Research Statement – Cynthia Vinzant

My research lies at the interface between real algebraic geometry, convexity, and combinatorics. This involves the study of real polynomials with special properties such as hyperbolicity. The goal of this research is to develop this theory both for its own sake as well as for applications in combinatorics, matrix theory, and convex optimization.

One of the benefits of studying polynomials and solutions to polynomial equations over the real numbers is that techniques and results lend themselves immediately to applications in several other fields. In my research, this has been particularly true for real algebraic sets defined by determinants or minors of matrices, which appear in convex optimization [38, 60], sums of squares [15, 24], numerical linear algebra [14, 40], and signal progressing [25, 61].

Hyperbolic polynomials and, even more generally, log-concave polynomials are real polynomials that share many of the useful functional properties of determinants. Hyperbolic polynomials were introduced in the mid-20th century by Petrovsky and Gårding, who established many of their basic properties in order to understand related partial differential equations. Güler [31] and Renegar [51] brought this theory into the field convex optimization and showed that hyperbolicity underlies the success of fundamental algorithms in the area. Borcea and Brändén [16] made several breakthroughs in the related theory of stable polynomials around the same time. Since then, hyperbolic and stable polynomials have found wide-spread applications in combinatorics [19, 32, 42], convex analysis [13], operator theory [34, 43], probability [17], and theoretical computer science [6, 41, 58]. My research in this area involves characterizing which hyperbolic polynomials can be written as determinants, constructing determinantal representations when they exist, investigating analogues in real varieties of higher codimension, and understanding their combinatorics via tropical geometry.

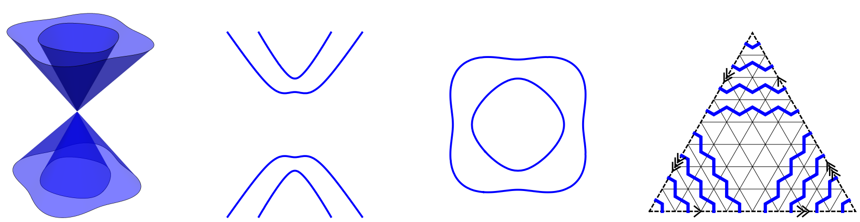

Figure 1. A quartic hyperbolic hypersurface, two affine slices, and tropicalization.

Recently, I have been working to extend this theory to a larger class of real polynomials, namely those that are log-concave as functions on the positive orthant. These polynomials are closely related to combinatorial structures called matroids. With coauthors Anari, Liu, and Oveis Gharan, I have used these polynomials to answer two long-standing open conjectures on matroids, namely Mason's conjecture on the ultra-log-concavity of numbers of independent sets [8] and the Mihail-Vazirani conjecture on the expansion of the basis-exchange graph [7].

My research program aims to use the powerful tools of real algebraic geometry and tropical geometry to better illuminate the algebraic, analytic, and combinatorial properties of determinantal, hyperbolic, and log-concave polynomials. This is a goal worthy in its own right as well as for its potential to impact other areas of mathematics, statistics, and computer science. The following three branches of this research are described in more detail below:

1. Hyperbolic polynomials, stable polynomials, and determinants

A homogeneous polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]_d$ is **hyperbolic** with respect to $e \in \mathbb{R}^n$ if $f(e) \neq 0$ and for every vector $v \in \mathbb{R}^n$ the univariate polynomial $f(te - v) \in \mathbb{R}[t]$ has only real zeroes. First explored by Gårding for his work in PDE's [30], hyperbolic polynomials later became useful in optimization as a very general context for interior point methods [31, 51]. We call f **stable** (or real stable) if it is hyperbolic with respect to every point in the positive orthant. This is equivalent to the more usual definition that $f(z) \neq 0$ whenever $\mathrm{Im}(z) \in \mathbb{R}^n_{>0}$.

