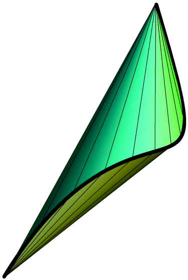


Ex: $\text{conv}\{(t, t^2, t^3) : t \in [-1, 1]\}$

$$= \left\{\mathbf{x} \in \mathbb{R}^3 : \begin{pmatrix} 1 \pm x_1 & x_1 \pm x_2 \\ x_1 \pm x_2 & x_2 \pm x_3 \end{pmatrix} \succeq 0\right\}$$

Extreme Points: Pataki range

$C = \{\mathbf{x} \in \mathbb{R}^d : A(\mathbf{x}) \succeq 0\}$, $\dim(C) = d$, $A_i \in \mathcal{S}^n$.

If $\mathbf{x}$ is an extreme point of C and r is the rank of $A(\mathbf{x})$ then

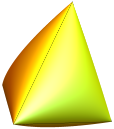
$$\binom{r+1}{2} + d \leq \binom{n+1}{2}$$

Furthermore if $A_0, \ldots, A_d$ are generic, then $d \geq \binom{n-r+1}{2}$.

The interval of $r \in \mathbb{Z}_+$ satisfying both $\leq$'s is the Pataki range.

Pataki range: examples

Example: $d = 3, n = 3$
Pataki range: $r = 1, 2$



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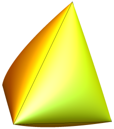

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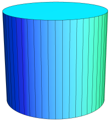

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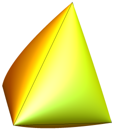

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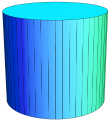

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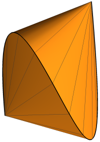

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Low-rank matrices on the elliptope

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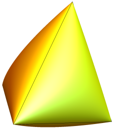

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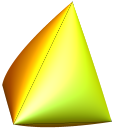

Counting rank-1 matrices:

$\{X : \text{rank}(X) \leq 1\}$ is variety of codim 3 and degree 4 in $\mathcal{S}^3$.

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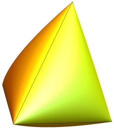

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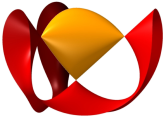
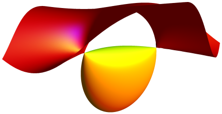
There must be ≥ 1 rank-1 matrix. Why? Topology!

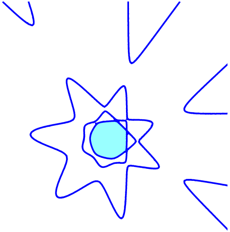
If ∂C has no rank-1 matrices, then the map $S^2 \cong \partial C \to \mathbb{P}^2(\mathbb{R})$ given by $\mathbf{x} \mapsto \ker(A(\mathbf{x}))$ is an embedding. $\Rightarrow\Leftarrow$





(For more see Friedland, Robbin, Sylvester,1984)

Another connection with topology



Suppose $A_0 = I$ and let $f(\mathbf{x}) = \det(A(\mathbf{x}))$.

$\Rightarrow$ f is hyperbolic, i.e.
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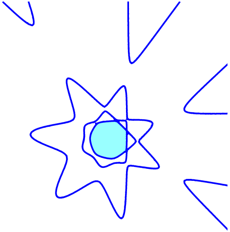

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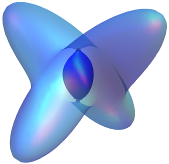

If $V_{\mathbb{R}}(f)$ is smooth, this implies that that $V_{\mathbb{R}}(f) \subset \mathbb{P}^{n-1}$ consists of $\lfloor d/2 \rfloor$ nested spheres.

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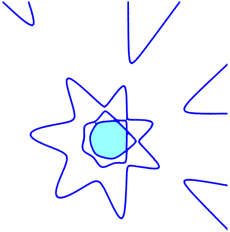

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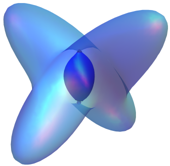

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Open (Generalized Lax Conjecture):
Is every hyperbolicity region a spectrahedron?

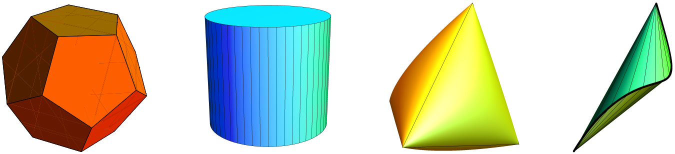
Some combinatorial questions on spectrahedra

What is the "f-vector" of a spectrahedron?

Extreme points and faces come with a lot of discrete data ...

dimension, matrix rank, dimension of normal cone, degree,
number of connected components, Betti #s, ...

Very open: What values are possible?



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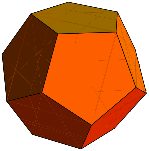
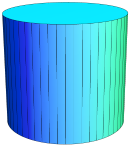
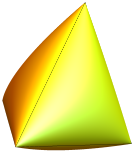
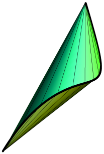

Thanks!