Non-commutative Algebraic Geometry

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Chapter 1

Categories

We assume the reader is comfortable using category theory as a language, or framework, in which to express basic results about algebraic objects such as groups, rings, modules, et cetera. At that level, the definitions of category theory take center stage and there is no need for deep results on abelian categories. Deeper results begin to play a role once one enters into the more modern parts of group representation theory, finite dimensional algebras, and the theory of $\mathcal{D}$-modules. More profound properties of abelian categories play a prominent role in algebraic geometry. The needs of algebraic goemetry were the stimulus for Grothendieck's Tohoku paper [94]. Thus one also expects that the foundations of non-commutative algebraic geometry will also require some of the deeper results on abelian categories. Much of this foundation was laid in Gabriel's thesis [88]. It is not unreasonable to view Gabriel's thesis is the first paper on non-commutative algebraic geometry.

This chapter and the next present the categorical results that provide the foundation for non-commutative algebraic geometry.

Since category theory provides a framework for a wide range of subjects it is necessarily abstract and technical. We will provide some relief by illustrating the ideas with examples relevant to non-commutative algebraic geometry.

1.1 Definitions and Examples

Definition 1.1 A category C consists of the following data:

- a set $\mathrm{Ob}(\mathsf{C})$ whose members are called the objects of C;
- for every pair of objects X and Y, a set $\mathrm{Hom}_{\mathsf{C}}(X,Y)$ whose elements are called morphisms from X to Y;
- for every triple of objects X, Y, Z, a map

$$\circ : \mathrm{Hom}_{\mathsf{C}}(Y,Z) \times \mathrm{Hom}_{\mathsf{C}}(X,Y) \to \mathrm{Hom}_{\mathsf{C}}(X,Z)$$

  called the composition law.

In addition these data are required to satisfy the following conditions:

- the composition law is associative;
- for every object X, there is an element $\mathrm{id}_X \in \mathrm{Hom}_{\mathsf{C}}(X, X)$ such that $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ h = h$ for all $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ and $h \in \mathrm{Hom}_{\mathsf{C}}(Y, X)$, and all objects Y. We call id_X the identity morphism. $\Diamond$

The zero category has a single object, denoted 0, and the identity morphism id_0 is the only morphism.

Definition 1.2 A subcategory of C is a category D such that $\mathrm{Ob}(\mathsf{D}) \subset \mathrm{Ob}(\mathsf{C})$ and for every pair $X, Y \in \mathrm{Ob}(\mathsf{D})$, $\mathrm{Hom}_{\mathsf{D}}(X, Y) \subset \mathrm{Hom}_{\mathsf{C}}(X, Y)$. It is also required that the composition law in D agree with that in C, and that the identity morphisms be the same. If $\mathrm{Hom}_{\mathsf{D}}(X, Y) = \mathrm{Hom}_{\mathsf{C}}(X, Y)$ for all $X, Y \in \mathrm{Ob}(\mathsf{D})$, we call D a full subcategory of C. $\Diamond$

Notation. It is usual to denote a morphism $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ by an arrow: either $f : X \to Y$ or

$$X \xrightarrow{f} Y.$$

It is permissible to drop the subscript from $\mathrm{Hom}_{\mathsf{C}}$ if the category in question is clear.

For most categories it is a routine matter to check that the axioms are satisfied.

Example 1.3 The category Set has as its objects all sets, and as its morphisms all maps between sets. The identity id_X is the map such that $\mathrm{id}_X(x) = x$ for all $x \in X$. The empty set, denoted ϕ, is an object in Set. We define $\mathrm{Hom}(\phi, X)$ to consist of a single morphism called the empty function. The empty function $\phi \to X$ is the inclusion of the empty set as a subset of X. If X is not empty we define $\mathrm{Hom}(X, \phi)$ to be empty. $\Diamond$

Foundations and Conventions. In order to avoid paradoxes involving *the set of all sets which are not members of themselves* (Example 3.5) we fix a universe, that is, some suitably large set of sets, in which to work—see [155, Chapter I, Section 6] for the definition of a universe. One axiom is that if a set belongs to the universe, then so does its power set. Sets belonging to our fixed universe, are called small sets. With this in mind, we modify the above definition of the category Set: its objects are all *small* sets. It is important to realize that the collection of all objects in Set is *not* itself a small set.

A category is said to be small if the set of all its objects and the set of all its morphisms are both small sets. When we wish to form the category of all categories we take care to form the category of all *small* categories.

When we discuss collections of objects in some category, we will often make the tacit assumption that the index set is small. This is especially relevant when we discuss products, coproducts, limits, colimits, and so on.

Example 1.4 The category of abelian groups, denoted Ab, has as its objects all abelian groups, and as its morphisms all group homomorphisms. Each $\mathrm{Hom}_{\mathsf{Ab}}(G, H)$ can be given an abelian group structure by using the group law in H and defining the addition pointwise. The composition of morphisms $(f, g) \mapsto f \circ g$ is now bilinear. A category with such a structure is called pre-additive. We say more about pre-additive categories in the next chapter. $\Diamond$

Example 1.5 If R is a ring, then we may form a category with one object, say $*$, and morphisms $\mathrm{Hom}(*, *) = R$ with composition being the product in R. The additive structure on R makes this a pre-additive category. Indeed, a pre-additive category with a single object is of this form. Thus a pre-additive category can be seen as a generalization of a ring. $\Diamond$

Example 1.6 If R is a $\mathbb{Z}$-graded ring, we may form a category with objects the set of integers, and morphisms $\mathrm{Hom}(m, n) = R_{m-n}$ with composition the product in R. This is a pre-additive category. One can picture the category as being laid out in the plane, with the homogeneous components of R distributed over the lattice points. The component R_i appears at the points $(n + i, n)$ on the shifted diagonal. $\Diamond$

Example 1.7 Let I be a set. An I-algebra A is a collection of abelian groups A_{ij} indexed by the elements of $I \times I$ together with an associative bilinear multiplication

$$A_{ij} \times A_{kl} \to A_{il}\delta_{jk},$$

and elements $e_i \in A_{ii}$ such that $e_i a = a = a e_j$ for all $a \in A_{ij}$. A shorter description of an I-algebra is that it is a pre-additive category with the objects indexed by $i \in I$. Thus A_{ij} is $\mathrm{Hom}(j, i)$ and e_i is the identity in $\mathrm{Hom}(i, i)$. The previous example shows how to associate to a $\mathbb{Z}$-graded ring an I-algebra with $I = \mathbb{Z}$. Thus I-algebras generalize graded rings. $\Diamond$

Example 1.8 The category of rings with identity, denoted Ring, has as objects all rings with identity, and as morphisms all ring homomorphisms that send the identity to the identity. The zero ring is allowed because the zero element functions also as the identity element. However, if R is not the zero ring, then $1 \neq 0$. For any ring R, $\mathrm{Hom}(R, 0)$ consists of a single element. If R is not the zero ring, then $\mathrm{Hom}(0, R)$ is empty because the map sending 0 to 0 does not send the identity to the identity. $\Diamond$

Example 1.9 Let k be a commutative ring with identity. A k-algebra is a ring R with identity together with a ring homomorphism $\eta : k \to R$ sending 1 to 1 such that the image of k is contained in the center of R. We call η the structure map. A homomorphism of k-algebras $\varphi : R \to S$ is a ring homomorphism that commutes with the structure maps. The category of k-algebras $\mathsf{Alg}(k)$ has objects the k-algebras and morphisms the k-algebra homomorphisms. The zero ring becomes a k-algebra in the obvious way. $\Diamond$

Example 1.10 Let R be a ring with identity. The unital right R-modules together with the R-module homomorphisms forms a category, denoted $\mathsf{Mod}R$. When R is right noetherian the full subcategory $\mathsf{mod}R$ consisting of the noetherian modules plays an important role. The category of left R-modules will be denoted $\mathsf{Mod}R^{\mathrm{op}}$; that is, it will be realized as the category of right modules over the opposite ring. The category of unital modules over the zero ring is the zero category.

We sometimes need to consider modules over rings without identity. A typical example is a ring R which is a union of subrings $R_1 \subset R_2 \subset \dots$ such that each R_i has an identity element e_i. For example, R_n might be the ring of $n \times n$ lower triangular matrices over the field k, or R might be a direct sum of rings. Then one might consider those R-modules M such that $M = \sum_i Me_i$; thus e_i acts as the identity on Me_i. $\Diamond$

Example 1.11 The category of graded vector spaces over a field k, denoted $\mathsf{GrMod}k$, has as objects all k-vector spaces, V say, which are endowed with a decomposition $V = \bigoplus_{n\in\mathbb{Z}} V_n$ as a direct sum of distinguished subspaces. A single vector space endowed with two different decompositions gives *two distinct objects* in $\mathsf{GrMod}k$. The elements of $\mathrm{Hom}(U,V)$ are the k-linear maps $f : U \to V$ such that $f(U_n) \subset V_n$ for all $n \in \mathbb{Z}$. $\Diamond$

Example 1.12 Fix a group G. A G-set is a set X endowed with an action of G such that $(gh).x = g.(h.x)$ for all $g, h \in G$ and all $x \in X$, and such that $1.x = x$ for all $x \in X$. The G-sets form a category Set_G with morphisms the maps $f : X \to Y$ satisfying $f(g.x) = g.f(x)$ for all $g \in G$ and all $x \in X$. The morphisms are called G-equivariant maps. $\Diamond$

Example 1.13 Let X be a topological space. The objects of $\mathsf{Open}X$ are the open subsets of X, including the empty set and X itself. Let U and V be open subsets of X. If $U \not\subset V$ then $\mathrm{Hom}(U,V)$ is empty. If $U \subset V$ then $\mathrm{Hom}(U,V)$ consists of a single morphism, namely the inclusion map $i_U^V : U \to V$. $\Diamond$

Example 1.14 Even if the objects in a category are sets, the morphisms need not be set maps. A correspondence from a set X to a set Y is a subset $C \subset X \times Y$ such that $\mathrm{pr}_1(C) = X$ where $\mathrm{pr}_1 : X \times Y \to X$ is the projection map. We say that $x \in X$ corresponds to those $y \in Y$ such that $(x,y) \in C$. The category Corres of correspondences has the same objects as Set, and the morphisms from X to Y, denoted $\mathsf{Corres}(X,Y)$, are the correspondences from X to Y. The composition of correspondences

$$X \xrightarrow{C} Y \xrightarrow{D} Z$$

is

$$D \circ C := \{(x,z) \mid \text{there exists } y \in Y \text{ such that } (x,y) \in C \text{ and } (y,z) \in D\}.$$

The identity correspondence id_X is the diagonal $\Delta_X = \{(x,x) \mid x \in X\}$. A map $f : X \to Y$ between two sets determines a correspondence $\Gamma_f := \{(x, f(x)) \mid x \in$

$X\}$ from X to Y called the graph of f. One has $\Gamma_{fg} = \Gamma_f \circ \Gamma_g$. In the language of section 1.4 this provides a functor from Set to Corres. We may view Set as a subcategory of Corres. $\Diamond$

Example 1.15 Let G be a group. Define a category with a single object $*$ and $\mathrm{Hom}(*, *) = G$, with composition of morphisms being the product in G. Thus, groups are categories with a single object in which every morphism is an isomorphism. More generally, a category with a single object is the same thing as a monoid. $\Diamond$

Example 1.16 A groupoid is a category in which every morphism has an inverse. For example, the category in the previous example is a groupoid. With that example in mind a groupoid may be thought of as "a group with several objects". If X denotes the set of objects in the groupoid, and A the set of arrows, then there are two maps from A to X, namely $s : A \to X$ which sends a morphism f to its domain, and $t : A \to X$ which sends a morphism f to its codomain. The letter t suggests "target" and s suggests the "start" of the arrow. One may define a groupoid by starting with the data A, X, s, and t, then imposing some axioms. For example, composition of morphism means there is a partially defined map $A \times A \to A$, and so on.

If X is a set and $R \subset X \times X$ an equivalence relation, then there is a corresponding groupoid, the objects of which are the elements of X, and the morphisms are the elements of R with composition law $(x, y) \circ (y, z) = (x, z)$ whenever (x, y) and (y, z) are in R, and inverses $(x, y)^{-1} = (y, x)$ if $(x, y) \in R$. Another important example of a groupoid arises from the action of a group G on a set X. In this case, the objects of the groupoid are the elements of X, and the morphisms are the pairs (g, x) with $g \in G$ and $x \in X$, such a pair being thought of as a morphism $gx \to x$; the inverse is $(g, x)^{-1} = (g^{-1}, gx)$ and the composition law is $(g, x) \circ (h, y) = (hg, x)$ if $gx = y$. $\Diamond$

Example 1.17 A category P in which all Hom-sets have at most one element is called a preorder. We may define a binary relation $\leq$ on the objects of P by saying $p \leq q$ if $\mathrm{Hom}_{\mathsf{P}}(p, q) \neq \phi$; i.e., if there is a morphism $p \to q$. Among the preorders are the partial orders, namely those preorders in which $p \leq q$ and $q \leq p$ implies $p = q$. $\Diamond$

Definition 1.18 The product of two categories C and D is denoted by $\mathsf{C} \times \mathsf{D}$ and is defined as follows. Its objects are ordered pairs (L, M) with L an object in C and M an object in D. A morphism $(L, M) \to (L', M')$ in $\mathsf{C} \times \mathsf{D}$ is by definition a pair (f, g) consisting of a morphism $f : L \to L'$ and a morphism $g : M \to M'$. Composition of morphisms is defined by $(f_1, g_1) \circ (f_2, g_2) = (f_1 f_2, g_1, g_2)$. $\Diamond$

Definition 1.19 Let C be a category. The dual or opposite category, denoted C^{op}, is defined by $\mathrm{Ob}(\mathsf{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathsf{C})$ and

$$\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathsf{C}}(Y, X).$$

The composition

$$\cdot : \mathrm{Hom}_{\mathsf{C}^{op}}(Y, Z) \times \mathrm{Hom}_{\mathsf{C}^{op}}(X, Y) \to \mathrm{Hom}_{\mathsf{C}^{op}}(X, Z)$$

is defined by $f \cdot g := g \circ f$, where $g \circ f$ is the composition in C. The identity morphisms remain the same. ◊

Example 1.20 The category of affine schemes may be defined as the opposite of the category of commutative rings. The objects are usually written as pairs $(\mathrm{Spec}\, R, \mathcal{O})$, where $\mathrm{Spec}\, R = \{\text{prime ideals of } R\}$ is endowed with the Zariski topology, and $\mathcal{O}$ is the sheaf of rings on $\mathrm{Spec}\, R$ whose stalks are the localizations $R_{\mathfrak{p}}$. The empty scheme is the pair $(\phi, 0)$. One is sometimes a little careless in using $\mathrm{Spec}\, R$ to denote both the underlying topological space and the pair.

The category of affine schemes is a subcategory of the category of ringed spaces, the objects of which are pairs $(X, \mathcal{R})$ consisting of a a topological space X and a sheaf of rings (with identity) $\mathcal{R}$ on X.

If k is a commutative ring one defines the category of affine k-schemes as the opposite of the category of commutative k-algebras. Thus an object is a pair consisting of an affine scheme X and a morphism $\alpha : X \to \mathrm{Spec}\, k$, the structure map. We call X a scheme over $\mathrm{Spec}\, k$. ◊

EXERCISES

1.1 Show that the identity morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathsf{C}}(X, X)$ is unique.

1.2 Show that the category $\mathsf{Open}X$ of open subsets of the topological space X, defined in Example 1.13, satisfies the category axioms.

1.3 Check that the composition of correspondences is associative, and hence that Corres is a category.

1.4 View an equivalence relation R on a set X as a correspondence from X to itself. Show that $R \circ R = R$. Does this property characterize the equivalence relations among the correspondences?

1.5 In the language of Definition 2.1, which correspondences are monics, epics, isomorphisms? Which correspondences have a left inverse and/or a right inverse?

1.2 Special types of morphisms and objects

Definition 2.1 A morphism $f : X \to Y$ in C is

- monic, or a monomorphism, if whenever $g_1, g_2 : W \to X$ are morphisms in C such that $fg_1 = fg_2$, then $g_1 = g_2$;
- epic, or an epimorphism, if whenever $g_1, g_2 : Y \to Z$ are morphisms in C such that $g_1 f = g_2 f$, then $g_1 = g_2$;

- an isomorphism if there exists $g \in \mathrm{Hom}_{\mathsf{C}}(Y,X)$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. If such a g exists it is unique, and is denoted by f^{-1}; we call it the inverse of f. Objects X and Y are isomorphic in C if there exists and isomorphism $f : X \to Y$ in C. ◊

Definition 2.2 A morphism $f : X \to Y$ is a split monic if there is a morphism $g : Y \to X$ such that $gf = \mathrm{id}_X$. A morphism $f : X \to Y$ is a split epic if there is a morphism $g : Y \to X$ such that $fg = \mathrm{id}_Y$. ◊

Example 2.3 A map $f : X \to Y$ in Set is a monomorphism (respectively, an epimorphism) if and only if it is injective (respectively, surjective). ◊

Example 2.4 An epimorphism in the category of topological Hausdorff spaces need not be surjective. The inclusion $f : \mathbb{Q} \to \mathbb{R}$ of the rationals in the reals, both being given their usual topology, is an epimorphism. To see this, suppose that $g_1, g_2 : \mathbb{R} \to Z$ are continuous maps such that $g_1 f = g_2 f$.

