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Always k will denote a field. I will frequently write $x_0, \dots, x_n$ for a system of homogeneous coordinates on $\mathbb{P}^n$ and write $S = k[x_0, \dots, x_n]$ for the associated polynomial ring.

(1) Let $R = k[x_1, \dots, x_n]$ be the polynomial ring. If I is an ideal in R show that R/I has finite length if and only if it has finite dimension.
(2) Let $\mathcal{S}$ and $\mathcal{T}$ be multiplicatively closed sets in the rings A and R respectively, and let $\theta : A \to R$ be a ring homomorphism such that $\theta(\mathcal{S}) \subset \mathcal{T}$. Show that θ extends to a homomorphism $A[\mathcal{S}^{-1}] \to R[\mathcal{T}^{-1}]$.
(3) Let $\mathcal{S}$ be a multiplicatively closed set in R and $\mathcal{T}$ its image in R/I where $I = \{a \in R \mid as = 0 \text{ for some } s \in \mathcal{S}\}$. Show that the natural map $R[\mathcal{S}^{-1}] \to (R/I)[\mathcal{T}^{-1}]$ is an isomorphism.
(4) Let $\mathcal{S}$ be a multiplicatively closed set in R and suppose that every element of $\mathcal{S}$ is a unit in R. Show that the map $R \to R[\mathcal{S}^{-1}]$ is an isomorphism.
(5) Suppose I and J are ideals in a ring R. Give an example to show that $I \cap J = 0 \not\Rightarrow \sqrt{I} \cap \sqrt{J} = 0$. Show that if $I + J = R$ and $I \cap J = 0$ then $\sqrt{I} \cap \sqrt{J} = 0$. Hence show that if $R = I \oplus J$, then $R = \sqrt{I} \oplus \sqrt{J}$.
(6) Let C and D be degree two curves in $\mathbb{P}^2$. Show that their scheme theoretic image can not be contained in line.
(7) Let $\mathbb{Z}_2$ act on $\mathbb{C}^2 \times \mathbb{C}^2$ by having the non-identity element σ act by $\sigma(p, q) = (q, p)$. Thus $\mathbb{Z}_2$ acts on $R = \mathcal{O}(\mathbb{C}^2 \times \mathbb{C}^2)$ by automorphisms. Determine the ring of invariants $R^{\mathbb{Z}_2} := \{f \in R \mid f^\sigma = f\}$. You might try to do this by placing a grading on R such that each homogenous component R_n is finite dimensional and stable under $\mathbb{Z}_2$. If you make the right choice it should not be hard to compute $R_n^{\mathbb{Z}_2}$. Now write $R = S/I$ where I is an ideal in a polynomial ring S and find a set of generators for I.

    How will you show that I is the whole kernel of the map $S \to R$? One way to do this is to use Hilbert series. If V is a graded vector space such that $\dim V_n < \infty$ for all n and $V_n = 0$ for $n \ll 0$ its Hilbert series is the formal Laurent series

$$H_V(t) := \sum (\dim V_n) t^n.$$

    If $0 \to U \to V \to W \to 0$ is an excat sequence of graded vector spaces in which the arrows preserve degree then

$$H_V(t) = H_U(t) + H_W(t)$$

    so, in particular, if S is a graded k-algebra and $f \in S_n$ is a regular element then $H_{S/fS}(t) = (1 - t^n) H_S(t)$. Also, a surjective degree-preserving map $V \to W$ is an isomorphism if and only if $H_V(t) = H_W(t)$.

    Finally, show that $\operatorname{Spec} R$ is singular (using the Jacobian criterion perhaps). What is the singular locus of $(\mathbb{C}^2 \times \mathbb{C}^2)/\mathbb{Z}_2$?
(8) Show that the localization $A[\mathcal{S}^{-1}] = 0$ if and only if $0 \in \mathcal{S}$.

(9) Let $A = k[x_1, \dots, x_n]$ and $S = k[x_0, \dots, x_n]$. Give S its standard grading, $\deg x_i = 1$ for all i. View S as the homogeneous coordinate ring of $\mathbb{P}^n$ and A as the coordinate ring of the copy of $\mathbb{A}^n$ that consists of the points $(1, a_1, \dots, a_n) \in \mathbb{P}^n$. Give both $\mathbb{A}^n$ and $\mathbb{P}^n$ their Zariski topologies. Show that the Zariski topology on $\mathbb{A}^n$ is the restriction of the Zariski topology on $\mathbb{P}^n$. If J is a graded ideal in S and $V(J)$ its zero locus in $\mathbb{P}^n$, what is "the" ideal I in A for which $V(I) = V(J) \cap \mathbb{A}^n$?

Consider the homogenization map

$$A \to S_{homog}, \qquad f \mapsto f^* := x_0^{\deg f} f(x_1/x_0, \dots, x_n/x_0).$$

If I is an ideal in A, define $I^* := (\{f^* \mid f \in I\})$. Show that

(a) I^* is a graded ideal in S by proving the more general result that an ideal generated by homogeneous elements is graded;

(b) $V(I^*)$ is the closure in $\mathbb{P}^n$ of $V(I)$, the zero locus of I in $\mathbb{A}^n$.

(10) Give either a proof or a counter example to the following statement: if I is a homogeneous ideal in S and $X \subset \mathbb{P}^n$ its zero locus, then the minimal primes over I are homogeneous and their zero loci are the irreducible components of X.

(11) Let X and Y be subvarieties of $\mathbb{P}^n$ cut out by the homogeneous ideals I and J respectively. Suppose that $X \cap Y$ does not meet the hyperplane at infinity, $x_0 = 0$. So we can think of X and Y as subvarieties of $\mathbb{A}^n = \mathbb{P}^n - \{x_0 = 0\}$. Determine their scheme-theoretic intersection in terms of I and J. Does this result suggest a definition of a projective scheme? Does it suggest a way to define the scheme-theoretic intersection of two projective varieties that does not involve using the affine definition?