Determinants form important examples of hyperbolic polynomials. Consider $d \times d$ Hermitian matrices $A_1, \ldots, A_n$ and the linear matrix polynomial $A(x) = x_1A_1 + \ldots x_nA_n$. If the matrix $A(e)$ is positive definite for some point $e \in \mathbb{R}^n$, then f is hyperbolic with respect to e. For example, if $A(e) = I$, then the roots of $f(te - v)$ are the eigenvalues of $A(v)$. Helton and Vinnikov showed that a polynomial in $n = 3$ variables is hyperbolic if and only if it has such a symmetric definite determinantal representation [34]. One of the main themes of my research has been to understand which hyperbolic polynomials are determinantal [38] and construct determinantal representations when they exist [14, 46, 47, 48, 49].

In 2013, Plaumann and I [49] extended a classical construction by Dixon [29] to produce Hermitian determinantal representations of hyperbolic plane curves. For $f \in \mathbb{R}[x_1, x_2, x_3]_d$ hyperbolic with respect to $e \in \mathbb{R}^3$, we give a method for constructing a $d \times d$ Hermitian matrix pencil $A(x) = \sum_i x_iA_i$ so that $f = \det(A(x))$ and $A(e)$ is definite. A main ingredient of this proof was the following strong connection between definiteness and interlacing.

Theorem (Plaumann, Vinzant [49]). *A matrix representation $f = \det(\sum_{i=1}^n x_iA_i)$ is definite (i.e. the linear span of $A_1, \ldots, A_n$ contains a definite matrix) if and only if the diagonal entries of the adjugate matrix $(\sum_{i=1}^n x_iA_i)^{\mathrm{adj}}$ are hyperbolic and interlace f.*

A follow-up paper studied convex cones of interlacers, certificates of hyperbolicity using sums of squares, and applications to basis generating polynomials of matroids [38].

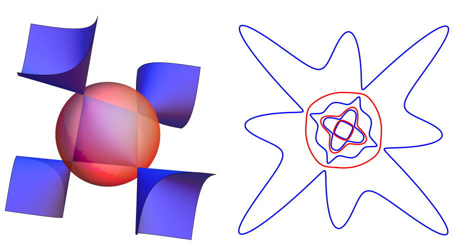

Figure 2. Invariant hyperbolic hypersurfaces and interlacers

My recently-graduated Ph.D. student, Faye Pasley Simon, built off of these techniques in her thesis [40, 57] to answer a question of Chien and Nakazato on the rotational invariance of the *numerical range* of a complex matrix [22]. This involves determinantal representations certifying invariance under finite group actions. Specifically, a unitary representation $\rho : \Gamma \to U(d)$ of the group $\Gamma \subset \mathrm{GL}(\mathbb{C}^n)$ acts on $\mathbb{C}^{d\times d}$ by conjugation. We say that a $d \times d$ linear matrix $A(x) = \sum_i x_iA_i$ is invariant with respect to γ and ρ if for every $\gamma \in \Gamma$, $A(\gamma \cdot x) = \rho(\gamma)A(x)\rho(\gamma^{-1})$. The determinant $f = \det(A(x))$ is then invariant under the action of Γ. For $n = 3$, curves invariant under the cyclic group of rotations by $2\pi/n$ always have such a representation.

Theorem (Simon, Vinzant, 2019+). *For $d \in n\mathbb{Z}_+$, every hyperbolic, invariant polynomial f in $\mathbb{R}[x_1, x_2, x_3]_d^{C_n}$ has a definite, invariant determinantal representation $f = \det(A(x))$.*

Studying determinantal representations in more variables requires understanding the appearance of low-rank matrices in subspaces of $d \times d$ Hermitian matrices. One application of this appears in signal processing, in which one wants to recover vector $v \in \mathbb{C}^d$ from its "measurements" $|\langle \phi_k, v\rangle|^2 = \mathrm{trace}(\phi_k\phi_k^*vv^*)$ given by $\phi_1, \ldots, \phi_n \in \mathbb{C}^d$. This relates to low-rank matrix completion and has many imaging-related applications: microscopy, optics, and diffraction imaging, among others. An important question in this field is how many measurements are necessary to guarantee that phase retrieval is possible [11, 33]. Using techniques from real algebraic geometry, coauthors and I prove a conjecture of [12] that $4d-4$ generic measurements suffice and that fewer measurements may not.

