Solutions.

6.4. First, check $\bar{f}$ is well-defined.
i.e. if $m+K = m'+K$, then $f(m) = f(m')$
But this is true since $K \subset \ker f$.

Then $\bar{f}(a\bar{m} + b\bar{n})$

$= \bar{f}(\overline{am+bn}) = f(am+bn)$

$= af(m) + bf(n)$

$= a\bar{f}(\bar{m}) + b\bar{f}(\bar{n})$

where $a, b \in R$, $\bar{m} = m+K$, $\bar{n} = n+K$, $m, n \in M$.

Hence $\bar{f}$ is an R-map.

6.5 (i) Let $\pi : R \to R/I$ be the canonical projecti
Then:

M is an R/I - module

$\Longleftrightarrow$ $\exists$ ring homo. $\rho : R/I \longrightarrow \mathrm{End}_{\mathbb{Z}}(M)$

$\Longleftrightarrow$ $\exists$ ring hom $\bar{\rho} : R \longrightarrow \mathrm{End}_{\mathbb{Z}}(M)$
s.t $\bar{\rho}(I) = 0$ and thus $\bar{\rho}$ factors through R/I.
(i.e. $\bar{\rho} = \rho \circ \pi$)

$\Longleftrightarrow$ M is an R-module and $IM = 0$.

(ii) M/JM is an R- module since JM is a submodule of M.
Obviously, M/JM, as an R- module, is killed by J. ($J.(M/JM) = JM/JM = 0$)
Now by (i), M/JM is an R/J module with the desired action.
The other statemenst follow readily.

(iii) Lemma; M, N are two moddles with submodules M', N', respectively.
Then $$\frac{M \oplus N}{M' \oplus N'} \cong (M/M') \oplus (N/N')$$

Now ~~denote~~ $\langle b_i \rangle$ by R_i,

Then $F/MF = \dfrac{\oplus R_i}{M(\oplus R_i)} = \dfrac{\oplus R_i}{\oplus (M \cdot R_i)}$

$\underset{\text{Lemma}}{=} \oplus \left(\dfrac{R_i}{M \cdot R_i} \right)$

Each $\dfrac{R_i}{M \cdot R_i} \cong R/M$ is an one-dim'l space over R/M ~~with~~ spanned by $b_i + MF$

Hence $(b_i + MF)$ form a basis for F/MF over R/M.

(You can also show this directly)

Now

$\#$ of summands of F as a free R-module

$= \dim_{R/M} (F/MF)$. $\quad (*)$

Since the RHS of $(*)$ is irrelevent to the choice of the free generators of F, we get the desired statement.

6.20. For any $x \in R$

$(x+x)^2 = x+x \Rightarrow 4x^2 = 2x \Rightarrow 4x = 2x$

$\Rightarrow 2x = 0 \Rightarrow x = -x$

Now $x+y = (x+y)^2 = x + xy + yx + y$

$\Rightarrow xy + yx = 0$

$\Rightarrow xy = -yx = yx$

6.21.

(i) Most proof is just standard verification, I will not list them.

Some people didn't show $\mathcal{F}(G, k)$ is closed under the convolution.

For any $\psi \in \mathcal{F}(G, k)$, let $\mathrm{Supp}\,\psi = \{s \in G;\ \psi(s) \neq 0\}$

For any $\varphi, \psi \in \mathcal{F}(G, k)$

$\varphi\psi(s) \neq 0 \Rightarrow \exists\, x \in G,\ \varphi(x) \neq 0$ and $\psi(x^{-1}s) \neq 0$

$\Rightarrow s = x(x^{-1}s) \in (\mathrm{Supp}\,\varphi) \cdot (\mathrm{Supp}\,\psi)$.

Hence $\mathrm{Supp}(\varphi\psi) \subset (\mathrm{Supp}\,\varphi) \cdot (\mathrm{Supp}\,\psi)$, which is finite.

Also, the identity of this ring is ϕ_e, which is

$$\phi_e(s) := \begin{cases} 1 & \text{if } s = e \\ 0 & \text{otherwise} \end{cases}$$

(ii) Define ϕ_g for any $g \in G$ as

$$\phi_g(h) = \begin{cases} 1 & \text{if } h = g \\ 0 & \text{otherwise.} \end{cases}$$

Claim: $\mathcal{F}(G, k)$ is a free k-module with basis $\{\phi_g : g \in G\}$

(For any $\psi \in \mathcal{F}(G, k)$.

$\psi = \sum_g \psi(g)\, \phi_g$ so $\{\phi_g : g \in G\}$ generate $\mathcal{F}(G, k)$. Also by definition it is easy to check $\{\phi_g : g \in G\}$ is a set of free generators).

By definition Φ sends g to ϕ_g and extends k-linearly to kG, that is to say for any $\sum a_g g \in kG$.

$$\Phi\left(\sum a_g g\right) = \sum a_g \Phi(g) = \sum a_g \phi_g \qquad (*)$$

$(*)$ shows that Φ is an isomorphism <u>as k-modules</u> (It sends a free k-basis to a free k-basis)

We still have to show Φ preserves the identity (which is obvious) and Φ preserves the multiplication.

Because Φ is k-linearly, to check that Φ is multiplicative, we only have to show

$\Phi(gh) = \Phi(g)\,\Phi(h)$ for any $g, h \in G$.

But this is easy to chek.