We put the product topology on products of spaces. The Hausdorff hypothesis ensures that the diagonal

$$\Delta := \{(z,z) \mid z \in Z\} \subset Z \times Z$$

is closed. The map $g := (g_1, g_2) : \mathbb{R} \times \mathbb{R} \to Z \times Z$ is continuous, so $g^{-1}(\Delta)$ is closed. By hypothesis $g^{-1}(\Delta)$ contains $\Delta_{\mathbb{Q}} := \{(q,q) \mid q \in \mathbb{Q}\}$. Hence g^{-1} contains the closure of $\Delta_{\mathbb{Q}}$ which is $\Delta_{\mathbb{R}} := \{(r,r) \mid r \in \mathbb{R}\}$. Thus $g_1(r) = g_2(r)$ for all $r \in \mathbb{R}$, so $g_1 = g_2$. The general principle illustrated by this example is that the inclusion of a dense subspace is an epimorphism. ◊

Example 2.5 An epimorphism in the category of rings need not be surjective. The inclusion $f : \mathbb{Z} \to \mathbb{Q}$ is an epimorphism. If $g_1, g_2 : \mathbb{Q} \to R$ and $g_1(n) = g_2(n)$ for all $n \in \mathbb{Z}$ then, for $n \neq 0$ we have

$$1 = g_1(n.\tfrac{1}{n}) = g_1(n)g_1(\tfrac{1}{n}),$$

from which it follows that $g_1(\frac{1}{n}) = g_2(\frac{1}{n})$ for all $n \neq 0$. From this, it follows that $g_1(m/n) = g_2(m/n)$ for all $m \in \mathbb{Z}$. That is, $g_1 = g_2$. Exercise 5 generalizes this example. The general principle illustrated by this example is that the inclusion of a commutative domain in it field of fractions is an epimorphism. ◊

The notion of isomorphism depends on the category in question: two objects in a subcategory $\mathsf{D} \subset \mathsf{C}$ may be isomorphic as objects of C but non-isomorphic as objects of D. This happens when $\mathrm{Hom}_{\mathsf{D}}(X,Y)$ or $\mathrm{Hom}_{\mathsf{D}}(Y,X)$ fails to contain the morphism implementing the isomorphism.

An isomorphism is both monic and epic (Exercise 6) but, as Examples 2.4 and 2.5 show, a morphism which is monic and epic need not be an isomorphism.

Definition 2.6 Let X be an object in a category C.

A subobject of X is an equivalence class of pairs (A,α) consisting of an object A and a monomorphism $\alpha : A \to X$; two such pairs (A,α) and (A',α') are equivalent if there is an isomorphism $\iota : A' \to A$ such that $\alpha' = \alpha\iota$.

A quotient object of X is an equivalence class of pairs (B,β) consisting of an object B and an epimorphism $\beta : X \to B$; two such pairs (B,β) and (B',α') are equivalent if there is an isomorphism $\iota : B \to B'$ such that $\beta' = \iota\beta$. $\diamondsuit$

Definition 2.7 An object Z in a category C is

- an initial object if $\mathrm{Hom}_{\mathsf{C}}(Z,X)$ is a singleton for all $X \in \mathrm{Ob}(\mathsf{C})$;
- a terminal object if $\mathrm{Hom}_{\mathsf{C}}(X,Z)$ is a singleton for all $X \in \mathrm{Ob}(\mathsf{C})$;
- a zero object if it is both an initial and a terminal object.

A zero object is denoted by 0 and, for every pair of objects X and Y, the composition of morphisms $X \to 0 \to Y$ is called the zero morphism and is denoted by 0, or 0_{XY} if necessary. $\diamondsuit$

Initial, terminal and zero objects are all unique up to unique isomorphism. Hence the definition of the zero morphism 0_{XY} does not depend on the choice of zero object.

The empty set is the unique initial object in Set. Any set consisting of a single element is a terminal object in Set. The ring of integers $\mathbb{Z}$ is an initial object in the category of rings with identity, and the zero ring is a terminal object. Thus the zero ring is not a zero object in the sense of Definition 2.7. More generally, a commutative ring k is an initial object in the category of k-algebras, and the zero ring is a terminal object.

Definition 2.8 Suppose that C has an initial object. An object in C is irreducible if it is not an intial object but its only subobjects are the initial object and itself. $\diamondsuit$

In the category of topological spaces the empty space is an initial object, and the singletons are the terminal objects. Thus the irreducible objects are the singletons. In the category of affine schemes $\mathrm{Spec}\,\mathbb{Z}$ is a terminal object, and the empty space is the only initial object. Hence the irreducible objects in the category of affine schemes are the spectra of fields $(\mathrm{Spec}\,k, k)$ (i.e., the reduced points).

EXERCISES

2.1 Show that the composition of two monomorphisms (respectively, epimorphisms) is a monomorphism (respectively, an epimorphism).

2.2 Let $f \in \mathrm{Hom}_{\mathsf{C}}(X,Y)$. Show that f is monic if and only if f, viewed as a morphism $f^{\mathrm{op}} : Y \to X$ in the dual category, is epic.

2.3 If $g : W \to X$ and $f : X \to Y$ are morphisms show that f is epic if fg is, and that g is monic if fg is.

2.4 Let $f : X \to Y$ be a morphism.

(a) Show that f is monic if and only if the induced map $\mathrm{Hom}(W, X) \to \mathrm{Hom}(W, Y)$, $g \mapsto fg$, is injective for all W.

(b) Show that f is epic if and only if the induced map $\mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$, $h \mapsto hf$, is injective for all Z.

2.5 Let R be a commutative ring and $\mathcal{S}$ a multiplicatively closed subset consisting of regular elements. Show that the inclusion $f : R \to R_{\mathcal{S}}$ of R in the localization is an epimorphism in the category of rings (cf. Example 2.5).

2.6 Show that an isomorphism is both a monomorphism and an epimorphism. Show the converse is false. [*Hint:* why are the maps f in Examples 2.4 and 2.5 not isomorphisms?]

2.7 Show that the inclusion $\mathbb{R} \to \mathbb{C}$ is not an epimorphism in the category of fields.

2.8 In the category $\mathsf{Open}X$ of open subsets of a topological space X, show that X is a terminal object, and that the empty set is an initial object.

2.9 Suppose that C has a zero object. Let $f : W \to X$ and $g : Y \to Z$ be morphisms in C. Show that $0_{XY} \circ f = 0_{WY}$ and $g \circ 0_{XY} = 0_{XZ}$.

2.10 Show that a functor need not send irreducible objects to irreducible objects. [*Hint:* consider the linearization functor $L : \mathsf{Set}_G \to \mathsf{Mod}kG$ in Example 6.8.]

1.3 Products and coproducts

Definition 3.1 Let $\{X_\alpha \mid \alpha \in I\}$ be an indexed set of objects in a category C. A product of the X_α is an object $\prod X_\alpha$ together with distinguished morphisms $p_\alpha : \prod X_\alpha \to X_\alpha$ called projections such that, if $Y \in \mathrm{Ob}(\mathsf{C})$ and $q_\alpha : Y \to X_\alpha$ are morphisms, then there is a unique morphism $f : Y \to \prod X_\alpha$ making the following diagram commute for all $\alpha \in I$:

$$\begin{tikzcd} Y \arrow[r, "f"] & \prod X_\alpha \arrow[d, "p_\alpha"] \\ & X_\alpha \end{tikzcd}$$

◊

Example 3.2 Suppose that $X \prod X$ exists. By its universal property there is a unique map $\Delta : X \to X \prod X$, called the diagonal, such that its composition with each projection $X \prod X \to X$ is the identity id_X. ◊

Definition 3.3 Let $\{X_\alpha \mid \alpha \in I\}$ be an indexed set of objects in a category C. A coproduct (or direct sum) of the X_α is an object $\coprod X_\alpha$ together with distinguished morphisms $i_\alpha : X_\alpha \to \coprod X_\alpha$ called injections such that, if $Y \in \mathrm{Ob}(\mathsf{C})$ and $j_\alpha : X_\alpha \to Y$ are morphisms, then there is a unique morphism $g : \coprod X_\alpha \to Y$ making the following diagram commute for all $\alpha \in I$:

$$\begin{tikzcd} X_\alpha \arrow[r, "i_\alpha"] & \coprod X_\alpha \arrow[d, "g"] \\ & Y \end{tikzcd} \tag{3-1}$$

◊

The product and coproduct of a family of objects M_α in C can be characterized by the existence of isomorphisms

$$\mathrm{Hom}_{\mathsf{C}}(N, \prod M_\alpha) \cong \prod \mathrm{Hom}_{\mathsf{C}}(N, M_\alpha)$$

and

$$\mathrm{Hom}_{\mathsf{C}}(\coprod M_\alpha, N) \cong \prod \mathrm{Hom}_{\mathsf{C}}(M_\alpha, N)$$

for all objects N in C.

The product of a collection of objects in C is the coproduct of those objects in C^{op}, and vice versa.

The definitions do not assert that products and coproducts exist.

Example 3.4 Small products and coproducts exist in Set. The product of a family of sets X_α is their cartesian product and the maps $p_\alpha : \prod X_\alpha \to X_\alpha$ are the obvious projections. The coproduct is the disjoint union and the maps $i_\alpha : X_\alpha \to \coprod X_\alpha$ are the obvious inclusions.

Products and coproducts with the empty set require care. Let I be any set. If Y is non-empty, then $\mathrm{Hom}(Y, \phi)$ is empty so $\mathrm{Hom}(Y, \phi \times I)$ must also be empty, whence $\phi \times I = \phi$. On the other hand, $\phi \coprod I = I$. If $*$ is a singleton set, then $* \times I = I$. Notice that the projection $\phi \times I \to I$ is not epic when I is non-empty. ◊

If a product or coproduct exists it is unique up to unique isomorphism, so we shall speak of *the* product and *the* coproduct. The uniqueness up to isomorphism may be proved directly, or as a consequence of Yoneda's Lemma (Example 5.7), or as a consequence of the fact that a product is a special case of a limit, and hence a terminal object in an appropriate category (Section 1.7).

Foundational issues arise when we try to form the product of all sets. Recall that we are working in a fixed universe, and the objects of Set are those sets belonging to the universe.

Example 3.5 The product of all non-empty sets does not exist in Set. Suppose to the contrary that $P \in \mathrm{Ob}(\mathsf{Set})$ is a product of all sets. Then the power set, Q say, of P also belongs to Set. Let $\pi : P \to Q$ be the associated projection. It is surjective (Exercise 2). The subset $X = \{x \in P \mid x \notin \pi(x)\}$ of P is a member of Q, so there is some $y \in P$ such that $\pi(y) = X$. If $y \in X$, then by definition of X, $y \notin \pi(y) = X$, a contradiction. However, if $y \notin X$, then by definition of X, $y \in \pi(y) = X$, a contradiction. Thus P cannot exist. ◊

The problem is that the index set for this product, Set itself, is too large: the collection of all objects in Set is not a small set.

Convention. From now on we tacitly assume that all index sets we use are small. This convention applies to products and coproducts, and later on will apply to limits, colimits, direct limits, and so on.

Definition 3.6 A category is complete if every small set of objects in it has a product, and is cocomplete if every small set of objects in it has a coproduct. We sometimes indicate this by saying that products, or coproducts, exist in C. $\Diamond$

Example 3.7 The category of groups has products and coproducts. The product of a collection of groups G_i is their Cartesian product endowed with the componentwise product. Their coproduct is the subgroup of the product consisting of all elements having only finitely many components that are not the identity.

The definition of products and coproducts imposes no restrictions on the index set. The index set could be empty. What is the product and coproduct in C of the empty collection of objects?

If I and J are disjoint sets indexing families X_α, $\alpha \in I$, and X_β, $\beta \in J$, then it follows from the universal properties that

$$\prod_I X_\alpha \quad \times \quad \prod_J X_\beta = \prod_{I \cup J} X_\gamma.$$

If I is empty, then

$$\prod_\phi X_\alpha \quad \times \quad \prod_J X_\beta = \prod_J X_\beta. \tag{3-2}$$

Suppose that T is an object in C such that $T \times X$ exists for some X, and the projection $T \times X \to X$ is an isomorphism. Then for all Y,

$$\operatorname{Hom}(Y, X) \cong \operatorname{Hom}(Y, T \times X) = \operatorname{Hom}(Y, T) \times \operatorname{Hom}(Y, X)$$

so $\operatorname{Hom}(Y, T)$ must be a singleton set. Hence T is a terminal object in C. But (3-2) shows that $T = \prod_\phi X_\alpha$ has this property provided the product exists. We conclude that the product over the empty set is a terminal object if one exists, and does not exist otherwise.

Passing to the dual category, one sees that the coproduct over the empty set is an initial object if one exists, and does not exist otherwise.

Example 3.8 Products and coproducts exist in $\mathsf{Mod}R$. The product is the cartesian product made into an R-module by $r.(x_\alpha) = (rx_\alpha)$, and the p_α are the projections. The coproduct is the submodule of $\prod X_\alpha$ consisting of those elements (x_α) for which x_α is non-zero for only finitely many α. That is, the coproduct is the direct sum of the X_α, denoted $\bigoplus X_\alpha$. If the index set is finite, then $\prod X_\alpha \cong \coprod X_\alpha$.

In the full subcategory of finitely generated R-modules products and coproducts do not always exist: for example, the product of infinitely many non-zero vector spaces is not finite dimensional. $\Diamond$

$\boxed{\text{Paul}}$ Give an example in $\mathsf{GrMod}A$ showing that the product is not the Cartesian product.

Example 3.9 Products and finite coproducts exist in the category $\mathsf{Alg}(k)$ of k-algebras and in the category of *commutative* k-algebras. In both cases the product is the cartesian product with component-wise addition and multiplication, and the k-algebra structure arising from the diagonal embedding of k in the product. In the category of commutative k-algebras the coproduct is the tensor product over k, whereas in $\mathsf{Alg}(k)$ the coproduct is the free coproduct, defined as follows. If $A = k\langle X\rangle/I$ and $B = k\langle Y\rangle/J$ are written as quotients of free algebras, then

$$A \coprod_k B := k\langle X \textstyle\coprod Y\rangle/(I, J).$$

(This is independent of the presentation of A and B.) ◊

Lemma 3.10 *A product of monics is monic.*

Proof. Suppose given a family of morphisms $f_\alpha : M_\alpha \to N_\alpha$, and suppose that the products $\prod M_\alpha$ and $\prod N_\alpha$ exist. For each β there is a composition $\prod M_\alpha \to M_\beta \to N_\beta$. Hence by the universal property of $\prod N_\alpha$, there is a map $\prod M_\alpha \to \prod N_\alpha$. This is called the product of the morphisms f_α and is denoted by $\prod f_\alpha$. It is elementary to show that $\prod f_\alpha$ is monic if each f_α is [241, Prop. 3.1, Ch. IV, pg. 85]. □

Corollary 3.11 *Let $f_* : \mathsf{B} \to \mathsf{A}$ be a fully faithful functor having a left adjoint f^*. If A is cocomplete, so is B.*

Proof. Because f_* is fully faithful the counit $\varepsilon : f^* f_* \to \mathrm{id}_\mathsf{B}$ is an isomorphism. Let M_i be a set of objects in B. By hypothesis $\coprod f_* M_i$ exists. If N is in B, then

$$\mathrm{Hom}_\mathsf{B}(f^*(\coprod f_*M_i), N) \cong \mathrm{Hom}_\mathsf{A}(\coprod f_*M_i, N) \cong \prod \mathrm{Hom}_\mathsf{A}(f_*M_i, f_*N) \cong \prod \mathrm{Hom}_\mathsf{B}(f^*f_*M_i, N).$$

However, $f^* f_* M_i \cong M_i$ because f_* is fully faithful, so the displayed isomorphisms show that $f^*(\coprod f_* M_i)$ is a coproduct of the M_i. □

Example 3.12 A product of epics need not be epic. Let Tors denote the full subcategory of Ab consisting of the torsion abelian groups. If M_α is a collection of torsion abelian groups, then their product in Tors is the torsion subgroup of their product in Ab (Exercise 8). To distinguish these two products we denote them by $\prod_\mathsf{Tors} M_\alpha$ and $\prod_\mathsf{Ab} M_\alpha$.

For each $m \geq 1$ there is an obvious epimorphism $f_m : \mathbb{Z}_{2^m} \to \mathbb{Z}_2$. Hence there is a commutative diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & \prod_\mathsf{Tors} \mathbb{Z}_{2^m} & \longrightarrow & \prod_\mathsf{Ab} \mathbb{Z}_{2^m} \\
& & \downarrow & & \downarrow \\
0 & \longrightarrow & \prod_\mathsf{Tors} \mathbb{Z}_2 & \longrightarrow & \prod_\mathsf{Ab} \mathbb{Z}_2
\end{array}$$

Consider $(1,1,\dots) \in \prod_{\mathsf{Tors}} \mathbb{Z}_2$. It is the image of an element $(a_1, a_2, \dots) \in \prod_{\mathsf{Ab}} \mathbb{Z}_{2^m}$ provided $f_m(a_m) = 1$ for all m. However, if $f_m(a_m) = 1$, then the order of a_m is 2^m, so $(a_1, a_2, \dots)$ is not torsion. Thus $(1,1,\dots)$ is not the image of any element in $\prod_{\mathsf{Tors}} \mathbb{Z}_{2^m}$. We conclude that $\prod f_m$ is not an epimorphism in Tors. $\Diamond$

Interpreting Lemma 3.10 and Example 3.12 in the opposite category, we conclude that a coproduct of epics is epic, but a coproduct of monics need not be monic.