Theorem (Conca, Edidin, Hering, Vinzant [25, 61])**.** *For generic vectors $\phi_1, \ldots, \phi_n \in \mathbb{C}^d$ with $n \geq 4d-4$, the real values* $\mathrm{trace}(\phi_k \phi_k^* v v^*)$ *uniquely determine the rank-one matrix vv^*. Moreover, this is tight for $d = 2^k + 1$ but not for $d = 4$.*

There are also several applications of determinantal polynomials in discrete probability and theoretical computer science comes from determinantal point processes [37, 53]. Polynomials of the form $f(x) = \det(\sum_i x_i v_i v_i^*) \in \mathbb{R}[x_1, \ldots, x_n]_d$ where $v_i \in \mathbb{C}^d$ have nonnegative coefficients. If they also sum to one, then these coefficients form a discrete probability distribution on the support $\{S : \mathrm{coeff}(f, x^S) \neq 0\}$ of f. The stability of f implies that this will be a *strongly Rayleigh distribution*, which exhibits nice properties such as negative dependence.

2. Log-concave polynomials and matroids

One important direction of my research concerns a generalization of hyperbolicity called complete log-concavity. For $v \in \mathbb{R}^n$, let $D_v f = \sum_{i=1}^n v_i \partial f / \partial x_i$ denote the directional derivative of f in direction v. A polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]_d$ is **completely log-concave** on an open convex cone $K \subset \mathbb{R}^n$ if for any vectors $v_1, \ldots, v_s \in K$, the polynomial $D_{v_1} \cdots D_{v_s} f$ is either identically zero or log-concave on K. One motivating example is a hyperbolic polynomial, which is log-concave on its hyperbolicity cone.

There is a beautiful connection between hyperbolic polynomials and combinatorial structures called matroids. A **matroid** M on ground set $[n]$ can be defined by its set of *bases*, $\mathcal{B}(M)$, which is a non-empty collection of subsets of $[n]$ satisfying the “exchange property”

$$A, B \in \mathcal{B}(M),\ a \in A \backslash B \quad \Rightarrow \quad \exists\, b \in B \ \text{ with } \ A \backslash \{a\} \cup \{b\} \in \mathcal{B}(M).$$

The *independent sets* of M are $\mathcal{I}(M) = \{I : I \subseteq B \text{ for some } B \in \mathcal{B}(M)\}$. Matroids model various types of independence, such as cyclic independence in graphs, linear independence in vector spaces, and algebraic independence in field extensions. Working with general matroids allows for proving statements about all of these independence structures simultaneously.

One can study properties of matroids using polynomials. Choe, Oxley, Sokal, and Wagner [23] show that the support $\{S \subset [n] : c_S \neq 0\}$ of a multiaffine stable polynomial $\sum_{S \in \binom{[n]}{d}} c_S x^S$ in $\mathbb{R}[x_1, \ldots, x_n]_d$ is the set of bases of a matroid. For example, if vectors $v_1, \ldots, v_n$ in $\mathbb{C}^d$ represent a matroid of rank d, the support of the stable, multiaffine polynomial $\det(\sum_i x_i v_i v_i^*)$ in $\mathbb{R}[x_1, \ldots, x_n]_d$ is exactly its set of bases. That is, the coefficient of the monomial x^S is nonzero if and only if the vectors $\{v_i : i \in S\}$ form a basis for $\mathbb{C}^d$. Some, but not all, matroids appear as the support of a real stable polynomial [18, 38, 63]. Completely log-concavity on the positive orthant, however, perfectly captures the combinatorial condition of being a matroid.

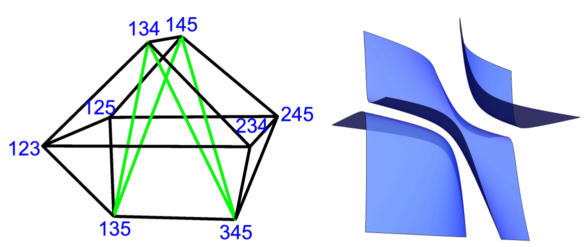

Figure 3. A matroid polytope and corresponding real stable hypersurface.

