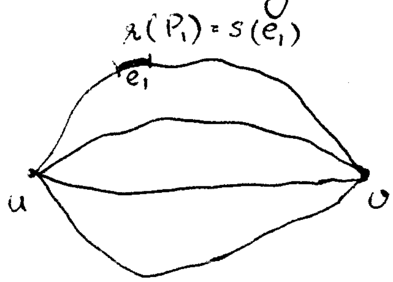
Application 1.7 Maximum Reliability Problem

Want to design a network that maximizes reliability (eg in energy or communications)

Given $G = (V, E)$ $\quad s : E \to \mathbb{R}_+$ "strength" function
$s(e)$ = strength of e

For a path P in G,
reliability $(P) := \min_{e \in EP} s(e)$ "strength of weakest edge in P"

for $u, v \in V$
reliability $r_G(u,v) := \max \{ r(P) : P$ path in G from u to $v \}$

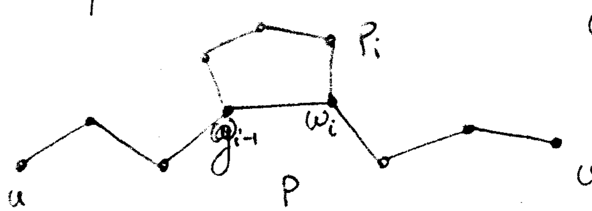


Let T be a max strength spanning tree in G

<u>Claim</u>: $r_T(u,v) = r_G(u,v) \quad \forall u, v \in V$

<u>Proof</u>: Fix $u, v \in V$
Let Q be the unique path in T from u to v
Then $r(Q) = r_T(u,v) \le r_G(u,v)$ since Q is a path from u to v.
Suppose P is another path from u to v with vertices $u = w_1, w_2, \ldots, w_m = v$

$\forall$ edge $(w_{i-1}, w_i) \in P$ $\exists!$ unique path P_i in T from w_{i-1} to w_i



Concatenating $P_2, \ldots, P_m$ get a walk from u to v.
Deleting cycles, get a path from u to v but this must be Q

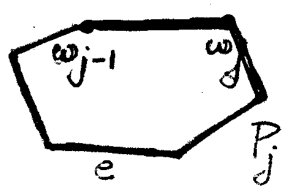
$\therefore$ Each edge of Q is in some P_j

NTS, the weakest edge in Q is stronger than the weakest edge in P.

Let e be the weakest edge in Q.
Suppose e lies in P_j

Case 1: $e \in P$ ($\Rightarrow P_j = e$). Then $s(e) \geq s(f)$ for the weakest edge f in P.

Case 2: $e \notin P$, $e \in P_j$. Then $s(\omega_{j-1}, \omega_j) \leq s(e)$ since otherwise we could swap e for (ω_{j-1}, ω_j) to get a bigger strength T.



So again $s(e) \geq s(f)$ for the weakest edge f in P.

$\therefore \lambda_T(u, v) \geq \lambda_P(u, v) \quad \forall\ P$ from u to v. $\blacksquare$