Linear and Nonlinear Scattering or Layer Stripping via the Riesz Transform

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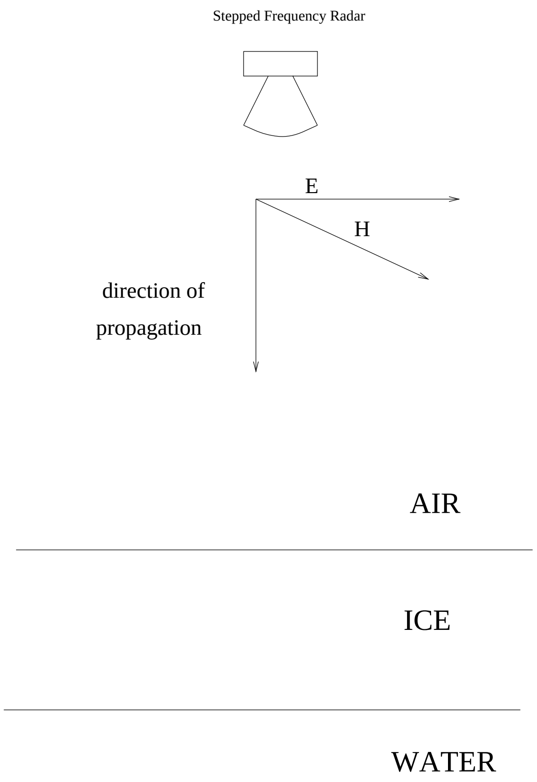
Remote Sensing Problem

Compute the wave speed, c, for a lossless, layered, medium from a single reflection coefficient at normal incidence.

$c(x)$ is real valued, depends only on depth, and independent of frequency

$r(\omega)$ is a complex valued (amplitude and phase) reflection coefficient assumed known at all frequencies (in a wide band .5Ghz - 10GHz)



The Mathematical Model
and
the Reflection Coefficient

$$\frac{d^2u}{dz^2} + \frac{\omega^2}{c(z)^2}u = 0$$

$$u = \mathrm{e}^{\frac{-i\omega(z-z_T)}{c_T}} + r(\omega)\mathrm{e}^{\frac{i\omega(z-z_T)}{c_T}}$$

AIR

C = 1

ICE

$$u = t(\omega)\mathrm{e}^{\frac{-i\omega(z-z_{)}}{c_B}}$$

WATER

The Variation in $c(x)$ and Travel Time

Only variations in the wavespeed produce reflections, so we introduce

$$\alpha = \frac{dc}{dz}$$

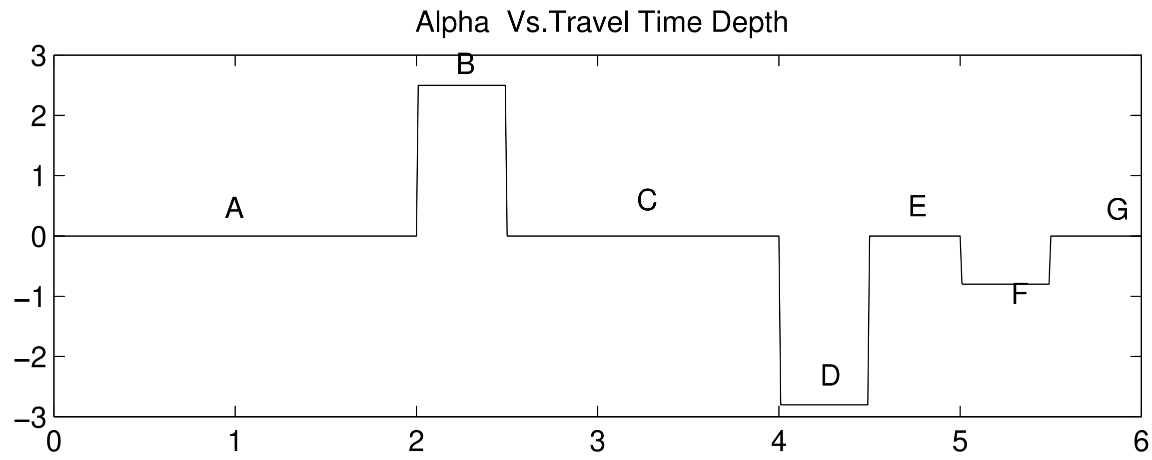
We *see* **these variations paramatrized by the time it takes the wave to reach them and return**. It is convenient to replace the physical depth by the travel time depth, x