Theorem (Anari, Liu, Oveis Gharan, Vinzant [8, 9])**.** *If a multiaffine polynomial is completely log-concave on $\mathbb{R}^n_{>0}$, then its support forms the bases of a matroid. Moreover, for any matroid M, the basis and independent set generating polynomials, $f_M = \sum_{B\in\mathcal{B}(M)}\prod_{i\in B} x_i$ and $g_M = \sum_{I\in\mathcal{I}(M)} y^{n-|I|}\prod_{i\in I} x_i$, are completely log-concave on $\mathbb{R}^n_{>0}$.*

We also have generalizations of this theorem for polynomials with arbitrary degree in each variable, but thus far, the strongest consequences appear using the multiaffine polynomials above. In particular, this implies the strongest form of Mason's conjecture [44].

Theorem (Anari, Liu, Oveis Gharan, Vinzant [8])**.** *For any matroid on n elements, the sequence $(\mathcal{I}_k)$ of the number independent sets of size k is ultra log-concave, i.e. for $0 < k < n$,*

$$\left(\frac{\mathcal{I}_k}{\binom{n}{k}}\right)^2 \geq \frac{\mathcal{I}_{k-1}}{\binom{n}{k-1}}\cdot\frac{\mathcal{I}_{k+1}}{\binom{n}{k+1}}.$$

The recent breakthroughs in combinatorial Hodge theory by Adiprasito, Huh, and Katz [1] resolved a weaker version of Mason's conjecture. Completely log-concavity is one distillation of conditions that appear in this work. Brändén and Huh independently introduce *Lorentzian polynomials* [20, 21] and prove the strongest form of Mason's conjecture.

Complete log-concavity of polynomials on $\mathbb{R}^n_{>0}$ also has remarkably close connections with the theory of *high dimensional expanders* developed by Dinur, Kaufman, Mass and Oppenheim [28, 35, 36]. These define random walks on the maximal faces of a simplicial complex that mix quickly. Mixing times of random walks are closely related to the *expansion* of the underlying graph. Mihail and Vazirani conjectured that the edge graph of any 0-1 polytope has expansion ≥ 1 [45]. Building off of this and the results above, we prove this conjecture for matroid polytopes, whose vertices are the indicator vectors of the bases of a matroid.

Theorem (Anari, Liu, Oveis Gharan, Vinzant [7])**.** *The basis-exchange graph of any matroid has expansion ≥ 1. Moreover, there is a Markov chain on the bases of a rank-r matroid M with uniform stationary distribution that mixes in time $O(r^2\log(n))$.*

Based on the techniques developed in this paper, Cryan, Guo, and Mousa recently obtained an improved bound on this mixing time [26]. There are several remaining open questions about the structure and applications of completely log-concave polynomials, such as a classification of preserving operations and the development of a meaningful analogue of the hyperbolicity cone. The theory of high dimensional expanders also suggests various generalizations of complete log-concavity whose combinatorial implications are entirely unknown. This is area rich with interesting research questions with a wide variety of applications.

3. Hyperbolic varieties and tropical geometry

Combinatorial structure also appears in higher codimensional analogues of hyperbolicity. Shamovich and Vinnikov extended the notion of hyperbolicity to general varieties [56]. Kummer and I introduced a natural analogue of stability in this context [39]. A real irreducible variety $X \subset \mathbb{P}^{n-1}(\mathbb{C})$ of codimension c is *positively hyperbolic* if it contains no points $[z]$ whose imaginary part $\mathrm{Im}(z)$ is non-zero and belongs to a positive linear space $L \in \mathrm{Gr}_+(c,n)$. Here a variety in $\mathbb{P}^{n-1}(\mathbb{C})$ is called real when invariant under complex conjugation.

Reciprocal linear spaces are fundamental examples. For a linear space $\mathcal{L} \subseteq \mathbb{P}^{n-1}(\mathbb{C})$, define

$$\mathcal{L}^{-1} = \overline{\left\{\left[x_1^{-1} : \ldots : x_n^{-1}\right] \ \text{ such that } x\in\mathcal{L} \text{ with } x_i\neq 0\right\}}.$$

This variety is closely related to the matroid of $\mathcal{L}$ [50] and appears in the study of central curves of linear programs [27] and hyperplane arrangements [54]. It is hyperbolic with respect to the orthogonal complement $\mathcal{L}^\perp$ [59]. Graduate student Georgy Scholten and I have studied the non-projective varieties obtained from $\mathcal{L}$ by inverting some subset of coordinates, [55].