EXERCISES

3.1 Prove that $(X \prod Y) \prod Z \cong X \prod (Y \prod Z)$ whenever these products exist.

3.2 Consider a product $\prod X_\alpha$. If the index β is such that $\operatorname{Hom}_{\mathsf{C}}(X_\beta, X_\alpha) \neq \phi$ for each α, show that the projection p_β is an epimorphism. For example, in Set, p_β is surjective if X_β is non-empty.

3.3 If T is a terminal object in C, prove that $X \prod T \cong T \prod X \cong X$ for all $X \in \operatorname{Ob}(\mathsf{C})$.

3.4 Suppose that $J \subset I$ are index sets. If the products exist, show there are morphisms $\theta : \prod_I X_\alpha \to \prod_J X_\alpha$ and $\varphi : \prod_J X_\alpha \to \prod_I X_\alpha$ such that $\theta\varphi = \mathrm{id}$. Hence show that if I is the disjoint union of J and K, then $\prod_I X_\alpha \cong (\prod_J X_\alpha) \prod (\prod_K X_\alpha)$.

3.5 Show that any finite length subobject of $\prod_I X_\alpha$ naturally embeds in $\prod_J X_\alpha$ for some finite subset $J \subset I$.

3.6 Verify the claims in Example 3.9 regarding the existence and description of the product and coproduct in the categories of commutative k-algebras and all k-algebras.

3.7 What is wrong with the following argument? Let C denote the category of commutative k-algebras, and let $A_1, \dots, A_n$ be in C. For each $\alpha = 1, \dots, n$ define $f_\alpha : A_\alpha \to \prod_\alpha A_\alpha$ by $f_\alpha(x) = (0, \dots, 0, x, 0, \dots, 0)$, where the x is in the A_α-position. By the universal property of the coproduct, there is a morphism

$$g : \coprod A_\alpha = \bigotimes_a A_\alpha \to \prod A_\alpha$$

such that $f_\alpha = g i_\alpha$, where $i_\alpha : A_\alpha \to \coprod A_\alpha$ is the map $i_\alpha(x) = 1 \otimes \dots 1 \otimes x \otimes 1 \dots \otimes 1$. Therefore $g(1 \otimes \dots \otimes x \otimes \dots \otimes 1) = (0, \dots, x, \dots, 0)$ for $x \in A_\alpha$. But this is ambiguous if $x = 1$.

3.8 Let Tors denote the category of torsion abelian groups. Verify that the product in Tors is the torsion subgroup of the product in Ab.

3.9 Is the product over all closed points $p \in \mathbb{P}^1$ of the epics $\mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_p$ epic?

3.10 Is every $\mathbb{P}^1$-module a subquotient of a direct product of copies of $\mathcal{O}_{\mathbb{P}^1}$?

1.4 Functors

A recurrent theme in mathematics is to assign to the objects being investigated objects in another category, the assigned object being in some sense an invariant of the original one. The classical example is the fundamental group of a topological space. It is even better if, in addition, one assigns to morphisms in the original category morphisms in the other category. This idea is formalized by the notion of a functor. Functors are the appropriate morphisms categories whose objects are themselves categories.

Definition 4.1 A covariant functor $F : \mathsf{C} \to \mathsf{C}'$ between two categories consists of the following data:

- a map $F : \mathrm{Ob}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{C}')$;
- for all $X, Y \in \mathrm{Ob}(\mathsf{C})$ a map $F_{XY} : \mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{C}'}(FX, FY)$, the image of $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ being denoted by $F(f)$.

This data is subject to the conditions:

- if f and g are morphisms in C, then $F(f \circ g) = F(f) \circ F(g)$ whenever $f \circ g$ is defined;
- $F(\mathrm{id}_X) = \mathrm{id}_{FX}$ for all $X \in \mathrm{Ob}(\mathsf{C})$.

A contravariant functor $F : \mathsf{C} \to \mathsf{D}$ is a covariant functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ from the dual category. That is, if $f : X \to Y$ then $F(f) : FY \to FX$, and the obvious analogues of the conditions for a covariant functor are satisfied. $\diamondsuit$

Example 4.2 Spec is a contravariant functor from the category of commutative rings to the category of topological spaces. If $\varphi : R \to S$ is a homomorphism of commutative rings and $\mathfrak{p}$ is a prime in S, then $\varphi^{-1}(\mathfrak{p}) = \{r \in R \mid \varphi(r) \in \mathfrak{p}\}$ is a prime in R. The functor sends the zero ring to the empty space. $\diamondsuit$

This functor does not extend to non-commutative rings because $\varphi^{-1}(\mathfrak{p})$ need not be a prime ideal; for example, if φ is the inclusion of the diagonal matrices $k^{\times n}$ in the full matrix algebra $M_n(k)$, then $\varphi^{-1}(0)$ is not prime.

Example 4.3 Let X be a scheme over $\mathbb{Z}$. For each commutative ring R, write

$$X(R) = \mathrm{Hom}(\mathrm{Spec}\, R, X)$$

for the set of morphisms of schemes $\mathrm{Spec}\, R \to X$. This gives a covariant functor from commutative rings to sets. We call $X(R)$ the R-valued points of X. By the Yoneda Lemma (see section 1.5), this functor completely determines X as a $\mathbb{Z}$-scheme. Hence we can think of schemes as certain types of functors from the category of commutative rings to Set. It is often an important problem to recognize whether a given functor is of the form $X(-)$, and if so to describe X as completely as possible. For further discussion see [78, ???]. $\diamondsuit$

Definition 4.4 A covariant functor $F : \mathsf{C} \to \mathsf{D}$ is

- faithful if all F_{XY} are injective;
- full if all F_{XY} are surjective;

- fully faithful if it is both full and faithful.

A fully faithful functor is called an embedding. ◊

If C is a subcategory of D, the inclusion $\mathsf{C} \to \mathsf{D}$ is faithful.

Example 4.5 The functor $\mathsf{Set} \to \mathsf{Corres}$ that sends each set to itself and sends a set map $f : X \to Y$ to its graph $\{(x, f(x) \;;|\; x \in X\}$ is faithful, but not full. ◊

Example 4.6 (Hom functors) Fix an object X in a category C. Define the covariant functor $\mathrm{Hom}(X, -) : \mathsf{C} \to \mathsf{Set}$ by $\mathrm{Hom}(X, -)(Y) = \mathrm{Hom}_{\mathsf{C}}(X, Y)$ and for $f \in \mathrm{Hom}_{\mathsf{C}}(Y_1, Y_2)$ define $\mathrm{Hom}(X, f) : \mathrm{Hom}_{\mathsf{C}}(X, Y_1) \to \mathrm{Hom}_{\mathsf{C}}(X, Y_2)$ by $\mathrm{Hom}(X, f)(g) = f \circ g$. There is a similar contravariant functor $\mathrm{Hom}_{\mathsf{C}}(-, X)$. Notice that $\mathrm{Hom}_{\mathsf{C}}(X, -)$ preserves terminal objects, and $\mathrm{Hom}_{\mathsf{C}}(-, X)$ sends initial objects to terminal objects. ◊

Example 4.7 Associated to a ring homomorphism $\varphi : R \to S$ are three functors. The extension of scalars functor

$$f^* = - \otimes_R S : \mathsf{Mod}R \to \mathsf{Mod}S$$

sends a right R-module M to the right S-module $M \otimes_R S$, and sends an R-module homomorphism α to $f^*(\alpha) := 1 \otimes \alpha$. The restriction functor

$$f_* = \mathrm{Hom}_S(S, -) : \mathsf{Mod}S \to \mathsf{Mod}R$$

sends an S-module N to N viewed as an R-module via the action $n.x = n\varphi(x)$. An S-module homomorphism is automatically an R-module homomorphism, so f_* sends an S-module map to the same map viewed as an R-module map. The third functor is

$$f^! = \mathrm{Hom}_R(S, -) : \mathsf{Mod}R \to \mathsf{Mod}S.$$

For more about these functors see Example 6.4. ◊

More generally, if ${}_R B_S$ is a bimodule over the rings R and S, then there are functors $- \otimes_R B : \mathsf{Mod}R \to \mathsf{Mod}S$ and $\mathrm{Hom}_S(B, -) : \mathsf{Mod}S \to \mathsf{Mod}R$. When $\varphi : R \to S$ we may take $B = S$ to obtain f^* and f_*.

Definition 4.8 The category Cat has as its objects the collection of all small categories, and as morphisms the functors between them. Actually it is better to think of Cat as a 2-category. See ???. ◊

Example 4.9 There is a functor $\mathsf{Ring} \to \mathsf{Cat}$ sending a ring R to $\mathsf{Mod}R$, the category of left R-modules, and a ring homomorphism $\varphi : R \to S$ to the functor f^*, defined in Example 4.7. There is also a contravariant functor $\mathsf{Ring} \to \mathsf{Cat}$ which sends R to $\mathsf{Mod}R$, and sends a homomorphism φ to f_*. ◊

Example 4.10 Let R be a ring. Let R be the category with one object $*$, and $\mathrm{Hom}(*, *) = R$. Let $F : \mathsf{R} \to \mathsf{Ab}$ be a covariant functor. Then $F(*)$ is an abelian group; let's call it M. If $r \in R$, then $F(r) : M \to M$ is an abelian group homomorphism; if we write rm for $F(r)(m)$, then $r(sm) = (rs)m$. If we also assume that F is an additive functor (Definition 2.**??**), then $F(r + s) = F(r) + F(s)$, so $(r + s)m = rm + sm$, whence M becomes a left R-module. Conversely, if we are given a left R-module M, then it determines an additive functor $\mathsf{R} \to \mathsf{Ab}$ in an obvious way. Thus left R-modules are the same things as additive functors $\mathsf{R} \to \mathsf{Ab}$; *right* R-modules correspond in a similar way to additive functors $\mathsf{R}^{\mathrm{op}} \to \mathsf{Ab}$. From this point of view R-module homomorphisms are the same things as natural transformations between the functors involved (Exercise **??**). $\Diamond$

The previous example suggests that for any category C it might be useful to consider the functors $\mathsf{C} \to \mathsf{Ab}$. One should think of this as the representation theory of the category C. This point of view is that taken in the representation theory of quivers. There the category has objects the vertices and morphisms the arrows of the quiver and the functors from C to $\mathsf{Mod}k$ are the k-valued representations of the quiver (see section 3.3.5. If G is a group and C is the category with one object $*$ and $\mathrm{Hom}(*, *) = G$, then the functors $\mathsf{C} \to \mathsf{Mod}k$ are the same things as the representations of G defined over k. At a more primitive level the functors from C to Set are the same things as G-sets.

Example 4.11 Let $\mathsf{Open}X$ be the category of open subsets of a topological space X (Example 1.13). Let $\mathcal{F} : \mathsf{Open}X \to \mathsf{Ab}$ be a contravariant functor Write $\rho_U^V = \mathcal{F}(i_U^V) : \mathcal{F}(V) \to \mathcal{F}(U)$ whenever $U \subset V$. Then $\mathcal{F}$ together with the maps ρ_U^V gives $\mathcal{F}$ the structure of a presheaf of abelian groups on X. Conversely a presheaf of abelian groups on X gives a contravariant functor $\mathcal{F} : \mathsf{Open}X \to \mathsf{Ab}$. Therefore such contravariant functors are the same things as presheaves of abelian groups on X. $\Diamond$

Definition 4.12 Let $F, G : \mathsf{A} \to \mathsf{B}$ be covariant functors. A natural transformation $\tau : F \to G$ is a class of morphisms $\tau_M : FM \to GM$, one for each object $M \in \mathsf{A}$, such that, for each $f \in \mathrm{Hom}_{\mathsf{A}}(M, M')$ the diagram

$$\begin{array}{ccc} FM & \xrightarrow{F(f)} & FM' \\ \downarrow{\scriptstyle \tau_M} & & \downarrow{\scriptstyle \tau_{M'}} \\ GM & \xrightarrow{G(f)} & GM' \end{array}$$

commutes. If each τ_M is an isomorphism, τ is said to be a natural equivalence or isomorphism, F and G are said to be naturally equivalent, and we write $F \cong G$. We write $\mathrm{Nat}(F, G)$ for the set of natural transformations from F to G.

Categories C and D are equivalent if there are covariant functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ such that $FG \cong \mathrm{Id}_{\mathsf{D}}$, and $GF \cong \mathrm{Id}_{\mathsf{C}}$.

There are similar definitions for contravariant functors, except that if $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ are contravariant functors such that $FG \cong \mathrm{id}_{\mathsf{D}}$ and $GF \cong \mathrm{id}_{\mathsf{C}}$, we say that F is a duality. $\diamondsuit$

Example 4.13 The standard example of a natural equivalence is duality of finite dimensional vector spaces. Let $\mathsf{mod}k$ be the category of finite dimensional vector spaces over the field k, and define $* : \mathsf{mod}k \to \mathsf{mod}k$ to be the contravariant functor sending a vector space to its dual, and a linear map $\varphi : V \to W$ to its transpose $\varphi^* : W^* \to V^*$ defined by $\varphi^*(f)(u) = f(\varphi(u))$. Composing this functor with itself yields a covariant functor $F : \mathsf{mod}k \to \mathsf{mod}k$ sending V to V^{**}, and sending φ to φ^{**} which is given by $\varphi^{**}(\alpha)(f) = \alpha(\varphi^*(f))$ where $u \in V$, $f \in W^*$ and $\alpha \in V^{**}$. It is an easy exercise to show that the rule $t_V : V \to V^{**}$, defined by $t_V(u)(f) = f(u)$ for $u \in V$ and $f \in V^*$, yields a natural equivalence $t : F \to \mathrm{id}_{\mathsf{mod}k}$.

The functor $V \mapsto V^*$ is a duality from $\mathsf{mod}k$ to itself. $\diamondsuit$

Natural equivalence allows one to recognize two categories of different sizes as being essentially the same. Before elaborating on this, we first define a skeleton of a category C to be a full subcategory D such that each object in C is isomorphic to a unique object in D. For example, the full subcategory of $\mathsf{mod}k$ consisting of all k^n is a skeleton. If D is a skeleton for C, then the inclusion $F : \mathsf{D} \to \mathsf{C}$ is an equivalence of categories. Since we do not usually want to distinguish between isomorphic objects in a category, equivalence is a more useful notion than isomorphism of categories: categories C and D are isomorphic if there exist functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ such that $F \circ G = \mathrm{id}_{\mathsf{D}}$ and $G \circ F = \mathrm{id}_{\mathsf{C}}$.

EXERCISES

4.1 Show that the rule which assigns to a set X the free k-algebra $k\langle X\rangle$ may be made into a functor from the category of sets to the category of k-algebras.

4.2 Show that the rule which assigns to a k-vector space V the tensor algebra $T(V)$, may be made into a functor from the category $\mathsf{Mod}k$ to the category of k-algebras. Do the same with the symmetric algebra $S(V)$ in place of $T(V)$

4.3 Can the rule which assigns to a vector space V the projective space $\mathbb{P}(V)$ be made into a functor from $\mathsf{Mod}k$ to the category of k-varieties?

4.4 (Compare with Exercise **??.??.??**.)

Let A, B, C, D be k-algebras. Show that $\mathrm{Hom}_k(-,-)$ gives a covariant functor

$$\mathrm{Hom}_k(-,-) : \mathsf{mod}(A \otimes B^{\mathrm{op}})^{\mathrm{op}} \times \mathsf{mod}(C \otimes D^{\mathrm{op}}) \to \mathsf{mod}(A^{\mathrm{op}} \otimes B \otimes C \otimes D^{\mathrm{op}}).$$

4.5 Show that a category is equivalent to any one of its skeletons, an equivalence being induced by the inclusion.

4.6 A group is the same thing as a category with a single object in which all morphisms are isomorphisms (see Example 1.15). Show that a functor from such a category to the category of vector spaces is the same thing as a representation of the group.

More generally a representation of a category is a functor to the category of vector spaces.

4.7 Let $\mathrm{id} = \mathrm{id}_{\mathsf{Mod}\,R}$ denote the identity functor on the category of left R-modules. Show that $\mathrm{Nat}(\mathrm{id}, \mathrm{id})$ is a ring isomorphic to the center of R.

4.8 In Example 4.10 show that module homomorphisms are the same things as natural transformations between the functors corresponding to the modules.

4.9 Fill in the details in Example 4.11.

4.10 Let $F : \mathsf{A} \to \mathsf{B}$ be a functor. Suppose that A_α are objects in A. Show there are natural morphisms $F(\prod A_\alpha) \to \prod F(A_\alpha)$, and $\coprod F(A_\alpha) \to F(\coprod A_\alpha)$. Give examples to show these morphisms need not be isomorphisms even when F is fully faithful.

1.5 Representable functors and Yoneda's lemma

Definition 5.1 Let C be a category. A functor $F : \mathsf{C} \to \mathsf{Set}$ is representable if there is an M in C such that F is naturally equivalent to $\mathrm{Hom}_{\mathsf{C}}(M, -)$. We say that F is represented by M, or that M is a representing object. $\Diamond$

Similarly, a contravariant functor from C to Set is representable if it is naturally equivalent to $\mathrm{Hom}_{\mathsf{C}}(-, N)$ for some N in C.