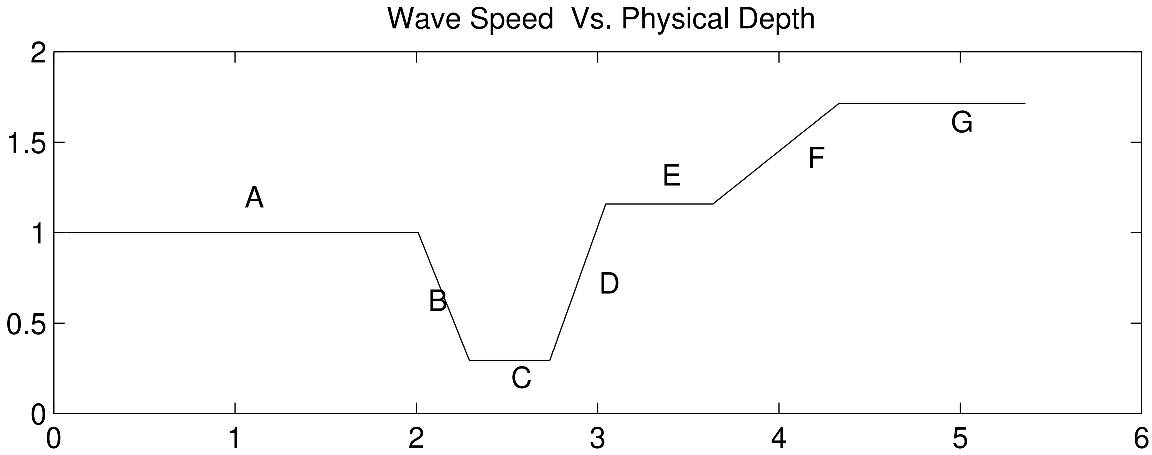
$$x(z) \;=\; \int_0^z \frac{dz}{c}$$

Scattering and Inverse Scattering

$$\alpha(x) \Leftrightarrow r(\omega)$$

The conversion between $\alpha(x)$ and $c(z)$ is easy.





There is an exact correspondance between piecewise linear $c(z)$ and piecewise constant $\alpha(x)$.

Linear Scattering

Mathematical Definition (Frechet Derivative at $\varepsilon = 0$)

$$\begin{aligned} \alpha &= 0 + \varepsilon a \\ r &= 0 + \varepsilon \rho + +O(\varepsilon^2) \end{aligned}$$

Linear Scattering and Inverse Scattering

$$a \Leftrightarrow \rho$$

Calculation

a and ρ are related by the Fourier Transform

$$\rho(\omega) = \int_{-\infty}^{0} \mathrm{e}^{-2i\omega x} a(x) dx$$

Linear Scattering is scattering weak enough that multiple reflections can be completely ignored.

Linear Scattering is exactly the Fourier Transform.

Linear and Nonlinear Scattering

Plancherel Equality

Linear

$$\int_{-\infty}^{\infty} |\rho(\omega)|^2 \, d\omega = \frac{\pi}{4} \int_{-\infty}^{0} |a|^2 dx$$

$$\|\rho\|_{L^2}^2 := \int_{-\infty}^{\infty} |\rho(\omega)|^2 d\omega$$

Nonlinear

$$-\int_{-\infty}^{\infty} \log\left(1 - |r(\omega)|^2\right) d\omega = \frac{\pi}{4} \int_{-\infty}^{0} |\alpha|^2 dx$$

$$E(r) \; := \; -\int_{-\infty}^{\infty} \log\left(1 - |r(\omega)|^2\right) d\omega$$

$$= \; \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \frac{|r(\omega)|^{2k}}{2k}$$