Just as a definite determinantal representation of a polynomial certifies its hyperbolicity, the hyperbolicity of a variety X follows from the existence of certain definite determinantal representations of its Chow form [56]. The *Chow form* of a $(d-1)$-dimensional variety $X \subset \mathbb{P}^{n-1}(\mathbb{C})$ is a polynomial in the Plücker coordinates of $V \in \mathrm{Gr}(d,n)$ that vanishes whenever the intersection $V^\perp \cap X$ in $\mathbb{P}^{n-1}(\mathbb{C})$ is nonempty.

Theorem (Kummer, Vinzant [39])**.** *The Chow form of $\mathcal{L}^{-1}$ has a determinantal representation given by a definite linear matrix pencil in Plücker coordinates.*

Real tropical geometry offers methods for understanding and constructing varieties and semialgebraic sets with intricate structure via combinatorial methods. This includes Viro's patchworking methods [62] for constructing real plane curves with prescribed topology, Brändén connections between stable polynomials and discrete convexity [18], and the work of Gaubert *et. al.* relating linear and semidefinite programs with mean payoff games [2, 3, 4, 5].

The definition of positive hyperbolicity extends directly to varieties over $\mathbb{C}\{\{t\}\}$. The tropicalization of a variety over $\mathbb{C}\{\{t\}\}$ is a polyhedral complex, which one can study via its maximal faces. For hypersurfaces $X = V(f)$, these correspond to the cones dual to the edges of the Newton polytope of f. One analogue of the Newton polytope for X of high codimension is the Chow polytope, which is the image of the Newton polytope of the Chow form of X under the map $\mathbb{R}^{\binom{n}{d}} \to \mathbb{R}^n$ given by $e_I \mapsto \sum_{i\in I} e_i$. Using the theory of non-crossing partitions, developed in [10] to study positroids, it is possible to give combinatorial conditions on the tropicalization of a positively hyperbolic variety.

Theorem (Rincón, Vinzant, Yu [52])**.** *If X is positively hyperbolic, then the linear space parallel to any maximal face of* $\mathrm{trop}(X)$ *is spanned by* 0-1 *vectors with disjoint, non-crossing supports. Moreover, the Chow polytope of X is a generalized permutohedron.*

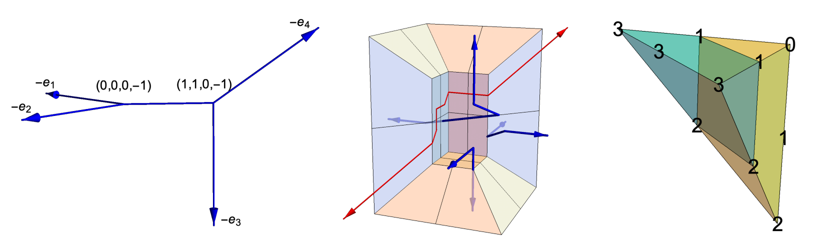

Figure 4. The complex and real tropicalization of a reciprocal line in $\mathbb{P}^3(\mathbb{C}\{\{t\}\})$ and induced subdivision of its Chow polytope.

For hypersurfaces, this recovers the theorems of [18, 23] on the discrete concavity of stable polynomials over $\mathbb{R}\{\{t\}\}$. Another motivation for developing this theory is that the tropical objects satisfying the combinatorial conditions for hyperbolicity may be interesting in their own right, even when not realized as the tropicalization of a hyperbolic variety. In tropical geometry, the difference between realizable matroids and all matroids appears as the difference between tropicalizations of linear spaces and tropical linear spaces, which merely satisfy the same combinatorial conditions. Tropical linear spaces provide geometric objects that can play the role of linear spaces, even for non-representable matroids. One goal for future work is to develop an analogous theory for tropical hyperbolic varieties.

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