Example 5.2 If $(X, \mathcal{O}_X)$ is a scheme, the global sections functor $\Gamma(X, -)$, defined on the category of quasi-coherent $\mathcal{O}_X$-modules is representable. It is naturally equivalent to $\mathrm{Hom}(\mathcal{O}_X, -)$ where $\mathcal{O}_X$ is the structure sheaf on X. $\Diamond$

Example 5.3 The duality $\mathsf{mod}k \to \mathsf{mod}k$ which sends a finite dimensional vector space to its dual and a morphism to its transpose is representable. The one-dimensional vector space is a representing object. For this reason k is called a dualizing object for $\mathsf{mod}k$. $\Diamond$

The functor category. Let C and D be categories. We define

$$\mathsf{Fun}(\mathsf{C}, \mathsf{D}),$$

the category of covariant functors from C to D to have objects the covariant functors $\mathsf{C} \to \mathsf{D}$, and morphisms the natural transformations $\mathrm{Nat}(F, G)$.

In order for $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ to be a category the Hom-sets $\mathrm{Nat}(F, G)$ are required to be small; since $\mathrm{Nat}(F, G)$ naturally embeds in $\prod_{C \in \mathrm{Ob}\,\mathsf{C}} \mathrm{Hom}_{\mathsf{C}}(FC, GC)$, the smallness of C ensures that $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ is a category.

Two functors F and G are isomorphic in $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ if and only if they are naturally equivalent.

Lemma 5.4 *Let $\tau : F \to G$ be a morphism in $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$. Then τ is monic if and only if $\tau_M : FM \to GM$ is monic for all M in C. And τ is epic if and only if $\tau_M : FM \to GM$ is epic for all M in C.*

The contravariant functor $\mathsf{C} \to \mathsf{Fun}(\mathsf{C}, \mathsf{Set})$. We extend the rule $M \mapsto \mathrm{Hom}_{\mathsf{C}}(M, -)$ to a functor. If $f \in \mathrm{Hom}_{\mathsf{C}}(M, N)$ the natural transformation

$$\hat{f} : \mathrm{Hom}_{\mathsf{C}}(N, -) \to \mathrm{Hom}_{\mathsf{C}}(M, -)$$

is defined as follows: for each $A \in \mathrm{Ob}(\mathsf{C})$ let

$$\hat{f}_A : \mathrm{Hom}_{\mathsf{C}}(N, A) \to \mathrm{Hom}_{\mathsf{C}}(M, A)$$

be the map $\hat{f}_A(h) := hf$. This is a natural transformation, and hence the rule

$$M \mapsto \mathrm{Hom}_{\mathsf{C}}(M, -) \qquad \text{and} \qquad f \mapsto \hat{f} \tag{5-1}$$

defines a contravariant functor $\mathsf{C} \to \mathsf{Fun}(\mathsf{C}, \mathsf{Set})$.

Theorem 5.5 (Yoneda's Lemma) *The functor* $\mathsf{C}^{\mathrm{op}} \to \mathsf{Fun}(\mathsf{C}, \mathsf{Set})$ *defined by* $M \mapsto \mathrm{Hom}_{\mathsf{C}}(M, -)$ *is fully faithful.*

Proof. Let $F : \mathsf{C} \to \mathsf{Set}$ be a functor, and let $M \in \mathrm{Ob}(\mathsf{C})$. For each $\xi \in FM$ and each $A \in \mathrm{Ob}(\mathsf{C})$, define $\tilde{\xi}_A : \mathrm{Hom}_{\mathsf{C}}(M, A) \to FA$ by $\tilde{\xi}_A(g) := (Fg)(\xi)$. It suffices to prove

- $\tilde{\xi} : \mathrm{Hom}_{\mathsf{C}}(M, -) \to F$ is a natural transformation, and
- the rule $\xi \mapsto \tilde{\xi}$ is a bijection between FM and $\mathrm{Nat}(\mathrm{Hom}_{\mathsf{C}}(M, -), F)$;
- in particular, for each $N \in \mathrm{Ob}(\mathsf{C})$, the rule $\xi \mapsto \tilde{\xi}$ is a bijection

$$\mathrm{Hom}_{\mathsf{C}}(N, M) \to \mathrm{Nat}(\mathrm{Hom}(M, -), \mathrm{Hom}(N, -)). \tag{5-2}$$

The details can be found in several books. $\square$

Corollary 5.6 *The functors* $\mathrm{Hom}_{\mathsf{C}}(M, -)$ *and* $\mathrm{Hom}_{\mathsf{C}}(N, -)$ *are naturally equivalent if and only if* $M \cong N$*. In particular, an object representing a functor is unique up to isomorphism.*

Example 5.7 Products and coproducts can be defined by using Yoneda's Lemma and the fact that products exist in Set. To see this, let N_α be an indexed set of objects in a category C. If the contravariant functor $\mathsf{C} \to \mathsf{Set}$ defined by

$$M \mapsto \prod \mathrm{Hom}_{\mathsf{C}}(M, N_\alpha)$$

is representable, we define $\prod N_\alpha$ to be a representing object; that is,

$$\mathrm{Hom}(M, \prod N_\alpha) \cong \prod \mathrm{Hom}(M, N_\alpha).$$

Similarly, if the covariant functor $N \mapsto \prod \mathrm{Hom}_{\mathsf{C}}(M_\alpha, N)$ is representable we define $\coprod M_\alpha$ to be a representing object; that is,

$$\mathrm{Hom}(\coprod M_\alpha, N) \cong \prod \mathrm{Hom}(M_\alpha, N).$$

Hence the uniqueness up to isomorphism of a product or coproduct follows from the uniqueness up to isomorphism of a representing object (Corollary 5.6). $\Diamond$

Proposition 5.8 *Let $f : L \to M$ be a morphism in* C. *Then*

1. *f is epic if and only if* $\mathrm{Hom}(f, N)$ *is injective for all N in* C*;*

2. *f is split monic if and only if* $\mathrm{Hom}(f, N)$ *is surjective for all N in* C.

Proof. The map $\mathrm{Hom}(f, N) : \mathrm{Hom}(M, N) \to \mathrm{Hom}(L, N)$ sends α to αf.

(1) By definition, f is epic if and only if αf and $\alpha' f$ are distinct whenever α and α' are distinct elements of $\mathrm{Hom}(M, N)$. This is precisely the condition that $\mathrm{Hom}(f, N)$ is injective.

(2) Suppose that f is split monic. Then there is a morphism $g : M \to L$ such that $gf = \mathrm{id}_L$. If $\beta \in \mathrm{Hom}(L, N)$, then $\beta = \beta g f$, which is the image of βg under $\mathrm{Hom}(f, N)$. Hence $\mathrm{Hom}(f, N)$ is surjective. Conversely, suppose that $\mathrm{Hom}(f, N)$ is surjective for all N. In particular, $\mathrm{Hom}(f, L)$ is surjective, so id_L is in its image. Thus $gf = \mathrm{id}_L$ for some $g \in \mathrm{Hom}(M, L)$, so f is split monic. $\square$

EXERCISES

5.1 Fill in the details to prove Yoneda's Lemma. Show that $\hat{f}$ really is a natural transformation, then show that $\widehat{f_1 f_2} = \hat{f}_2 \hat{f}_1$, to prove that (5-1) does define a contravariant functor.

5.2 Let F be the functor from commutative k-algebras to groups such that $F(R) = SL_n(R)$, the group of $n \times n$ matrices, with entries in R, which have determinant 1. Show that F is a representable functor, represented by the k-algebra

$$A = k[x_{11}, \ldots, x_{nn}]/(\det - 1)$$

where det denotes the determinant of the $n \times n$ generic matrix $X = (x_{ij})$.

5.3 Let $\mathcal{F} \subset k\langle X \rangle$ be a set of non-commutative polynomials, and let F be the functor from the category of k-algebras to the category of sets given by

$$F(S) = \{\text{solutions in } S \text{ to the system of equations } \mathcal{F} = 0\}.$$

Show that F is represented by $k\langle X \rangle / (\mathcal{F})$.

1.6 Adjoint pairs of functors

Definition 6.1 Let $f : \mathsf{C} \to \mathsf{D}$ and $g : \mathsf{D} \to \mathsf{C}$ be covariant functors. We say that f is a left adjoint of g and that g is a right adjoint of f if the covariant functors $\mathrm{Hom}_{\mathsf{C}}(-, g-)$ and $\mathrm{Hom}_{\mathsf{D}}(f-, -)$, taking $\mathsf{C}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{Set}$, are naturally equivalent. For brevity we call (f, g) an adjoint pair; we will always use the convention that the left adjoint is written first. $\Diamond$

Proposition 6.2 *Let $f : \mathsf{C} \to \mathsf{D}$ and $g : \mathsf{D} \to \mathsf{C}$ be functors. Then (f, g) is an adjoint pair if and only if for all M in C and N in D there are bijections*

$$\nu_{MN} : \mathrm{Hom}_{\mathsf{C}}(M, gN) \to \mathrm{Hom}_{\mathsf{D}}(fM, N) \tag{6-1}$$

such that if $\alpha \in \mathrm{Hom}_{\mathsf{C}}(M, M')$ *and* $\beta \in \mathrm{Hom}_{\mathsf{D}}(N, N')$, *the diagram*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{C}}(M', gN) & \xrightarrow{\nu_{M'N}} & \mathrm{Hom}_{\mathsf{D}}(fM', N) \\ {\scriptstyle (-)\circ\alpha}\downarrow & & \downarrow{\scriptstyle (-)\circ(f\alpha)} \\ \mathrm{Hom}_{\mathsf{C}}(M, gN) & \xrightarrow{\nu_{MN}} & \mathrm{Hom}_{\mathsf{D}}(fM, N) \\ {\scriptstyle (g\beta)\circ(-)}\downarrow & & \downarrow{\scriptstyle \beta\circ(-)} \\ \mathrm{Hom}_{\mathsf{C}}(M, gN') & \xrightarrow{\nu_{MN'}} & \mathrm{Hom}_{\mathsf{D}}(fM, N') \end{array} \tag{6-2}$$

commutes.

The commutativity of (6-2) is equivalent to the condition that

$$\nu(\lambda \circ \alpha) = \nu(\lambda) \circ f\alpha \tag{6-3}$$

and

$$\nu(g\beta \circ \lambda) = \beta \circ \nu(\lambda) \tag{6-4}$$

for all $\lambda : M' \to gN$. There are similar identities involving ν^{-1}.

The maps ν_{MN} give a morphism of bifunctors

$$\nu : \mathrm{Hom}_{\mathsf{C}}(-, g-) \to \mathrm{Hom}_{\mathsf{D}}(f-, -).$$

The commutativity of (6-2) says that this morphism is a natural transformation in each variable.

The paradigmatic algebraic example of an adjoint pair is provided by the tensor and Hom functors.

Example 6.3 If ${}_RB_S$ is an R-S-bimodule, then $- \otimes_R B$ is a left adjoint to $\mathrm{Hom}_S(B, -)$. In particular, if M is a right R-module and N is a right B-module, then the map that sends λ to the map $m \otimes b \mapsto (\lambda(m))(b)$ is an isomorphism

$$\mathrm{Hom}_R(M, \mathrm{Hom}_S(B, N)) \longrightarrow \mathrm{Hom}_S(M \otimes_R B, N).$$

One checks that the diagrams in Definition 6.1 commute by using the explicit form of the map. ◊

Example 6.4 A ring homomorphism $f : R \to S$ induces an adjoint triple of functors $(f^*, f_*, f^!)$ as defined in Example 4.7. We mean that (f^*, f_*) and $f_*, f^!)$ are both adjoint pairs. The fact that (f^*, f_*) is an adjoint pair is a special case of Example 6.3 with the bimodule being ${}_RS_S$. The fact that (f_*, f^*) is an adjoint pair is also a special case of Example 6.3 with the bimodule being ${}_SS_R$. We call f^* and f_* the inverse image and direct image functors associated to f.

If $g : S \to T$ is another ring homomorphism, then $g_* \circ f_* = (g \circ f)_*$ and $f^* \circ g^* \cong (g \circ f)^*$. ◊

We isolate an important special case of this. Recall that an element y in a ring R is normal if $yR = Ry$. Conjugation by y induces an auto-equivalence of $\mathsf{Mod}R$.

Lemma 6.5 *Let y be a normal regular element in a ring R. Define $\sigma \in \operatorname{Aut} R$ by*

$$yr = r^\sigma y$$

for all $r \in R$. Let σ^* and σ_* be the inverse and direct image functors associated to σ. If M is an R-module, then $\sigma_* M$ is equal to M as an abelian group but with the R-action defined by

$$m.r = mr^\sigma,$$

where the right-hand action is the original one. Multiplication by y is an R-module homomorphism $M \to \sigma^* M$.

Proof. The description of $\sigma_* M$ is simply a restatement of the definition of the direct image functor (see Example 4.7). Since σ is an automorphism of R, it is easy to see that $\sigma^* \cong (\sigma_*)^{-1} \cong (\sigma^{-1})_*$. A simple calculation shows that right multiplication by y is a homomorphism $M \to (\sigma^{-1})_*$, so the result follows. □

A forgetful functor is one which simply forgets some of the structure on the objects in a category. The next few examples illustrate the adage that left adjoints to forgetful functors solve universal problems. (See Theorem 2.18.2 for the case of sheaves and presheaves.)

Example 6.6 The left adjoint to the forgetful functor $\mathsf{Mod}R \to \mathsf{Set}$ sending a module to its underlying set of elements is the functor F sending a set to the free R-module with that set as basis. That is, for an R-module M and set X, $\operatorname{Hom}_{\mathsf{Set}}(X, M) \cong \operatorname{Hom}_R(FX, M)$. ◊

Example 6.7 If R is a k-algebra, we may send R to the k-Lie algebra which is R itself as a k-vector space endowed with the Lie bracket $[a, b] = ab - ba$. The left adjoint to this functor sends a Lie algebra to its universal enveloping algebra. ◊

Example 6.8 Let G be a group, kG its group algebra, and Set_G the category of G-sets (Example 1.12). The forgetful functor $F : \mathsf{Mod}kG \to \mathsf{Set}_G$ which forgets the linear structure has a left adjoint, namely the linearization functor L sending a G-set X to the vector space with basis the elements of X endowed with the kG-action linearly extending the G-action. That is $\operatorname{Hom}_G(X, FV) \cong \operatorname{Hom}_{kG}(LX, V)$. ◊

Example 6.9 The forgetful functor sending an abelian group to its underlying semigroup has a left adjoint. Applying the left adjoint to the semigroup of finitely generated projective modules over a ring R (with direct sum as the operation) produces the Grothendieck group $K_0(R)$. The adjoint functor can either be constructed explicitly, or obtained as a consequence of the Adjoint Functor Theorem. ◊

Example 6.10 Let $\mathcal{S}h(X)$ and $\mathsf{Pre}\mathcal{S}h(X)$ be the categories of sheaves and presheaves of abelian groups on a topological space X. The left adjoint of the forgetful functor $\mathcal{S}h(X) \to \mathsf{Pre}\mathcal{S}h(X)$ assigns to a presheaf its sheafification. $\Diamond$

Example 6.11 Let D denote the category of commutative domains with morphisms being the *injective* ring homomorphisms. There is a forgetful fully faithful embedding $F : \mathsf{F} \to \mathsf{D}$ of the category of fields into D. This has a left adjoint $Q : \mathsf{D} \to \mathsf{F}$ which sends a domain D to its field of fractions. $\Diamond$

$\boxed{\text{Paul}}$ – what happens to this example if we take non-commutative domains — why is there is no left adjoint???

Proposition 6.12 *Let (f, g) be an adjoint pair of functors with $f : \mathsf{C} \to \mathsf{D}$ and $\nu : \mathrm{Hom}_{\mathsf{C}}(-, g-) \to \mathrm{Hom}_{\mathsf{D}}(f-, -)$ the associated isomorphism of bifunctors. There are natural transformations*

$$\varepsilon : fg \to \mathrm{id}_{\mathsf{D}} \qquad \eta : \mathrm{id}_{\mathsf{C}} \to gf$$

defined as follows. If M is in C and N is in D, then

$$\eta_M = \nu^{-1}(\mathrm{id}_{fM}) : M \to gfM \tag{6-5}$$

and

$$\varepsilon_N = \nu(\mathrm{id}_{gN}) : fgN \to N. \tag{6-6}$$

If $\alpha \in \mathrm{Hom}_{\mathsf{C}}(M, gN)$ and $\beta \in \mathrm{Hom}_{\mathsf{D}}(fM, N)$ then

$$\nu(\alpha) = \varepsilon_N \circ f(\alpha) \tag{6-7}$$

and

$$\nu^{-1}(\beta) = g(\beta) \circ \eta_M. \tag{6-8}$$

Proof. The details are left to the reader. $\Box$

Two special cases of (6-9) and (6-10) yield

$$\varepsilon_{fM} \circ f(\eta_M) = \mathrm{id}_{fM} \tag{6-9}$$

and

$$g(\varepsilon_N) \circ \eta_{gN} = \mathrm{id}_{gN}\,. \tag{6-10}$$

Definition 6.13 Let (f, g) be an adjoint pair of functors with $f : \mathsf{C} \to \mathsf{D}$. The counit associated to (f, g) is the natural transformation $\varepsilon : fg \to \mathrm{id}_{\mathsf{D}}$ and the unit is the natural transformation $\eta : \mathrm{id}_{\mathsf{C}} \to gf$. $\Diamond$

Define natural transformations $\mathrm{id}_f * \eta : f \to fgf$ by $(\mathrm{id}_f * \eta)_M := f(\eta_M)$ and $\varepsilon * \mathrm{id}_f : fgf \to f$ by $(\varepsilon * \mathrm{id}_f)_M := \varepsilon_{fM}$ Composing these gives a natural transformation

$$f \xrightarrow{\mathrm{id}_f * \eta} fgf \xrightarrow{\varepsilon * \mathrm{id}_f} f.$$

The next result says that this and a similar natural transformation

$$g \xrightarrow{\eta * \mathrm{id}_g} gfg \xrightarrow{\mathrm{id}_g * \varepsilon} g$$

are identities.