Characterization and Causality

<u>Linear</u>

$$H^2(C^+) \;=\; \left\{ \begin{array}{c} \|\rho\|_{L^2} < \infty \\ \rho \text{ holomorphic in complex} \\ \text{upper half plane} \end{array} \right.$$

$$\;=\; \begin{array}{l} \text{Fourier transforms of } \mathrm{L}^2 \text{ functions} \\ \text{which vanish on x} > 0 \end{array}$$

$$H^2(C^-) \;=\; \text{acausal functions}$$

<u>Nonlinear</u>

$$H^E(C^+) \;=\; \left\{ \begin{array}{c} E(r) < \infty \\ r \text{ holomorphic in complex} \\ \text{upper half plane} \end{array} \right.$$

$$\;=\; \begin{array}{l} \text{Scattering transforms of } \mathrm{L}^2 \text{ functions} \\ \text{which vanish on x} > 0 \end{array}$$

$$H^E(C^-) \;=\; \text{acausal functions}$$

Separation into Causal and Acausal Parts

Linear The Riesz Transform

Any $f \in L^2(R)$ can be decomposed uniquely into a sum of causal and acausal parts; $f \mapsto f^+$ is called the Riesz Transform.

$$f = f^+ + f^-$$

Orthogonality

$$\|f\|_{L^2}^2 = \|f^+\|_{L^2}^2 + \|f^-\|_{L^2}^2$$

$$\langle f^+, f^- \rangle = 0$$

Nonlinear

Any $f \in L^E(R)$,which is close to $H^E(C^+)$, can be decomposed uniquely into a ‘conformal composition” of causal and acausal parts; $f \mapsto f^+$ is the (nonlinear) Riesz Transform.

$$f = f^+ \circ f^- \quad := \quad \frac{f^+ + f^-}{1 + \overline{f^+} f^-}$$

Orthogonality

$$E(f) = E(f^+) + E(f^-)$$

$$E(f^+, f^-) = 0$$

A Metric on H^E

$$
\begin{aligned}
D_E(r,0) &:= E(r) \\
&= -\int_{-\infty}^{\infty} \log\left(1-|r(\omega)|^2\right) d\omega
\end{aligned}
$$

$$
\begin{aligned}
D_E(r,s) &= E(-r \circ s) \\
&= E(F^0_{-r}(s))
\end{aligned}
$$

where

$$
a \circ b = \frac{a+b}{1+\overline{a}b}
$$

$$
\begin{aligned}
F^\theta_r(z) &= \frac{r + \mathrm{e}^{i\theta} z}{1+\overline{r}\mathrm{e}^{i\theta} z} \\
&= \text{conformal map of the disk} \\
&\quad \text{onto itself, mapping} \\
&\quad 0 \mapsto r \\
&\quad -\mathrm{e}^{-i\theta} r \mapsto 0
\end{aligned}
$$

Continuous Dependence

- H^E is a complete metric space

- The Scattering map is bicontinuous i.e.

$$\alpha_n \overset{L^2}{\to} \alpha \quad \leftrightarrow \quad r_n \overset{D_E}{\to} r$$

- D_E is a conformally invariant metric i.e.

$$D_E(F_t^\theta(r), F_t^\theta(s)) = D_E(r, s)$$

- Cauchy Schwartz and Triangle Inequalities

$$\begin{aligned} E(r,s) &:= -\int_{-\infty}^{\infty} \log\left(1 - \overline{r}s\ \right) d\omega \\ E(r,s) &\leq E(r)^{1/2} E(s)^{1/2} \\ D_E(r,s) &\leq D_E(r,t) + D_E(t,s) \end{aligned}$$

Adding a Layer

Adding a layer on top amounts to compositon with a conformal map.

Layer 1

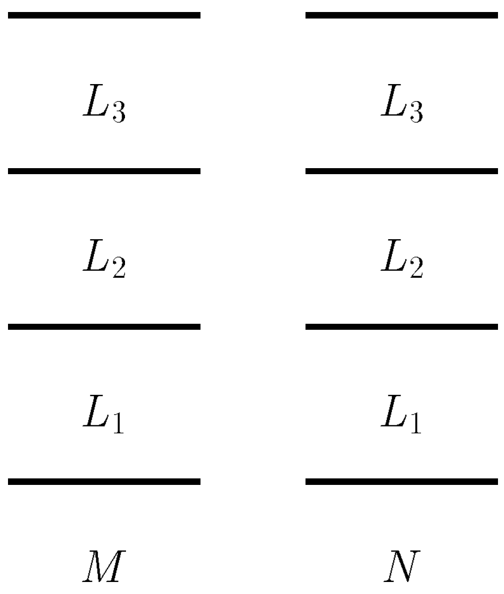
Layer 2

$$
\begin{aligned}
r_{12} &= r_1 \circ \frac{t_1}{\bar{t}_1}\, r_2 \\
&= F_{r_1}^{\theta_1}(r_2)
\end{aligned}
$$

where

$$i\theta_1 = log(\frac{t_1}{\bar{t}_1})$$

Adding Lots of Layers



$$D_E(r_{321M}, r_{321N}) = D_E(r_M, r_N)$$

Moral: Conformal invariance implies that you can't hide a feature by burying it in this metric.