(12) Suppose that I and J are homogeneous ideals in S. Find an algebraic condition on I and J that is equivalent to the condition that the projective varieties $V(I)$ and $V(J)$ are equal. Hint: Look first at when $V(I) = \phi$ and also do the case $I \subset J$ first. Is there a largest ideal J such that $V(I) = V(J)$? If so, what is J in terms of I?

(13) Let k be an algebraically closed field and R a finitely generated commutative k-algebra. Show there is no k-algebra homomorphism $\theta : k[x, y] \to R$ such that the induced map $\operatorname{Max} R \to \operatorname{Max} k[x, y]$ between the sets of maximal ideals sends $\operatorname{Max} R$ homeomorphically onto $\operatorname{Max}(k[x, y]) - \{(x, y)\}$. This shows that $\mathbb{A}^2 - \{0\}$ is not an affine scheme.

(14) Let $S = k[x_0, \dots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n$. Suppose that $I \subset J$ are homogeneous ideals in S. Find an algebraic condition on I and J that is equivalent to the condition that the natural morphism $\operatorname{Proj}(S/J) \to \operatorname{Proj}(S/I)$ is an isomorphism. In other words, find conditions that are equivalent to the condition that the maps

$$\frac{S}{I}[x_i^{-1}]_0 \to \frac{S}{J}[x_i^{-1}]_0$$

are isomorphisms for $i = 0, \dots, x_n$.

(15) Let I and J be graded ideals in a graded ring S. Assume $S_0 = k$ and S is generated by S_1 as a k-algebra. Let $z \in S_1$ and define $R = S[z^{-1}]_0$. Define $I_* := I[z^{-1}]_0$ and J_* similarly. Either give proofs or counter-examples to the following statements: $I_* + J_* = (I+J)_*$, $I_* J_* = (IJ)_*$, $I_* \cap J_* = (I \cap J)_*$.

(16) Now let z, S, and R be as in the previous question and define

$$R_{\leq n} := S_n z^{-n}.$$

If $f \in R_{\leq n} - R_{\leq n-1}$ let $f^* \in S_n$ be the unique element such that $f = f^* z^{-n}$. Let I and J be ideals in R and define I^* to be the ideal generated by $\{f^* \mid f \in I\}$. Either give proofs of, or counter-examples to, the following statements: $I^* + J^* = (I + J)^*$, $I^* J^* = (IJ)^*$, $I^* \cap J^* = (I \cap J)^*$.

(17) Let k be an algebraically closed field and $k[x, y]$ the polynomial ring in 2 variables. Let f be an irreducible polynomial, C the curve in $\mathbb{A}^2$ it cuts out, $p \in C$, and $\mathfrak{m}_p$ the maximal ideal in $\mathcal{O}(C)$ vanishing at p.

Terminology. We call p a simple point of C if either

$$\frac{\partial f}{\partial x}(p) \neq 0 \quad \text{or} \quad \frac{\partial f}{\partial y}(p) \neq 0.$$

If p is not a simple point it is called a multiple point and its multiplicity is defined as the minimal n such that $f \in \mathfrak{m}_p^n$. We write $m_p(C)$ for the multiplicity of $p \in C$.

Thus p is a singular point of C if and only if $m_p(C) \geq 2$.

(a) Show that p is a simple point if and only if $\dim_k(\mathfrak{m}_p/\mathfrak{m}_p^2) = 1$. Hint: expand f in a Taylor series around p. It might make the notation simpler to assume that p is chosen so that $p = (0, 0)$.

(b) How can the Taylor series expansion of f at p be used to compute $m_p(C)$? Is there a more algebraic way of saying this if we choose coordinates such that $p = (0, 0)$?

(18) (Continuation of the previous exercise.) Let L be a line in $\mathbb{A}^2$ passing though p. Suppose that $p = (\alpha, \beta)$ is a simple point of C. Show that $I(L, C, p) > 1$ if and only if L is the line given by

$$\frac{\partial f}{\partial x}(p)(x - \alpha) + \frac{\partial f}{\partial y}(p)(y - \beta) = 0.$$

In other words, the tangent line to C at p is given by this equation.

(19) (Continuation of the previous exercises.) Show for every line L through p that $I(C, L, p) \geq m_p(C)$.

(20) (Continuation of the previous exercises.) Show for every other curve D through p that $I(C, D, p) \geq m_p(C) m_p(D)$.

(21) Let $C \subset \mathbb{P}^2$ be the zero locus of a positive degree irreducible homogeneous polynomial $F \in k[X, Y, Z]$. Show that p is a simple point of C if and only if at least one of the partial derivatives of F does not vanish at p, and in that case the line

$$\frac{\partial F}{\partial X}(p)X + \frac{\partial F}{\partial Y}(p)Y + \frac{\partial F}{\partial Z}(p)Z = 0$$

is the tangent line to C at p.

(22) ($\operatorname{char} k = 0$) This is a rather open-ended question. Let $f \in k[t]$. There is a simple criterion for whether or not f has a multiple zero: it has no multiple zeroes if and only if $\gcd\{f, f'\} = 1$. Is there a similar simple criterion for whether or not a form $F \in k[x, y]$ has a multiple zero on $\mathbb{P}^1$. You might need to think about (i) the relation between $\gcd\{F, G\}$ and $\gcd\{F_*, G_*\}$; (ii) the relation between derivatives of F with respect to x and y and derivatives of f_* with respect to t where t is a ratio of two linear forms.