Corollary 6.14 *Let (f, g) be an adjoint pair of functors. With the previous notation*

$$(\varepsilon * \mathrm{id}_f) \circ (\mathrm{id}_f * \eta) = \mathrm{id}_f$$

and

$$(\mathrm{id}_g * \varepsilon) \circ (\eta * \mathrm{id}_g) = \mathrm{id}_g .$$

Proof. These follow immediately from (6-9) and (6-10). □

Theorem 6.15 *Let (f, g) be an adjoint pair of functors with associated counit $\varepsilon : fg \to \mathrm{id}$. Then*

1. *g is full if and only if every ε_M is split monic;*
2. *g is faithful if and only if every ε_M is epic;*
3. *g is fully faithful if and only if every ε_M is an isomorphism.*

Proof. Suppose that $f : \mathsf{C} \to \mathsf{D}$. Let M and N be in D. The composition

$$\mathrm{Hom}_{\mathsf{D}}(M, N) \xrightarrow{\ g\ } \mathrm{Hom}_{\mathsf{C}}(gM, gN) \xrightarrow{\ \nu\ } \mathrm{Hom}_{\mathsf{D}}(fgM, N) \tag{6-11}$$

is equal to $\mathrm{Hom}(\varepsilon_M, N)$ because it sends $\beta \in \mathrm{Hom}_{\mathsf{D}}(M, N)$ to

$$\nu(g\beta) = \beta \circ \nu(\mathrm{id}_{gM}) = \beta \circ \varepsilon_M.$$

Since ν is bijective, $\mathrm{Hom}(\varepsilon_M, N)$ is injective or surjective for all M and N exactly when g is faithful or full. However, for a fixed M, $\mathrm{Hom}(\varepsilon_M, N)$ is injective or surjective for all N exactly when ε_M is epic or split monic (Proposition 5.8). □

A short proof of part (2) is obtained as follows. If we set $\lambda = \mathrm{id}_{gM}$ in (6-4), then $g(\beta_1) = g(\beta_2)$ if and only if $\beta_1 \circ \nu(\mathrm{id}_{gM}) = \beta_2 \circ \nu(\mathrm{id}_{gM})$, so g is injective on morphisms if and only if $\nu(\mathrm{id}_{gM})$ is epic. But $\nu(\mathrm{id}_{gM}) = \varepsilon_M$.

Theorem 6.16 *Let (f, g) be an adjoint pair of functors with associated unit $\eta : \mathrm{id} \to gf$. Then*

1. *f is full if and only if every η_N is split epic;*
2. *f is faithful if and only if every η_N is monic;*

3. *f is fully faithful if and only if every η_N is an isomorphism.*

Proof. Suppose that $f : \mathsf{C} \to \mathsf{D}$. Let M and N be in C. The composition

$$\mathrm{Hom}_{\mathsf{C}}(M,N) \xrightarrow{\mathrm{Hom}(M,\eta_N)} \mathrm{Hom}_{\mathsf{C}}(M,gfN) \xrightarrow{\ \nu\ } \mathrm{Hom}_{\mathsf{D}}(fM,fN) \tag{6-12}$$

is equal to f because it sends $\alpha \in \mathrm{Hom}_{\mathsf{C}}(M,N)$ to

$$\nu(\eta_N \circ \alpha) = \nu(\eta_N) \circ f(\alpha) = f(\alpha).$$

Since ν is bijective, $\mathrm{Hom}_{\mathsf{C}}(M,\eta_N)$ is injective or surjective for all M and N exactly when f is faithful or full. However, for a fixed M, $\mathrm{Hom}_{\mathsf{C}}(M,\eta_N)$ is injective or surjective for all N exactly when η_N is monic or split epic (cf. Proposition 5.8). □

A short proof of part (2) is obtained as follows. If we set $\lambda = \nu^{-1}(\mathrm{id}_{fN})$ in (6-3), then $f(\alpha_1) = f(\alpha_2)$ if and only if $\nu^{-1}(\mathrm{id}_{fN}) \circ \alpha_1 = \nu^{-1}(\mathrm{id}_{fN}) \circ \alpha_2$, so f is injective on morphisms if and only if $\nu^{-1}(\mathrm{id}_{fN})$ is monic. But $\nu^{-1}(\mathrm{id}_{fN}) = \eta_N$.

Corollary 6.17 *Let $f_* : \mathsf{B} \to \mathsf{A}$ be a functor having a left adjoint f^*, and a right adjoint $f^!$. The following are equivalent:*

1. *f_* is faithful;*
2. *$M \to f^! f_* M$ is monic for all M in B;*
3. *$f^* f_* M \to M$ is epic for all M in B.*

Theorem 6.18 *Let A be an abelian category. Let $i_* : \mathsf{B} \to \mathsf{A}$ be the inclusion of a full subcategory.*

1. *Suppose that i_* has a left adjoint i^*. Then*
   - *(a) $i^* i_* \to \mathrm{id}_{\mathsf{B}}$ is a natural equivalence, and*
   - *(b) if B is closed under quotients, then $M \to i_* i^* M$ is epic for all M in A.*
2. *Suppose that i_* has a right adjoint $i^!$. Then*
   - *(a) $\mathrm{id}_{\mathsf{B}} \to i^! i_*$ is a natural equivalence, and*
   - *(b) if B is closed under subobjects, then $i_* i^! M \to M$ is monic for all M in A.*

Proof. By hypothesis, i_* is full and faithful. Therefore (1a) follows from part (3) of Theorem 6.15, and (2a) follows from part (3) of Theorem 6.16.

(1b) Let M be an A-module, and write C for the cokernel in the exact sequence

$$M \to i_* i^* M \to C \to 0.$$

Since i_*i^*M belongs to B, the hypothesis that B is closed under quotients ensures that C is also in B. Since i^* is right exact,

$$i^*M \to i^*i_*i^*M \to i^*C \to 0$$

is exact. It follows from (1a) that the natural transformation $i^*i_*i^* \to i^*$ is an isomorphism. But the composition $i^* \to i^*i_*i^* \to i^*$ is the identity by Corollary 6.14, so $i^* \to i^*i_*i^*$ is an isomorphism. Therefore $i^*C = 0$. Hence

$$0 = \operatorname{Hom}_{\mathsf{B}}(i^*C, C) \cong \operatorname{Hom}_{\mathsf{A}}(C, i_*C) = \operatorname{Hom}_{\mathsf{A}}(C, C).$$

It follows that $C = 0$.

(2b) The proof is similar, starting from an exact sequence $0 \to K \to i_*i^!M \to M$. □

Example 6.19 Let Tors denote the full subcategory of Ab consisting of torsion abelian groups. Let $i_* : \mathsf{Tors} \to \mathsf{Ab}$ be the inclusion functor, and $i^!$ the functor sending a group to its torsion subgroup. Then $(i_*, i^!)$ is an adjoint pair. Although i_* is exact, it does not have a left adjoint because if it did, then it would commute with products; but it does not because the product of all $\mathbb{Z}/n\mathbb{Z}$ in $\mathsf{Tors}\mathbb{Z}$ must be torsion, whereas, $\mathbb{Z}$ embeds in $\prod_n \mathbb{Z}/n\mathbb{Z}$ in $\mathsf{Mod}\mathbb{Z}$. Also see Exercise 1.8. ◊

Proposition 6.20 *A functor $f : \mathsf{C} \to \mathsf{D}$ has a right adjoint if and only if the functor $X \mapsto \operatorname{Hom}_{\mathsf{D}}(fX, Y)$ is representable for each Y in D.*

Proof. If g is a right adjoint of f then the functor is represented by gY.

Conversely, suppose that the functor is representable. For each Y in D let gY be a representing object (gY is only determined up to isomorphism, so we just make some choice) and let $\varphi_Y : \operatorname{Hom}_{\mathsf{C}}(-, gY) \to \operatorname{Hom}_{\mathsf{D}}(f-, Y)$ be a natural equivalence. If $f : Y \to Y'$ is a morphism in D, we define $gf : gY \to gY'$ to be the unique morphism making the following diagram commute:

$$\begin{tikzcd}
\operatorname{Hom}_{\mathsf{C}}(-, gY) \arrow[r, "\varphi_Y"] \arrow[d, "\operatorname{Hom}_{\mathsf{C}}(-,gf)"'] & \operatorname{Hom}_{\mathsf{D}}(f-, Y) \arrow[d, "\operatorname{Hom}_{\mathsf{D}}(f-,f)"] \\
\operatorname{Hom}_{\mathsf{C}}(-, gY') \arrow[r, "\varphi_{Y'}"] & \operatorname{Hom}_{\mathsf{D}}(f-, Y').
\end{tikzcd}$$

It is now straightforward to check that g is a right adjoint to f. □

Theorem 6.21 *Let $f : \mathsf{C} \to \mathsf{D}$ be a functor. The following are equivalent:*

1. *f is an equivalence of categories;*
2. *f is fully faithful and every object of D is isomorphic to an object of the form fX;*
3. *f is fully faithful, and has a fully faithful left adjoint;*

4. f is fully faithful, and has a fully faithful right adjoint.

Proof. [155, Theorem 1, page 99] or [182, Chapter 1, Theorem 5.3]. □

EXERCISES

6.1 Let (f, g) be an adjoint pair. If $\theta \in \mathrm{Hom}(fM, N)$, show that

$$\varepsilon_N \circ fg(\theta) \circ f(\eta_M) = \theta.$$

6.2 Let (f, g) be an adjoint pair with $f : \mathsf{B} \to \mathsf{A}$. By replacing B and A by their opposite categories, show that Theorem 6.16 is a consequence of Theorem 6.15.

6.3 Let $f_1 : \mathsf{C} \to \mathsf{D}$ and $f_2 : \mathsf{D} \to \mathsf{E}$ be functors. Suppose that (f_1, g_1) and (f_2, g_2) are adjoint pairs. Show that $(f_2 f_1, g_1 g_2)$ is an adjoint pair.

6.4 If (f, g) is an adjoint pair, show that f preserves initial objects, and g preserves terminal objects.

6.5 Let $f : R \to S$ be a ring homomorphism, and (f^*, f_*) the associated adjoint pair (Example 6.4). Show that f_*, which is $\mathrm{Hom}_S(S, -)$ has a *right* adjoint, namely $f^! = \mathrm{Hom}_R(S, -)$.

6.6 Show that the right adjoint of a functor is only determined up to natural equivalence.

6.7 Prove Proposition 6.12.

6.8 Let B be a set. Show that as functors from Set to Set, $- \times B$ is left adjoint to $\mathrm{Hom}(B, -)$. That is,

$$\mathrm{Hom}(M \times B, N) \cong \mathrm{Hom}(M, \mathrm{Hom}(B, N)).$$

1.7 Limits and colimits

The notions of limit and colimit subsume some familiar ideas. For example, products, kernels, pullbacks, and inverse limits are special types of limits, and coproducts, cokernels, pushouts, and direct limits are special types of colimits.

We begin this section by formalizing the notion of a diagram in a category—a diagram will be a certain sort of functor, but it is really just a fancy way of saying that we have a collection of objects and morphisms between them satisfying certain commutativity rules. The formal definition of a diagram is similar to the functorial definition of a representation of a quiver.

Definition 7.1 A directed graph consists of a set of vertices and, for each ordered pair of vertices (α, β), a set E^α_β of edges from α to β.

A directed graph is essentially the same thing as a category. The graph determines a category in which the objects are the vertices, and the morphisms from α to β are the directed paths from α to β; we also define the identity morphisms id_α as the empty paths e_α. Composition of morphisms is concatenation of paths.

We denote a graph by $G = (V, E)$ where V is the set of vertices, and E the set of edges. All our graphs will be small, meaning that both V and E are small sets.

If e is an edge from α to β, we call α the start of e and β the end of e, and indicate this by writing

$$\alpha \xrightarrow[e]{} \beta.$$

A directed path from a vertex α to a vertex β is a finite sequence of edges $e_1, \ldots, e_n$ such that the start of e_1 is α, the end of e_n is β and, for each $i = 1, \ldots, n-1$, the end of e_i is the start of e_{i+1}; we write $e_n \cdots e_2 e_1$ for the path. For each vertex α, we define the empty path e_α which begins and ends at α; we declare that $ee_\alpha = e$ for any edge e starting at α, and $e_\alpha e' = e'$ for any edge e' ending at α.

Definition 7.2 Let C be a category. Let G be a graph with associated category $\mathcal{G}$. A diagram in C of shape G is a functor $D : \mathcal{G} \to \mathsf{C}$.

Let D be a diagram in C. A tuple (Z, ψ_α) consisting of an object Z in C, and morphisms $\psi_\alpha : Z \to D(\alpha)$, one for each vertex α, is called a cone over D, or a cone from Z to D, if, for all edges $e : \alpha \to \beta$, $D(e) \circ \psi_\alpha = \psi_\beta$. A morphism between two cones, say $\theta : (Z', \psi'_\alpha) \to (Z, \psi_\alpha)$, is a morphism $\theta \in \mathrm{Hom}_{\mathsf{C}}(Z', Z)$ such that $\psi_\alpha \circ \theta = \psi'_\alpha$ for all vertices α. The collection of all cones over D is a category $\mathsf{Cone}(D)$.

A limit of the diagram D is a terminal object in $\mathsf{Cone}(D)$; we denote it by $\lim D$, or $\lim_\alpha D(\alpha)$, if it exists; being a terminal object, it is unique up to isomorphism.

A small diagram is one arising from a small graph. A small limit is one associated to a small diagram.

Although a limit consists of an object together with morphisms, we will often refer to the object itself as the limit.

The terminology 'cone over D' should evoke a picture of the diagram lying in a horizontal plane, with Z sitting above the plane, and maps ψ_α down to each vertex of the diagram in such a way that each triangle having Z as a vertex commutes.

Example 7.3 (Products) If $G = (V, \phi)$ is a graph with no edges, then a diagram, D say, in C of shape G is just a collection of objects D_α indexed by V. The limit of this diagram is therefore the product of the objects,

$$\lim D \cong \prod_\alpha D_\alpha.$$

Now let D be an arbitrary diagram in C; since there are morphisms $\lim D \to D_\alpha$, the universal property of the product implies there is a morphism

$$\lim D \to \prod_\alpha D_\alpha.$$

Example 7.4 (Equalizers and kernels) A diagram of shape

$$\begin{tikzcd} \bullet \arrow[d, shift left] \arrow[d, shift right] \\ \bullet \end{tikzcd}$$

consists of a pair of objects X and Y, and two morphisms $f_1, f_2 : X \to Y$. The limit of this is a triple (L, g, h) consisting of an object L and two morphisms $g : L \to X$ and $h : L \to Y$ such that $f_1 \circ g = f_2 \circ g = h$. Notice that the morphism h carries no extra information: we could say that the limit is a pair (L, g) where $g : L \to X$ satisfies $f_1 \circ g = f_2 \circ g$ and the appropriate universal property holds. We call L the equalizer of f_1 and f_2. In Set the equalizer exists and is the subset of X where f_1 and f_2 agree. In $\mathsf{Mod}R$ the equalizer exists and is the kernel of $f_1 - f_2$; in particular, the kernel of a module homomorphism f is a special type of limit, namely the equalizer of the pair $(f, 0)$. That is, if $f : X \to Y$, then $\ker f$ is the pair (L, g) where L is an R-module, and $g : L \to X$ satisfies $fg = 0$ and, if $g' : L' \to X$ satisfies $fg' = 0$, then there is a unique $h : L' \to L$ such that $g' = gh$.

Example 7.5 (Pullbacks) A pullback is a limit over a diagram of shape

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

The pullback or fiber product of a diagram

$$\begin{tikzcd} & X \arrow[d, "f_1"] \\ Y \arrow[r, "f_2"'] & S \end{tikzcd}$$

in C is a triple $(X \times_S Y, g_1, g_2)$ consisting of an object $X \times_S Y$ in C, and morphisms $g_1 : X \times_S Y \to X$ and $g_2 : X \times_S Y \to Y$ such that the diagram

$$\begin{tikzcd} X \times_S Y \arrow[r, "g_1"] \arrow[d, "g_2"'] & X \arrow[d, "f_1"] \\ Y \arrow[r, "f_2"'] & S \end{tikzcd}$$

commutes and, whenever (Z, h_1, h_2) is another triple in C with $h_1 : Z \to X$, $h_2 : Z \to Y$ and $f_1 \circ h_1 = f_2 \circ h_2$, there is a unique map $\theta : Z \to X \times_S Y$ such that $h_1 = g_1 \circ \theta$ and $h_2 = g_2 \circ \theta$. The requirement that $\lim D$ be a terminal object in $\mathsf{Cone}(D)$ coincides with the universal property the pullback is required to possess.