Paley-Wiener and Sampling Theorems

Linear

Paley-Wiener theorem detects the (convex hull of the) support of a from the growth rate of its Fourier transform, ρ, along the imaginary axis.

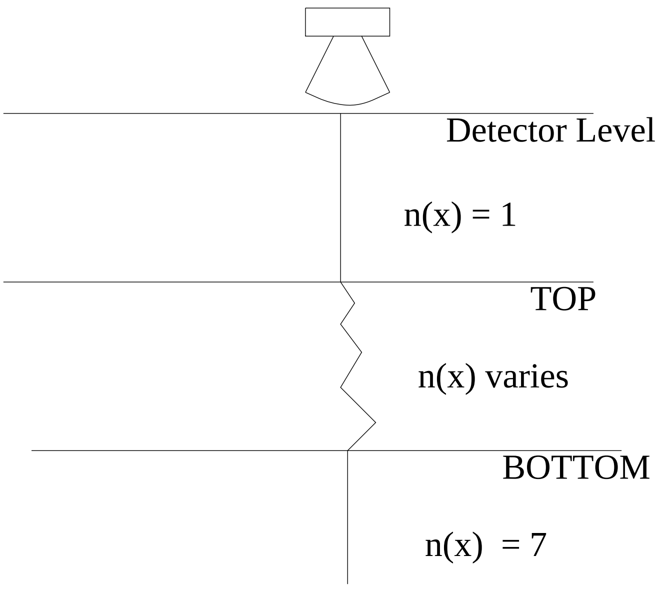
Sampling Theorem gives a formula from which a function with compactly supported Fourier transform can be exactly recovered from its sampled values.

Nonlinear

Paley-Wiener theorem detects the support of α from the growth rate of $\dfrac{r}{1-|r|^2}$ and/or $\dfrac{1}{1-|r|^2}$. Note that, due to multiple reflections, a Paley-Wiener theorem involving the growth rate of the reflection coefficient r itself couldn't possibly be true.

Sampling Theorem gives a formula for exactly recovering the reflection coefficient, $r(\omega)$, from its sampled values. How fine the sampling must be depends on the width of the suport of α, as in the linear case.

The Paley Weiner Theorem

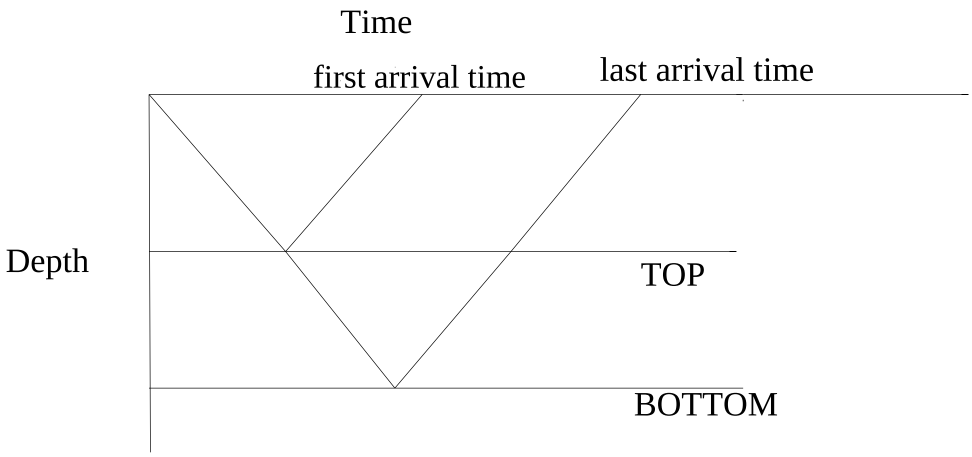


TOP Question: Can you find the top from the reflection data?

BOTTOM Question: Can you find the bottom from the reflection data?

