Solutions.

6.60 (i) Show that if R/I has IBN for some ideal I then so does R.

Pf: Suppose $R^m \cong R^n$ for some m, n.

We want to show $m = n$.

Denote the isomorphism (as R-modules) by

$$\phi : R^m \longrightarrow R^n .$$

Since ϕ is an R-module isom, we have

$$\phi(I . R^m) = I . R^n .$$

Therefore ϕ induces an (R/I)-module isom

$$\begin{array}{ccc} R^m / I.R^m & \xrightarrow{\bar{\Phi}} & R^n / I . R^n . \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ (R/I)^m & & (R/I)^n \end{array}$$

But R/I has IBN, we have $m = n$.

(ii) We can do this by a direct construction. But I'd like to do it with a slightly more theoretical way.

As a $\mathbb{Z}$-module $F_\infty \cong \bigoplus_{i=1}^{\infty} R_i$ where $R_i \cong \mathbb{Z}$ as $\mathbb{Z}$-module.

Claim: $\mathrm{Hom}_{\mathbb{Z}}\left(\bigoplus_{i=1}^{\infty} R_i , M \right)$

$$= \prod_{i=1}^{\infty} \mathrm{Hom}_{\mathbb{Z}}(R_i . M)$$ for any $\mathbb{Z}$-module M.

direct product.

By taking $M = \bigoplus_{i=1}^{\infty} R_i$

we have
$$\mathrm{End}(F_\infty) \cong \mathrm{Hom}_{\mathbb{Z}}\left(\bigoplus_{j=1}^{\infty} R_j , \bigoplus_{i=1}^{\infty} R_i\right)$$
$$\cong \prod_{j=1}^{\infty} \mathrm{Hom}_{\mathbb{Z}}\left(R_j , \bigoplus_{i=1}^{\infty} R_i\right)$$
$$\cancel{\cong \prod_{j=1}^{\infty} \mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z} , \bigoplus_{i=1}^{\infty} R_i\right)}$$
$$\cancel{\cong \prod_{j=1}^{\infty}}$$

For each j,
$$\mathrm{Hom}_{\mathbb{Z}}\left(R_j , \bigoplus_{i=1}^{\infty} R_i\right) \cong \mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z} , \bigoplus_{i=1}^{\infty} R_i\right) \cong \bigoplus_{i=1}^{\infty} R_i \cong \{\aleph_0\text{-dimensional column vectors}/\mathbb{Z} \text{ with finite many non-zero entries}\}.$$

Hence the result follows.

Explicitly,
$$\mathrm{End}(F_\infty) \cong \prod_{j=1}^{\infty} \mathrm{Hom}_{\mathbb{Z}}\left(R_j , \bigoplus_{i=1}^{\infty} R_i\right)$$
$$\cong \underbrace{\left(\bigoplus_{i=1}^{\infty} R_i\right) \times \left(\bigoplus_{i=1}^{\infty} R_i\right) \times \cdots}_{\aleph_0\text{ - copies.}}$$

1st column   2nd column   . . . .

(iii) Denote the ring of $\aleph_0 \times \aleph_0$ column finite matrices by $M_{\aleph_0}(\mathbb{Z})$.

As a left $M_{\aleph_0}(\mathbb{Z})$ - modules,

$$M_{\aleph_0}(\mathbb{Z}) \cong M_{\aleph_0}^{even}(\mathbb{Z}) \oplus M_{\aleph_0}^{odd}(\mathbb{Z})$$

where $M_{\aleph_0}^{even}(\mathbb{Z}) \subset M_{\aleph_0}(\mathbb{Z})$ is the set of matrices with odd colums $= 0$.

$M_{\aleph_0}^{odd}(\mathbb{Z}) \subset M_{\aleph_0}(\mathbb{Z})$ is the set of matrices with even columns $= 0$

But $M_{\aleph_0}(\mathbb{Z}) \cong M_{\aleph_0}^{even}(\mathbb{Z}) \cong M_{\aleph_0}^{odd}(\mathbb{Z})$

Hence $M_{\aleph_0}(\mathbb{Z})$ doesn't have IBN.

6.62. Consider the exact sequence.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_n \longrightarrow 0.$$

where f is multiplication by n.

Since $\mathrm{Hom}_{\mathbb{Z}}(-, G)$ is left exact. we have.

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, G) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, G) \xrightarrow{f^*} \mathrm{Hom}(\mathbb{Z}, G).$$

(i.e. $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, G) \cong \ker f^*$)

But, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, G) \arrow[r, "f^*"] \arrow[d, phantom, "\cong" rotate=-90] & \mathrm{Hom}(\mathbb{Z}, G) \arrow[d, phantom, "\cong" rotate=-90] \\
G \arrow[r, "g"] & G
\end{tikzcd}$$

where $g : G \longrightarrow G.$

$$x \longmapsto nx.$$

Hence $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, G) \cong \ker f^* \cong \ker g$

$= G[n]$