Theorem 7.6 Set *has all small limits.*

Proof. Let $\mathcal{G}$ be the category associated to a small graph G, and let $D : \mathcal{G} \to \mathsf{Set}$ be a diagram. Fix a singleton set $*$, and let L be the set of all cones over D of the form $(*, \varphi_\alpha)$. Note that L is small since G is, so L is an object in Set. For each vertex α, define $\psi_\alpha : L \to D(\alpha)$ by $\psi_\alpha((*, \varphi_\alpha)) = \varphi_\alpha(*)$. Thus L is a cone over D. We will show it is the limit of D.

Suppose that (Z, ρ_α) is a cone over D. For each $z \in Z$, define $\iota_z : * \to Z$ by $\iota_z(*) = z$. It is easy to see that $(*, \rho_\alpha \circ \iota_z)$ is a cone over D, and hence an element of L, so we may define $\theta : Z \to L$ by $\theta(z) = (*, \rho_\alpha \circ \iota_z)$. Therefore

$$(\psi_\alpha \circ \theta)(z) = \psi_\alpha((*, \rho_\alpha \circ \iota_z)) = \rho_\alpha(i_z(*)) = \rho_\alpha(z),$$

whence $\rho_\alpha = \psi_\alpha \circ \theta$, thus showing that (L, ψ_α) is a terminal object in $\mathsf{Cone}(D)$, and hence a limit of D. $\square$

Colimits. Colimits are like limits, except that they are initial objects defined in terms of cones having vertex below a horizontal plane containing a diagram and morphisms going down from the diagram to the vertex of the cone.

Definition 7.7 Let D be a diagram in C. A tuple (Z, φ_α) consisting of an object Z in C, and morphisms $\varphi_\alpha : D(\alpha) \to Z$, one for each vertex α, is called a cone under D, or a cone from D to Z, if for all edges $e : \alpha \to \beta$, $\varphi_\alpha = \varphi_\beta \circ D(e)$. A morphism between two cones, say $\theta : (Z', \varphi'_\alpha) \to (Z, \varphi_\alpha)$, is a morphism $\theta \in \mathrm{Hom}_{\mathsf{C}}(Z', Z)$ such that $\varphi_\alpha = \theta \circ \varphi'_\alpha$ for all vertices α. The collection of all cones under D is a category $\mathsf{Cone}(D)$.

A colimit of the diagram D is an initial object in $\mathsf{Cone}(D)$, and we denote it by $\operatorname{colim} D$ if it exists.

Remark 7.8 *The terminology 'cone under D' should evoke a picture of the diagram lying in a horizontal plane, with Z sitting below the plane, and maps φ_α down to Z from each vertex of the diagram in such a way that each triangle with Z as one vertex commutes.*

The arguments showing that products, kernels, and pullbacks are special types of limits have analogues showing that coproducts, cokernels, and pushouts are special types of colimits. For example, if $f : X \to Y$, $\operatorname{coker} f$ is the colimit over the diagram

$$\begin{tikzcd} X \arrow[d, shift right, "f"'] \arrow[d, shift left, "0"] \\ Y \end{tikzcd}$$

and a pushout is a colimit over a diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

A coproduct is a colimit over a diagram with no edges; for any diagram D, there is a morphism $\coprod_\alpha D(\alpha) \to \operatorname{colim} D$.

Example 7.9 Let R be an equivalence relation on a set X. Write $\alpha, b : R \to X$ for the projections onto the first and second components, then the set of equivalence classes X/R is the pushout in the following diagram:

$$\begin{array}{ccc} R & \xrightarrow[\alpha]{} & X \\ {\scriptstyle\beta}\downarrow & & \downarrow \\ X & \longrightarrow & X/R \end{array}$$

Example 7.10 Let G be a group acting on a set X. Then the quotient space X/G is the pushout in the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow[\alpha]{} & X \\ {\scriptstyle\beta}\downarrow & & \downarrow \\ X & \longrightarrow & X/G \end{array}$$

where $\alpha(g, x) = x$ and $\beta(g, x) = gx$.

Notice that the quotient spaces X/R and X/G in the previous two examples arise from groupoids (see Example 1.16).

Definition 7.11 A covariant functor F

- preserves, or commutes with, limits if $F(\lim X_\alpha) \cong \lim FX_\alpha$ whenever $\lim X_\alpha$ exists;
- preserves, or commutes with colimits if $F(\operatorname{colim} X_\alpha) = \operatorname{colim} FX_\alpha$ whenever $\operatorname{colim} X_\alpha$ exists;

Theorem 7.12 *The functor* $\mathrm{Hom}_{\mathsf{C}}(X, -)$ *preserves limits, and* $\mathrm{Hom}_{\mathsf{C}}(-, X)$ *sends colimits to limits.*

Proof. Let $D : \mathcal{G} \to \mathsf{C}$ be a diagram with limit (L, ψ_α). If (Z, φ_α) is a cone in Set over the diagram $\mathrm{Hom}(X, -) \circ D$, then each φ_α is a set map $Z \to \mathrm{Hom}_{\mathsf{C}}(X, D(\alpha))$. Thus, for each $z \in Z$, the maps $\varphi_\alpha(z) : X \to D(\alpha)$ make $(X, \varphi_\alpha(z))$ a cone over D (if $e : \alpha \to \beta$, then $\varphi_\beta(z) = D(e) \circ \varphi_\alpha(z)$ because $\varphi_\beta = D(e) \circ \varphi_\alpha$). Hence there is a unique morphism $\theta_z : X \to L$ in C such that $\varphi_\alpha(z) = \psi_\alpha \circ \theta_z$ for all vertices α. Now, defining $\theta : Z \to \mathrm{Hom}_{\mathsf{C}}(X, L)$ by $\theta(z) = \theta_z$, we obtain $\varphi_\alpha = \psi_\alpha \circ \theta$ for all α. The uniqueness of θ follows from the uniqueness of θ_z. Hence $(\mathrm{Hom}_{\mathsf{C}}(X, L), \mathrm{Hom}(X, \psi_\alpha))$ is a limit of $\mathrm{Hom}(X, -) \circ D$.

If we write $F = \mathrm{Hom}_{\mathsf{C}}(-, X)$, then $\bar{F} : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is the covariant functor $\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(X, -)$, which commutes with limits. But the limit of a diagram $D : \mathcal{G} \to \mathsf{C}^{\mathrm{op}}$ is the colimit of the 'same' diagram $D : \mathcal{G} \to \mathsf{C}$. Hence the result. $\square$

Using Yoneda's Lemma, the existence of limits in Set (Theorem 7.6), and Theorem 7.12, we can define limits and colimits as the objects representing suitable functors.

Corollary 7.13 *If X is an object in C, and $D : \mathcal{G} \to \mathsf{C}$ is a diagram, then*

1. $\lim D$ *represents the functor* $X \mapsto \lim(\mathrm{Hom}(X, -) \circ D)$, *and*
2. $\operatorname{colim} D$ *represents the functor* $Y \mapsto \lim(\mathrm{Hom}(-, Y) \circ D)$.

Corollary 7.14 *If (F, G) is an adjoint pair of functors, then F preserves colimits, and G preserves limits.*

Proof. Let $F : \mathsf{C} \to \mathsf{D}$, and let $D : \mathcal{G} \to \mathsf{C}$ be a diagram such that $\operatorname{colim} D$ exists. We must show that $F(\operatorname{colim} D) \cong \operatorname{colim}(F \circ D)$. For an arbitrary Y in D, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{D}}(F(\operatorname{colim} D), Y) &\cong \mathrm{Hom}_{\mathsf{C}}(\operatorname{colim} D, GY) && \text{by adjointness,}\\
&\cong \lim \mathrm{Hom}_{\mathsf{C}}(-, GY) \circ D \\
&= \lim \mathrm{Hom}_{\mathsf{C}}(D(\alpha), GY) \\
&\cong \lim \mathrm{Hom}_{\mathsf{D}}(F(D(\alpha)), Y) && \text{by adjointness,}\\
&\cong \mathrm{Hom}_{\mathsf{D}}(\operatorname{colim}(F \circ D), Y) && \text{by Corollary 7.13.}
\end{aligned}$$

Therefore, by Yoneda's Lemma $\operatorname{colim}(F \circ D) \cong F(\operatorname{colim} D)$.

The proof that G commutes with limits is similar (also see Exercise **??**). □

Definition 7.15 A category C is

- $\mathsf{complete}$ if it has all limits;
- $\mathsf{cocomplete}$ if it has all colimits.

(Remembering our convention that all index sets be small, it might be better to say "small-complete" rather than "complete".)

Definition 7.16 A category C is $\mathsf{filtering}$ if

- for any two objects $i, j \in \mathsf{C}$, there exists an object $k \in \mathsf{C}$ and morphisms $i \to k$ and $j \to k$, and
- given any two morphisms $\alpha_1, \alpha_2 : i \to j$, there exists an object k and a morphism $\beta : j \to k$ such that $\beta \circ \alpha_1 = \beta \circ \alpha_2$.

In particular, a directed category if filtering, but not conversely, since there may be more than one morphism between two objects.

The basics can be found in the appendix to the book of Artin and Mazur [13].

EXERCISES

7.1 Consider the empty graph having one vertex and no edges. Show that a limit over this graph is a terminal object, and a colimit is an initial object.

Pullbacks and pushouts in Mod$R$. Let A, B, C be R-modules.

(a) Show that the pushout of the diagram

$$\begin{tikzcd} A \arrow[r, "g_1"] \arrow[d, "g_2"'] & B \\ C & \end{tikzcd}$$

is the cokernel of the map $g_1 \coprod g_2 : A \to B \coprod C$, together with the obvious maps from B and C to it.

(b) Show that the pullback of the diagram

$$\begin{tikzcd} . & C \arrow[d, "g_2"'] \\ A \arrow[r, "g_1"] & B \end{tikzcd}$$

is the kernel of the map $g_1 \prod g_2 : A \prod C \to B$, together with the obvious maps to A and C from it.

Chapter 2

Abelian categories

This chapter lays out the machinery and results concerning abelian categories that we need later. More complete information can be found in the papers of Gabriel [88] and Grothendieck [94], and in the books by Popescu [182] and Stenstrom [241].

The standard example of an abelian category is the category of modules over a ring. Abstracting the important properties of this category leads to the definition of an abelian category. Every abelian category can be embedded as a full subcategory of a module category, so the intuition one has from module categories carries over to abelian categories. There are some differences, and therefore some pitfalls. For example, an abelian category need not have many projective or injective modules. Nor must it have arbitrary direct limits.

A Grothendieck category is a special kind of abelian category that is closer still to a module category. Not only can it be realized as a full subcategory of a module category, but this can be done in such a way that the embedding functor has an exact left adjoint. In other words, every Grothendieck category is a localization of a module category. A Grothendieck category has enough injectives, meaning that every object embeds in an injective object. One of the axioms for a Grothendieck category is that it be cocomplete. In particular, it has direct limits. It turns out that a Grothendieck category is also complete. The sheaves of abelian groups on a topological space form a Grothendieck category, and so do the quasi-coherent $\mathcal{O}_X$-modules on a reasonable scheme X.

2.1 Additive categories

Definition 1.1 A category is

- pre-additive if all its Hom sets are abelian groups, and composition of morphisms is bilinear,
- additive if it is pre-additive and has finite products and coproducts, and contains a zero object.

A functor $F : \mathsf{A} \to \mathsf{B}$ between pre-additive categories is additive if each map $\operatorname{Hom}_{\mathsf{A}}(M, N) \to \operatorname{Hom}_{\mathsf{B}}(FM, FN)$ is a group homomorphism. $\Diamond$

Remark. It follows from the remarks after Example 1.3.4 that in an additive category a coproduct of the empty family is an initial object and a product of the empty family is a terminal object. Therefore the axiom that an additive category has a zero object follows from the other axioms.

Example 1.2 The module category $\mathsf{Mod}R$ is additive: the abelian group structure on $\mathrm{Hom}_R(M,N)$ is that induced from the abelian group structure on N, and the existence of products and coproducts is explained in Example 1.3.8. The full subcategory of finitely generated modules is also additive since only *finite* products and coproducts are required to exist. $\Diamond$

The group operation in the Hom-sets of a pre-additive category will be written additively, and their identity elements will be denoted by 0. There is some potential for confusion with many different zeroes, so we will write 0_{MN} for the zero element in $\mathrm{Hom}(M,N)$ when necessary. The reader may check that the composition of a morphism with a zero morphism is zero. If a pre-additive category has a zero object, then the bilinearity of the composition

$$\mathrm{Hom}(0,N)\times\mathrm{Hom}(M,0)\to\mathrm{Hom}(M,N)$$

implies that the composition $M\to 0\to N$ is equal to 0_{MN}.

In a pre-additive category each $\mathrm{Hom}(M,M)$ is a ring with identity. We call it the endomorphism ring of M. The associativity and trilinearity of the composition maps

$$\mathrm{Hom}(M,N)\times\mathrm{Hom}(L,M)\times\mathrm{Hom}(K,L)\to\mathrm{Hom}(K,N),$$

endow each $\mathrm{Hom}(L,M)$ with a $\mathrm{Hom}(M,M)$-$\mathrm{Hom}(L,L)$-bimodule structure.

Example 1.3 The categories Set, Group and Ring are not pre-additive. Although there is no obvious abelian group structure on the Hom spaces, a little thought is required to see that none can be imposed so as to make composition of morphisms bilinear.

Recall that a ring homomorphism is required to send the identity in one to the identity in the other. There are many rings R and S such that $\mathrm{Hom}_{\mathsf{Ring}}(R,S)$ is empty. For example, take R to be the 2×2 matrices over a field k, and $S=k$. But a group is non-empty, so Ring is not pre-additive.

Suppose that Set were pre-additive. Let M be any set, and $N=\{s,t\}$ a set with two distinct elements. Since $\mathrm{Hom}_{\mathsf{Set}}(M,\{s\})$ has one element it is the trivial group. Since composition is bilinear $M\to\{s\}\to N$ must be the identity element in the group $\mathrm{Hom}_{\mathsf{Set}}(M,N)$. But the same argument applies to $M\to\{t\}\to N$, whence $\mathrm{Hom}_{\mathsf{Set}}(M,N)$ has two distinct identity elements! This is absurd. $\Diamond$

Proposition 1.4 *Let $\{M_i \mid i\in I\}$ be a small set of objects in an additive category, and suppose that their product and coproduct exist. Let*

$$\alpha_j : M_j \to \coprod M_i \qquad \textit{and} \qquad \rho_j : \prod M_i \to M_j$$

be the morphisms guaranteed by the definitions. For each pair of indices (i,j) *define* $\delta^i_j : M_i \to M_j$ *by*

$$\delta^i_j = \begin{cases} 0 & \text{if } i \neq j, \\ \mathrm{id}_{M_i} & \text{if } i = j. \end{cases}$$

Then

1. *there are unique maps* $\varepsilon_j : M_j \to \prod M_i$ *such that* $\rho_i \varepsilon_j = \delta^j_i$ *for all* $i, j \in I$*;*
2. *there are unique maps* $\gamma_j : \coprod M_i \to M_j$ *such that* $\gamma_j \alpha_i = \delta^i_j$ *for all* $i, j \in I$*;*
3. *there is a unique map*
$$\Psi : \coprod M_i \to \prod M_i$$
such that $\Psi \alpha_i = \varepsilon_i$ *for all* $i \in I$*;*
4. $\rho_j \Psi = \gamma_j$ *for all* $j \in I$*;*
5. *if* I *is finite, then* Ψ *is an isomorphism.*

Proof. The universal property of the product provides the morphisms ε_j in (1), and the universal property of the coproduct provides the morphisms γ_j in (2). The map Ψ in (3) is ensured by the universal property of $\coprod M_j$ applied to the diagram

$$\begin{array}{ccc} M_i & \xrightarrow{\alpha_i} & \coprod M_j \\ {\scriptstyle \varepsilon_i}\big\downarrow & & \\ \prod M_j & & \end{array}$$

It follows that $\rho_j \Psi \alpha_i = \rho_j \varepsilon_i = \delta^i_j$, so by the uniqueness of the maps γ_j, we conclude that $\rho_j \Psi = \gamma_j$; thus (4) holds.

(5) Now suppose that I is finite.

For each i,

$$\left(\sum_j \alpha_j \gamma_j \right) \circ \alpha_i = \sum_j \alpha_j \delta^i_j = \alpha_i$$

for all i. Therefore the uniqueness of the map making the diagram

$$\begin{array}{ccc} M_i & \xrightarrow{\alpha_i} & \coprod M_j \\ {\scriptstyle \alpha_i}\big\downarrow & & \\ \coprod M_j & & \end{array}$$

commute implies that

$$\sum_j \alpha_j \gamma_j = \mathrm{id}_{\coprod M_j} . \tag{1-1}$$

Similarly, $\rho_i \circ \left(\sum_j \varepsilon_j \rho_j\right) = \rho_i$ for all i, from which it follows that

$$\sum_j \varepsilon_j \rho_j = \mathrm{id}_{\prod M_j}\,. \tag{1-2}$$

(Alternatively, (1-2) is equivalent to (1-1) in the opposite category.)