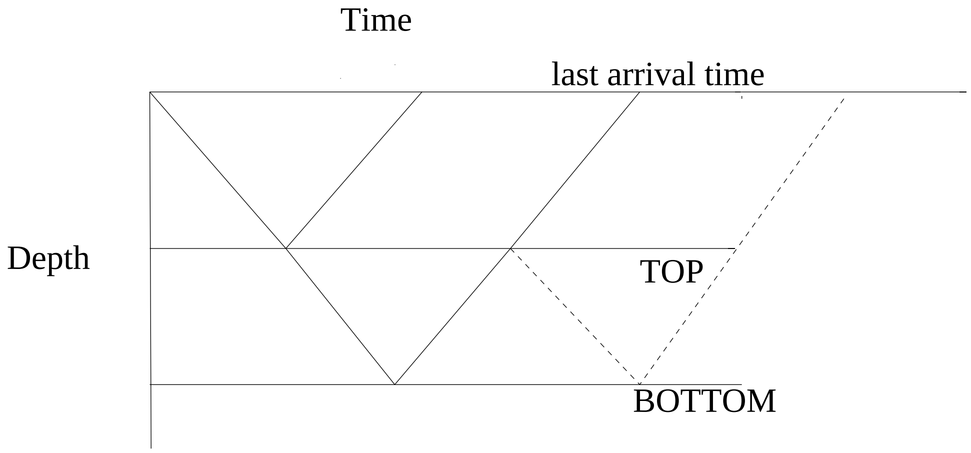
Linear Scattering

The travel time depths of the top and the bottom are easily recognized as the first and last arrival times — *although we must compute α (or estimate it) to convert to physical depth.*

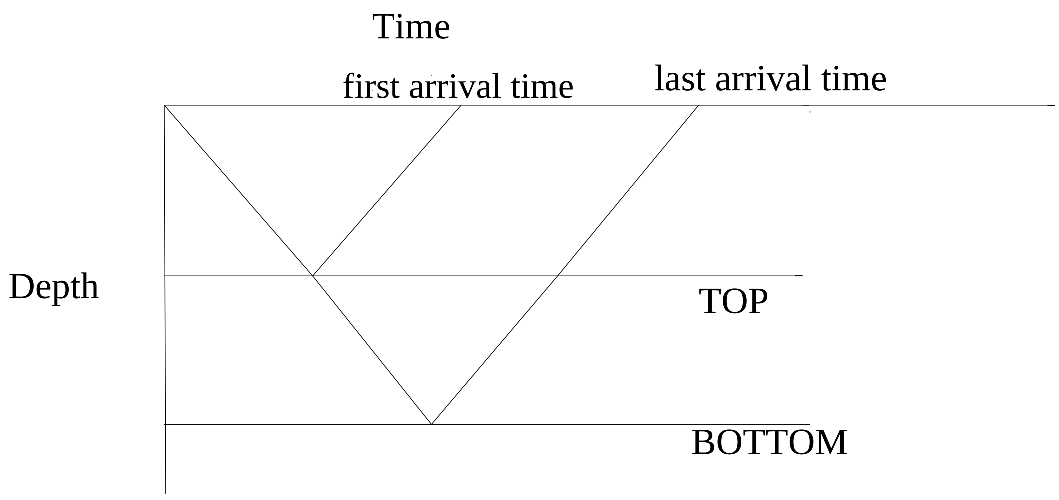


Nonlinear Scattering

Multiple reflections obscure the “Last Arrival Time”.



Finding the Bottom for Nonlinear Scattering



Paley Weiner Theorem Even though the time trace of $\mathbf{r}$ has no last arrival time, the *time trace* of $\frac{\mathbf{r}}{\mathbf{1}-|\mathbf{r}|^{\mathbf{2}}}$ exhibits a last arrival time, precisely as in the linear case.

Width and Intensity Data The *time trace* of $\frac{|r|^2}{\mathbf{1}-|\mathbf{r}|^{\mathbf{2}}}$ exhibits a last arrival time, which is precisely the Width of the inhomogeneity

In the linear case, the same is true of $|\rho|^2$.