Now we show that Ψ is both monic and epic. If $f : L \to \coprod M_i$ is such that $\Psi f = 0$, then $0 = \rho_j \Psi f = \gamma_j f = 0$ for all j, so $f = (\sum_j \alpha_j \gamma_j) f = 0$; thus Ψ is monic. If $g : \prod M_i \to N$ is such that $g\Psi = 0$, then $0 = g\Psi\alpha_j = g\varepsilon_j$ for all j, so $g = g(\sum_j \varepsilon_j \rho_j) = 0$; thus Ψ is epic. Hence Ψ is an isomorphism when I is finite. □

It follows from (1) and (2) that α_i and ε_i are monic, and that ρ_i and γ_i are epic.

Warning. The map $\Psi : \coprod M_i \to \prod M_i$ need not be monic in general (Example 5.3), but is when the ambient category satisfies Ab5 (Proposition 5.13). When Ψ fails to be monic, there is a non-zero map $f : K \to \coprod M_i$ such that $\gamma_i f = 0$ for all i.

Corollary 1.5 *Let f be a morphism in an additive category. Then*

1. *f is a monomorphism if and only if $fg = 0$ implies $g = 0$, and*
2. *f is a epimorphism if and only if $gf = 0$ implies $g = 0$.*

Proof. Exercise 3. □

Definition 1.6 Let $f : M \to N$ be a morphism in an additive category A.

A kernel of f is a pair (A, α), consisting of an object A and a morphism $\alpha : A \to M$ such that $f\alpha = 0$ and, if $\alpha' : A' \to M$ is a morphism for which $f\alpha' = 0$, then there is a unique morphism $\rho : A' \to A$ such that $\alpha' = \alpha\rho$.

A cokernel of f is a pair (B, β), consisting of an object B and a morphism $\beta : N \to B$ such that $\beta f = 0$ and, if $\beta' : N \to B'$ is a morphism for which $\beta' f = 0$, then there is a unique morphism $\rho : B \to B'$ such that $\beta' = \rho\beta$. ◊

The uniqueness of kernels and cokernels up to isomorphism follows from the uniqueness of limits and colimits up to isomorphism. For example, the kernel of a morphism $f : M \to N$ is the limit of the system

$$\begin{tikzcd} & 0 \arrow[d] \\ M \arrow[r, "f"'] & N. \end{tikzcd}$$

One could also define *the* kernel of $f : M \to N$ as the equivalence class of all kernels (A, α) under the equivalence relation $(A, \alpha) \equiv (A', \alpha')$ if there is an isomorphism $\iota : A' \to A$ such that $\alpha' = \alpha\iota$.

Lemma 1.7 *Let $f : M \to N$ be a morphism in an additive category. Then*

1. *f is a monomorphism if and only if* $\ker f = (0 \to M)$*;*

2. *f is an epimorphism if and only if* $\operatorname{coker} f = (N \to 0)$.

Proof. (1) ($\Rightarrow$) Let $\alpha' : A' \to M$ be a morphism such that $f\alpha' = 0$. Since f is a monomorphism, $\alpha' = 0$. Hence there is a unique morphism $\rho : A' \to 0$ such that $\alpha' = \rho \circ 0$, namely $\rho = 0$. Thus the pair $(0, 0 : 0 \to M)$ satisfies the required universal property to be a kernel.

($\Leftarrow$) If $g_1, g_2 : W \to M$ are morphisms such that $fg_1 = fg_2$ then $f(g_1 - g_2) = 0$ so, by the universal property of a kernel, there is a (unique) morphism $\rho : W \to 0$ such that $g_1 - g_2 = 0 \circ \rho$. That is, $g_1 - g_2 = 0$, showing that f is a monomorphism.

(2) Exercise. □

Proposition 1.8 *Let $f : M \to N$ be a morphism in an additive category* A*. Then*

1. *if* $\ker f$ *exists, it is a subobject of M;*

2. *if* $\operatorname{coker} f$ *exists, it is a quotient object of N;*

3. *there is a natural morphism* $\operatorname{coker} \ker f \to \ker \operatorname{coker} f$*, if these objects exist.*

Proof. (1) Let $(A, \alpha) = \ker f$ and suppose there exist morphisms $\rho_1, \rho_2 : W \to A$ such that $\alpha\rho_1 = \alpha\rho_2$. Since $f\alpha\rho_1 = 0$, the uniqueness of ρ in the definition of the kernel implies that $\rho_1 = \rho_2$.

(2) Let $(B, \beta) = \operatorname{coker} f$ and suppose there exist morphisms $\rho_1, \rho_2 : B \to Z$ such that $\rho_1\beta = \rho_2\beta$. Since $\rho_1\beta f = 0$, the uniqueness of ρ in the definition of the cokernel implies that $\rho_1 = \rho_2$.

(3) Retain the earlier notation. Provided all the required objects exist, there is a diagram:

$$\begin{tikzcd}
A \arrow[r, "\alpha"] & M \arrow[r, "f"] \arrow[d, "\gamma"'] & N \arrow[r, "\beta"] & B \\
& \operatorname{coker} \alpha & \ker \beta \arrow[u, "\delta"'] &
\end{tikzcd}$$

We will construct a morphism $\mu : \operatorname{coker} \alpha \to \ker \beta$ making the rectangle commute. Since $f\alpha = 0$, the defining property of $\operatorname{coker} \alpha$ guarantees the existence of a morphism $\rho : \operatorname{coker} \alpha \to N$ such that $f = \rho\gamma$. Thus $\beta\rho\gamma = \beta f = 0$. But γ is an epimorphism, so $\beta\rho = 0$. The defining property of $\ker \beta$ guarantees the existence of a morphism $\mu : \operatorname{coker} \alpha \to \ker \beta$ such that $\rho = \delta\mu$. Therefore the morphism μ makes the diagram commute. □

Definition 1.9 Let f be a morphism in an additive category. The image and coimage of f are

$$\operatorname{im} f := \ker \operatorname{coker} f$$

and

$$\operatorname{coim} f = \operatorname{coker} \ker f,$$

whenever these objects exist. ◊

There is an obvious generalization of pre-additive categories in which each Hom-set is required to be a module over a fixed commutative ring k, and the composition of morphisms is required to be k-bilinear. The following is a more formal way of saying this.

Definition 1.10 Let k be a commutative ring. A category C is k-linear there is a ring homomorphism from k to the ring of natural transformations of the identity functor id_{C}. ◊

Let C be a k-linear category. Since each element of k acts as a natural transformation of the identity functor, there is a ring homomorphism $k \to \operatorname{Hom}_{\mathsf{C}}(M, M)$ for each object M in C. Composition of morphisms

$$\operatorname{Hom}_{\mathsf{C}}(M, N) \times \operatorname{Hom}_{\mathsf{C}}(M, M) \to \operatorname{Hom}_{\mathsf{C}}(M, N)$$

then gives each $\operatorname{Hom}_{\mathsf{C}}(M, N)$ a k-module structure. Composition of morphisms

$$\operatorname{Hom}_{\mathsf{C}}(N, N) \times \operatorname{Hom}_{\mathsf{C}}(M, N) \to \operatorname{Hom}_{\mathsf{C}}(M, N)$$

gives each $\operatorname{Hom}_{\mathsf{C}}(M, N)$ another k-module structure. It is easily checked that these two module structures are the same. The composition of morphisms $(f, g) \to f \circ g$ is k-bilinear.

A functor $F : \mathsf{A} \to \mathsf{B}$ between k-linear categories is k-linear if all the maps $\operatorname{Hom}_{\mathsf{A}}(M, N) \to \operatorname{Hom}_{\mathsf{B}}(FM, FN)$ are k-linear. We will almost always work within some k-linear category, and all our functors will be k-linear.

EXERCISES

1.1 Show that in a pre-additive category $\operatorname{Hom}_{\mathsf{A}}(M, M)$ is a ring.

1.2 Let R be a ring with identity, and let R be the category associated to R as in Example 3.1. Show that the category of additive functors from R to Ab is equivalent to the category of left R-modules.

1.3 Prove Corollary 1.5 and Lemma 1.7(2).

1.4 Show that the kernel and cokernel of a morphism $f : M \to N$ are, respectively, a subobject of M, and a quotient of N.

1.5 In an additive category show that

(a) $\ker(0 : M \to N) = (M, \mathrm{id}_M)$;

(b) $\operatorname{coker}(0 : M \to N) = (N, \mathrm{id}_N)$;

(c) $f = 0$ if and only if $\operatorname{im} f = 0$.

2.2 Abelian categories

Proposition 2.1 *A morphism $f : M \to N$ in a pre-additive category factors as*

$$M \to \text{coker } \ker f \to \ker \text{ coker } f \to N. \tag{2-1}$$

Definition 2.2 An additive category is abelian if every morphism has a kernel and a cokernel, and the natural morphism coker $\ker f \to \ker$ coker f is an isomorphism for all morphisms f. ◊

Terminology. The objects in an abelian category A will be called A-modules.

It follows from (2-1) that every morphism $f : M \to N$ in an abelian category may be factored as $f = \beta \circ \alpha$ with α an epimorphism and β a monomorphism.

If $f : M \to N$ is monic and epic, then its kernel and cokernel are zero, so coker $\ker f = M$ and ker coker $f = N$, whence $M \cong N$. Therefore f is an isomorphism if and only if it is both a monomorphism and an epimorphism.

Theorem 2.3 *The category of right modules over a ring is abelian.*

Proof. Let R be a ring. Kernels and cokernels exist in Mod$R$. If $f : M \to N$ then $\ker f = \{m \in M \mid f(m) = 0\}$ (together with its natural inclusion in M) and coker $f = N/\{f(m) \mid m \in M\}$ (together with the natural surjection from N). Hence coker $\ker f = M/\ker f$, and

$$\ker \text{ coker } f = \{n \in N \mid n = f(m) \text{ for some } m \in M\}.$$

The first isomorphism theorem for modules says that the natural morphism from coker $\ker f$ to ker coker f is an isomorphism. □

The standard example of an abelian category that is not of the form Mod$R$ is the category of presheaves of abelian groups on a topological space.

Example 2.4 Let R be a filtered ring. The category Filt$R$ of filtered right R-modules is an additive category having kernels and cokernels, but is not abelian. This can be seen by observing that the identity map on a filtered module M may be interpreted as a map in the filtered category from M to the filtered module $M(1)$ that is defined to be equal to M as an unfiltered module, but with filtration defined by $F_iM(1) = F_{i+1}M$; this map is monic and epic, but not an isomorphism in Filt$R$.

The problem is this. Let $f : M \to N$ be a morphism of filtered modules. The filtration on the cokernel of f is given by $F_i(\text{coker } f) = F_iN + f(M)/f(M)$, and the filtration on $X = \ker$ coker f is given by $F_iX = F_iN \cap f(M)$. The filtration on $Y =$ coker $\ker f$ is given by $F_iY = f(F_iM)$. But the natural isomorphism $X \to Y$ of unfiltered modules does not always respect the filtrations on X and Y, so ker coker f is not isomorphic to coker $\ker f$ in general.

A morphism f in an additive category is strict if the natural morphism $\operatorname{coim} f \to \operatorname{im} f$ is an isomorphism. Strict morphisms play an important role in categories of filtered modules. $\Diamond$

The sum and intersection of submodules. Suppose that $\{M_i \mid i \in I\}$ is a small set of submodules of an A-module M. If $\bigoplus_{i\in I} M_i$ exists, we define the sum of the submodules to be the image of the canonical map $\bigoplus_{i\in I} M_i \to M$; it is denoted by

$$\sum_{i\in I} M_i.$$

If $\prod_{i\in I} M_i$ exists, we define the intersection of the submodules to be the kernel of the canonical map $M \to \prod_{i\in I} M/M_i$, and it is denoted by

$$\bigcap_{i\in I} M_i.$$

The pre-image. If $f : M \to N$ is a morphism in an abelian category and N' is a submodule of N we define the pre-image of N' to be

$$f^{-1}(N') := \ker(M \to N \to N/N').$$

Proposition 2.5 *If $f : M \to N$ is a homomorphism of A-modules and $\{M_i \mid i \in I\}$ are submodules of M and $\{N_j \mid j \in J\}$ are submodules of N, then*

$$\sum_{i\in I} f(M_i) = f\Big(\sum_{i\in I} M_i\Big)$$

and

$$\bigcap_{j\in J} f^{-1}(N_j) = f^{-1}\Big(\bigcap_{j\in J} N_j\Big).$$

Proof. Let C and D be the categories of submodules of M and N respectively; the only morphisms are the inclusions. There is an adjoint pair of functors $(f_*, f^!)$ with $f_* : \mathsf{C} \to \mathsf{D}$ defined by $f_*M' = f(M')$ and $f^!N' = f^{-1}(N')$. It turns out that $\sum M_i$ is the coproduct of the M_i in C and that $\bigcap N_j$ is the product of the N_j in D. By **??**, f_* commutes with colimits and $f^!$ commutes with limits, so the result follows. $\square$

See section 2.6 of Popescu for further properties of f^{-1}.

Definition 2.6 A sequence $L \to M \to N$ of A-modules is exact if $\operatorname{im}(L \to M) = \ker(M \to N)$. $\Diamond$

If $f : M \to N$ is a map and N' is a submodule of N, then there is an exact sequence

$$0 \to f^{-1}(N') \to M \to f(M)/f(M) \cap N' \to 0.$$

Simple Modules. A dominant theme in almost every branch of algebra is the classification and understanding the simple objects in the various abelian categories that arise within that branch. The ur-example is the problem of determining the irreducible representations of a finite group.

Definition 2.7 Let A be an abelian category. An A-module M is simple or irreducible if it is non-zero and its only submodules are 0 and M. A module is semisimple if it is isomorphic to a direct sum of simple modules. $\Diamond$

The next example provides a warning that our intuition from categories of modules over a ring might lead us astray in an arbitrary abelian category.

Example 2.8 A sum of simple modules need not be semsimple. Let A be the opposite of the category $\mathsf{Mod}R$ where R is a polynomial ring in $n \geq 1$ variables over a field. We will exhibit an A-module that is a quotient of a direct sum of simple A-modules, and therefore a sum of simple modules, that is not isomorphic to a direct sum of simples.

Simple A-modules coincide with simple R-modules. The intersection of all the maximal ideals is zero, so R embeds in a direct product of simple modules, namely the product of all the simple quotients of R. Reinterpreting this in A, there is an epimorphism from a direct sum of simple A-modules to R. However, if R were a direct sum of simple A-modules, then in $\mathsf{Mod}R$, R would be a direct product of simple modules. We leave it to the reader to verify that R is not a direct product of simple modules in $\mathsf{Mod}R$. Thus, we deduce that in A a quotient of a direct sum of simples need not be simple. $\Diamond$

The situation is better than this example suggests. If an abelian category satisfies Grothendieck's condition Ab5 (Section 2.5), then every sum of simple modules in it is isomorphic to a direct sum of simples (Proposition 5.12).

Many of the arguments that work for simple modules over a ring work for simple modules in any abelian category. For example, one has a version of Schur's Lemma: The endomorphism ring of a simple module is a division ring because a non-zero endomorphism of a simple module must be monic and epic, hence an isomorphism.

Lemma 2.9 *If $M = \prod M_\alpha$ is a product of simple modules, then every simple submodule of M is isomorphic to one of the M_αs.*

Proof. Since the restriction of at least one of the projections $p_\alpha : M \to M_\alpha$ to the non-zero submodule is non-zero, that restriction implements the claimed isomorphism. $\square$

EXERCISES

2.1 Show that a sequence $0 \to L \to M \to N$ of A-modules is exact if and only if the sequence $0 \to \mathrm{Hom}_{\mathsf{A}}(B,L) \to \mathrm{Hom}_{\mathsf{A}}(B,M) \to \mathrm{Hom}_{\mathsf{A}}(B,N)$ is exact for all A-modules B.

2.2 [241, ???] Show that the short exact sequences over a ring R form a pre-additive category that is not abelian. The morphisms are triples (α,β,γ) such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "\alpha"] & M \arrow[r] \arrow[d, "\beta"] & N \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & L' \arrow[r] & M' \arrow[r] & N' \arrow[r] & 0 \end{tikzcd}$$

commutes. Find criteria for (α,β,γ) to be monic and/or epic.

2.3 Check the adjointness claim in the proof of Proposition 2.5.

2.4 Let $p_i : \prod M_j \to M_i$ be the projections. Show that if N is a non-zero submodule of $\prod M_j$, then the restriction of some p_i to N is non-zero. Hence show that if each M_j is simple, then every simple submodule of $\prod M_j$ is isomorphic to some M_i.

2.3 Functors to abelian categories

Throughout this section A will denote an abelian category.

If C and A are categories, we write

$$\mathsf{Fun}(\mathsf{C},\mathsf{A})$$

for the category of covariant functors from C to A. The morphisms in $\mathsf{Fun}(\mathsf{C},\mathsf{A})$ are the natural transformations.

As the next three examples suggest, many of the abelian categories one first encounters may be realized as categories of functors to the ur abelian category Ab. In this section we show that the functors from a category C to a fixed abelian category form an abelian category.

Example 3.1 Let R be a ring. Define the category R to have a single object $*$ and morphisms the elements of R, with composition being the multiplication in R. The addition in R gives R the structure of a pre-additive category. The category of additive functors from R to Ab is equivalent to the category of right R-modules. Such a functor assigns to $*$ an abelian group, say M, and assigns to each element of $R = \mathrm{Hom}_{\mathsf{R}}(*,*)$ a group homomorphism $M \to M$. (Some details need to be checked). ♢

Example 3.2 Let G be an abelian group, and R a G-graded ring. Define the category C to have objects the elements of G, and define $\mathrm{Hom}_{\mathsf{C}}(i,j) = R_{j-i}$. The composition of morphisms is given by the multiplication in R. Thus C is a pre-additive category. The identity element of R is the identity morphism on each element of G.