A Thin Layer Approximation

$$u = \mathrm{e}^{-i\omega(x-x_T)} + r(\omega)\mathrm{e}^{i\omega(x-x_T)}$$

$$\frac{d^2u}{dx^2} + \alpha\frac{du}{dx} + \omega^2 u = 0$$

$$u = t(\omega)\mathrm{e}^{-i\omega(x-x_B)}$$

$$\begin{aligned} t &\sim \mathrm{e}^{i\omega\Delta} \\ r &\sim \hat{\alpha}(\omega) \end{aligned}$$

A Stack of Thin Layers

Layer 0

Layer 1

Layer 2

Layer 3

$$r = \hat{\alpha}_0 \circ \mathrm{e}^{2i\omega\Delta}\hat{\alpha}_1 \circ \mathrm{e}^{4i\omega\Delta}\hat{\alpha}_2 \circ \mathrm{e}^{6i\omega\Delta}\hat{\alpha}_3 \circ \ldots$$

converges in the D_E metric if and only if

$$E(r) = \sum_{j=0}^{\infty} E(\hat{\alpha}_j) < \infty$$

Subtracting a Thin Layer via the Riesz Tranform

Layer 0

Layer 1

$$
\begin{aligned}
r &= \hat{\alpha}_0 \circ \mathrm{e}^{2i\omega\Delta} r_1 \\
\mathrm{e}^{-2i\omega\Delta} r &= \mathrm{e}^{-2i\omega\Delta} \hat{\alpha}_0 \circ r_1 \\
\mathrm{e}^{-2i\omega\Delta} \hat{\alpha}_0 &\in H^E(C^-) \\
r_1 &\in H^E(C^+)
\end{aligned}
$$

A Reconstruction from Experimental Data

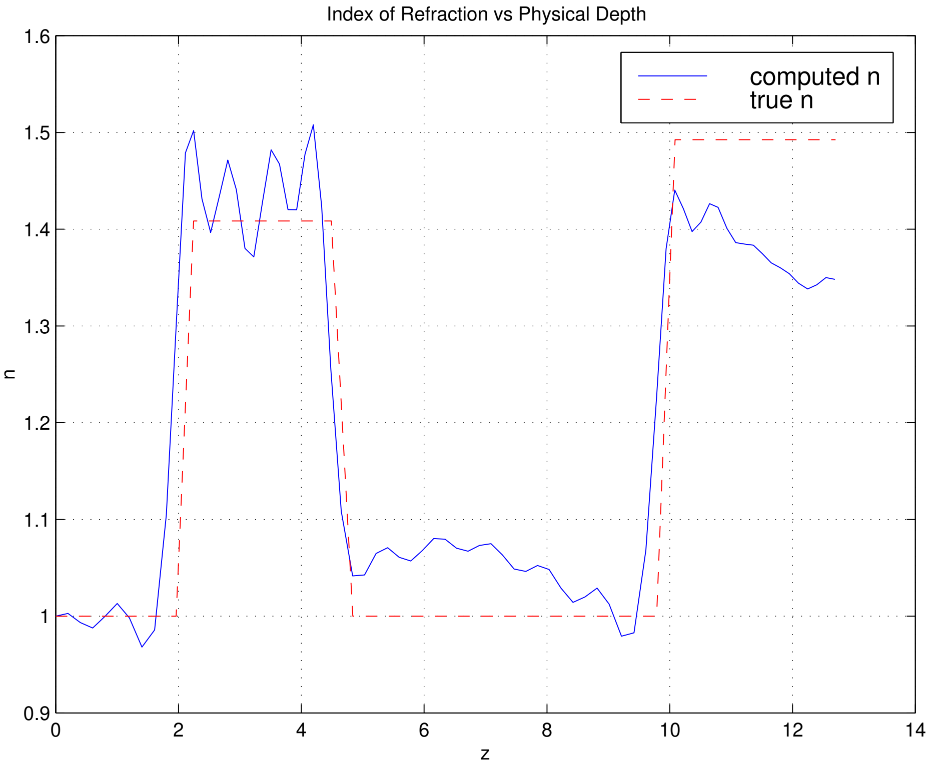
The Target

Air	Drywall	Polystyrene	Sand
2cm	2.5cm	5.5cm	
$c = 1$	$c = 0.71$	$c = 1$	$c = 0.67$

The Data

Complex Reflection coefficient from 1Ghz to 17Ghz, spaced .05 Ghz — courtesy of David Borup and Steve Johnson, University of Utah

The Reconstruction



Alpha and Abs(R) in the Layerstrip GUI