The category of additive functors from C to Ab is equivalent to the category $\mathsf{GrMod}_G R$ of G-graded right R-modules. Such a functor assigns to each $i \in G$

an abelian group, say M_i, and assigns to each element of R_j a collection of group homomorphisms $M_i \to M_{i+j}$, one for each $i \in G$, each one coming from the fact that $R_j = \mathrm{Hom}_{\mathsf{R}}(i, i+j)$. Conversely, one may associate to a G-graded R-module M the functor $\mathsf{C} \to \mathsf{Ab}$ that sends $i \in G$ to M_i and sends morphism $x \in \mathrm{Hom}_{\mathsf{C}}(i, i+j) = R_j$ to the map $x : M_i \to M_{i+j}$. (Some details need to be checked). $\lozenge$

The category of presheaves of abelian groups on a topological space X is equivalent to the category of contravariant functors from $\mathsf{Open}(X)$, the category of open subspaces of X, to Ab (Example 1.4.11).

Let A be an abelian category. If F and G are functors from C to A there is an abelian group structure on $\mathrm{Hom}(F, G)$ defined as follows. Natural transformations $\tau, \mu : F \to G$ yield morphisms $\tau_M, \mu_M \in \mathrm{Hom}_{\mathsf{A}}(FM, GM)$ for each M in C, so $\tau_M + \mu_M$ is defined. We define $\tau + \mu$ by setting

$$(\tau + \mu)_M := \tau_M + \mu_M.$$

The zero element in $\mathrm{Hom}(F, G)$ is the natural transformation that associates to each M in C the zero map $FM \to GM$. This makes $\mathsf{Fun}(\mathsf{C}, \mathsf{A})$ a pre-additive category.

In fact, $\mathsf{Fun}(\mathsf{C}, \mathsf{A})$ is an additive category. The product of two objects F and G in $\mathsf{Fun}(\mathsf{C}, \mathsf{A})$ is given by

$$(F \prod G)(M) = FM \prod GM$$

and

$$(F \prod G)(f) = Ff \prod Gf.$$

The coproduct of two objects is defined in a similar way. The zero object in $\mathsf{Fun}(\mathsf{C}, \mathsf{A})$ is the functor that sends every object of C to the zero object in A.

Theorem 3.3 *Let C be an arbitrary category and A an abelian category. Then $\mathsf{Fun}(\mathsf{C}, \mathsf{A})$ is an abelian category.*

Proof. Let $\tau : F \to G$ be a morphism in $\mathsf{Fun}(\mathsf{C}, \mathsf{A})$.

First we show that a kernel of τ is the functor $K : \mathsf{C} \to \mathsf{A}$ defined as follows. On an object M,

$$KM := \ker(\tau_M : FM \to GM).$$

More precisely, for each M in C we fix a pair $(KM, \iota_M : KM \to FM)$ that is a kernel of τ_M. The value of K on a morphism $f : M \to N$ in C is defined as follows. There is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & KM \arrow[r] & FM \arrow[r] \arrow[d, "Ff"] & GM \arrow[d, "Gf"] \\ 0 \arrow[r] & KN \arrow[r] & FN \arrow[r] & GN \end{tikzcd}$$

with exact rows. The unique map $KM \to KN$ making the diagram commute is defined to be Kf. Because $(Ff)\circ\iota_M = \iota_N \circ (Kf)$, it follows that the morphisms ι_M determine a natural transformation $\iota : K \to F$. Clearly $\tau\iota = 0$.

To see that (K, ι) is the kernel of τ, suppose that $\mu : K' \to F$ is such that $\tau\mu = 0$. Then, for every M in C, the composition $\tau_M \circ \mu_M : K'M \to FM \to GM$ is zero. By the universal property of the kernel, there is a unique morphism $\rho_M : K'M \to KM$ such that $\mu_M = \iota_M \circ \rho_M$. The uniqueness of each ρ_M means that these morphisms give a natural transformation $\rho : K' \to K$, and $\mu = \iota \circ \rho$. Finally, the uniqueness of each ρ_M ensures that ρ is unique. Hence (K, ι) is a kernel.

Cokernels are constructed in a similar way. We might say that kernels and cokernels are defined "pointwise".

Hence $P := \operatorname{coker}\ker\tau$ and $Q := \ker\operatorname{coker}\tau$ are also determined pointwise. That is, $PM = \operatorname{coker}\ker(\tau_M : FM \to GM)$ and $QM = \ker\operatorname{coker}(\tau_M : FM \to GM)$. But the natural map $PM \to QM$ is an isomorphism since A is abelian. Therefore the natural natural transformation $\eta : P \to Q$ has the property that $\eta_M : PM \to QM$ is an isomorphism for all M. This is exactly what is required to show that η is a natural equivalence. □

Just as the kernel and cokernel of a natural transformation are determined pointwise, so is the exactness of a sequence of functors determined pointwise.

Corollary 3.4 *A sequence $F \to G \to H$ in $\mathsf{Fun}(\mathsf{C}, \mathsf{A})$ is exact at G if and only if the sequence $FM \to GM \to HM$ is exact in A for all objects M in C.*

If C and A are k-linear categories we write $\mathsf{Hom}(\mathsf{C}, \mathsf{A})$ for the category of k-linear functors from C to A.

Corollary 3.5 *If C and A are k-linear categories then $\mathsf{Hom}(\mathsf{C}, \mathsf{A})$ is an abelian subcategory of $\mathsf{Fun}(\mathsf{C}, \mathsf{A})$.*

$\boxed{\text{Paul}}$ What are the properties of the inclusion functor?

Corollary 3.6 *If G is an abelian group and R is a G-graded ring, then the category of G-graded R-modules is abelian.*

Definition 3.7 Let $F : \mathsf{A} \to \mathsf{B}$ be an additive functor between abelian categories. Then F is

1. left exact if for every exact sequence $0 \to L \to M \to N$, the sequence $0 \to FL \to FM \to FN$ is exact;
2. right exact if for every exact sequence $L \to M \to N \to 0$, the sequence $FL \to FM \to FN \to 0$ is exact;
3. exact if for every exact sequence $L \to M \to N$, the sequence $FL \to FM \to FN$ is exact. ◊

A functor is exact if and only if it is both left and right exact.

Left (respectively, right) exact functors between abelian categories preserve finite limits (respectively, colimits). Left (respectively, right) exact contravariant functors change finite colimits (respectively, limits) to limits (respectively, colimits).

Proposition 3.8 *Let P be an object in an abelian category A and let R denote the ring $\mathrm{Hom}_{\mathsf{A}}(P,P)$. Then $\mathrm{Hom}_{\mathsf{A}}(P,-)$ and $\mathrm{Hom}_{\mathsf{A}}(-,P)$ are left exact functors taking values in $\mathsf{Mod}R$ and $\mathsf{Mod}R^{\mathrm{op}}$ respectively.*

Proof. We will just treat the covariant functor $F = \mathrm{Hom}_{\mathsf{A}}(P,-)$. By considering the contravariant functor $G = \mathrm{Hom}_{\mathsf{A}}(-,P)$ as a covariant functor on the opposite category A^{op} the result for F implies that for G.

Let M be an A-module. The map

$$\mathrm{Hom}_{\mathsf{A}}(P,M) \times \mathrm{Hom}_{\mathsf{A}}(P,P) \to \mathrm{Hom}_{\mathsf{A}}(P,M),$$

given by composition of morphisms endows FM with the structure of a right R-module. If $f \in \mathrm{Hom}_{\mathsf{A}}(M,N)$ then $Ff : FM \to FN$, which is given by $Ff(\alpha) = f\alpha$, is a right R-module map because composition of morphisms is associative (and A is a $\mathbb{Z}$-linear category).

It remains to show that F is left exact. Let

$$0 \longrightarrow L \xrightarrow{\ f\ } M \xrightarrow{\ g\ } N$$

be exact in A. Then

$$0 \longrightarrow FL \xrightarrow{\ Ff\ } FM \xrightarrow{\ Fg\ } FN$$

is a complex of right R-modules. Since $Ff(\alpha) = f\alpha$, the fact that f is a monomorphism implies that Ff is injective. This proves exactness at FL. Now suppose that $Fg(\beta) = 0$. That is, $g\beta = 0$. Since (L,f) is the kernel of g, there exists a unique $\rho : P \to L$ such that $\beta = f\rho$; that is, $\beta \in \mathrm{im}(Ff)$, which proves exactness at FM. □

Proposition 3.9 *If (f,g) is an adjoint pair of functors between two abelian categories, then f is right exact and g is left exact.*

Proof. This is a special case of Corollary 1.7.14. □

EXERCISES

3.1 Show that the category of additive functors $\mathrm{Hom}(\mathsf{B},\mathsf{A})$ from one additive category to another is additive.

3.2 If A is an additive category show that the ring of natural transformations of the identity functor id_{A} is commutative with identity. If A is k-linear, show that this ring a a k-algebra.

3.3 If A is an abelian category show that the $\mathrm{Hom}(\mathsf{A},\mathsf{A})$ is a monoidal category with $\otimes$ given by composition.

2.4 Direct and inverse limits

Direct and inverse limits are special kinds of colimits and limits. A *direct limit* (respectively, an *inverse limit*) is a colimit (respectively, a limit) taken over a direct set. Our convention that an index set is small remains in force.

Definition 4.1 A set I with a reflexive and transitive binary relation $\leq$ is said to be quasi-ordered. (It is possible for $i \leq j$ and $j \leq i$ with $i \neq j$, so I is not necessarily partially ordered.) If, in addition, for each pair $i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$, we say that I is directed. A subset $J \subset I$ is cofinal in I if, for each $i \in I$, there exists $j \in J$ with $i \leq j$. ◊

We often prefer to treat a quasi-ordered set as a category. The objects of the category I are the elements of the quasi-ordered set, and the morphisms are

$$\mathrm{Hom}_{\mathsf{I}}(i,j) = \begin{cases} \text{singleton} & \text{if } i \leq j, \\ \emptyset & \text{otherwise.} \end{cases}$$

If $i \leq j$ we write ι_j^i for the unique morphism $i \to j$, and define the composition $\iota_k^j \circ \iota_j^i = \iota_k^i$ whenever $i \leq j \leq k$. Conversely, if I is a category such that there is at most one element in $\mathrm{Hom}(i,j) \cup \mathrm{Hom}(j,i)$ for all i and j, then I is equivalent to a category arising from a quasi-ordered set; we then say that I is a quasi-ordered category.

Definition 4.2 Let I be a directed set. A directed system in a category C, with index set I, consists of objects $\{M_i \mid i \in I\}$ and morphisms $\varphi_j^i : M_i \to M_j$ whenever $i \leq j$, which satisfy:

- $\varphi_i^i = \mathrm{id}_{M_i}$, and
- $\varphi_k^j \circ \varphi_j^i = \varphi_k^i$ whenever $i \leq j \leq k$.

More succinctly, if I is a quasi-ordered category, a directed system over I in C is a covariant functor $\mathsf{I} \to \mathsf{C}$.

Given such a directed system, a collection of morphisms $\psi_i : M_i \to N$, $i \in I$, such that $\psi_i = \psi_j \varphi_j^i$ whenever $i \leq j$ is said to be compatible with the directed system. ◊

The directed systems over I in C form a category $\mathsf{Fun}(\mathsf{I}, \mathsf{C})$. If C is abelian, so is $\mathsf{Fun}(\mathsf{I}, \mathsf{C})$.

Definition 4.3 Let (M_i, φ_j^i) be a directed system in C, indexed by I. A direct limit of this system is an object $\varinjlim M_i$ in C together with morphisms

$$\varphi_i : M_i \to \varinjlim M_i$$

such that

- $\varphi_i = \varphi_j \varphi_j^i$ whenever $i \leq j$, and
- if $\psi_i : M_i \to N$, $i \in I$, is a set of compatible morphisms

$$\begin{array}{ccc} M_i & \xrightarrow{\varphi_i} & \varinjlim M_i \\ {\scriptstyle \psi_i}\downarrow & & \\ N & & \end{array}$$

then there exists a unique morphism $\rho : \varinjlim M_i \to N$ such that $\psi_i = \rho \circ \varphi_i$ for all i.

If every directed system in C has a direct limit in C we say that C has direct limits. $\diamondsuit$

The next result says that a direct sum is a special case of a direct limit.

Lemma 4.4 *Suppose that the direct sum $\bigoplus_{i \in I} M_i$ exists. For each finite subset $F \subset I$ let $M_F = \bigoplus_{i \in F} M_i$. Then*

$$\bigoplus_{i \in I} M_i = \varinjlim M_F,$$

where the limit is taken over all finite subsets $F \subset I$.

Proof. Throughout this proof F, G, and H, denote finite subsets of I.

For each $i \in F$, let $\alpha_F^i : M_i \to M_F$ be the canonical inclusion existing by virtue of the fact that M_F is the direct sum of the M_is for $i \in F$. In particular $\alpha_{\{i\}}^i = \mathrm{id}_{M_i}$. When $i \in F \subset G$ the existence of the maps $\alpha_G^i : M_i \to M_G$ and the universal property of M_F imply that there is a unique map $\alpha_G^F : M_F \to M_G$ such that $\alpha_G^i = \alpha_G^F \alpha_F^i$ for all $i \in F$. Taking $F = \{i\}$ this gives $\alpha_G^i = \alpha_G^{\{i\}} \alpha_{\{i\}}^i = \alpha_G^{\{i\}}$. It is now straightforward to check that the M_F form a directed system: the equality $\alpha_H^G \alpha_G^F = \alpha_H^F$ for $F \subset G \subset H$ follows from the uniqueness of the α_G^F.

Let $\alpha_i : M_i \to \bigoplus M_j$ be the canonical inclusions. There is a unique map $\alpha_F : M_F \to \bigoplus M_j$ such that $\alpha_i = \alpha_F \alpha_F^i$ for all $i \in F$. We will show that the maps α_F give $\bigoplus M_j$ the appropriate universal property to be the direct limit of the M_Fs. To this end, suppose given maps $\beta_F : M_F \to N$ such that $\beta_F = \beta_G \alpha_G^F$ when $F \subset G$. We must show there is a unique map $\beta : \bigoplus M_j \to N$ such that $\beta_F = \beta \alpha_F$ for all F.

By the universal property of the direct sum there is a map $\beta : \bigoplus M_j \to N$ such that $\beta \alpha_i = \beta_{\{i\}}$ for all i. If $i \in F$, then $\beta \alpha_F \alpha_F^i = \beta \alpha_i = \beta_{\{i\}} = \beta_F \alpha_F^i$. It follows that $\beta \alpha_F = \beta_F$, as required.

It remains to check the uniqueness of β. Suppose that $\beta' : \bigoplus M_j \to N$ is such that $\beta_F = \beta' \alpha_F$ for all F. Then $\beta_{\{i\}} = \beta' \alpha_i$ for all i so $\beta' = \beta$. $\square$

Example 4.5 Let x be a point in a topological space X. Then the open sets containing x form a directed system if we declare that $U \leq V$ whenever V is a subset of U. Let $\mathcal{F}$ be a presheaf of abelian groups on X. The stalk of $\mathcal{F}$ at x is

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U),$$

the direct limit of the sections of $\mathcal{F}$ over the open sets containing x. ◊

Example 4.6 Let R be a commutative domain, and $\mathfrak{p}$ a prime ideal. Then the localization $R_\mathfrak{p}$ may be realized as the direct limit of the localizations $R[u^{-1}]$, where the limit is indexed by the elements u in $R \backslash \mathfrak{p}$, and $u \leq v$ if $v \in uR$. Indeed, $R_\mathfrak{p}$ is the stalk at $\mathfrak{p}$ of the structure sheaf for $\operatorname{Spec} R$. ◊

Example 4.7 Although colimits need not exist in Set (Exercise **??**), direct limits do. The direct limit of (M_i, φ_j^i) may be constructed as follows. Define an equivalence relation $\sim$ on the disjoint union $\coprod M_i$ as follows: if $x_i \in M_i$ and $x_j \in M_j$, then $x_i \sim x_j$ if $\varphi_k^i(x_i) = \varphi_k^j(x_j)$ for some k such that $i \leq k$ and $j \leq k$. The hypothesis that I is directed is needed to prove the transitivity of $\sim$. Now define $\varinjlim M_i = \coprod M_i / \sim$, and define $\varphi_i : M_i \to \varinjlim M_i$ to be the composition $M_i \to \coprod M_i \to \varinjlim M_i$. ◊

The construction in the previous example is similar to the construction of the direct limit in the next result.

Proposition 4.8 *Let (M_i, φ_j^i) be a directed system. Let $\alpha_j : M_j \to \bigoplus M_i$ be the canonical injections. Define*

$$S = \{(i, j) \in I \times I \mid i \leq j\}.$$

Then there is an exact sequence

$$0 \longrightarrow \textstyle\sum_{(i,j) \in S} (\alpha_j \varphi_j^i - \alpha_i)(M_i) \longrightarrow \bigoplus_{i \in I} M_i \xrightarrow{\Psi} \varinjlim M_i \longrightarrow 0$$

provided these modules exist.

Proof. For each $(i, j) \in S$ write

$$\alpha_{ij} : M_i \to \bigoplus_{\{h \mid h \leq j\}} M_h$$

for the canonical inclusion. If $i \leq j$ there is a map $\alpha_j \varphi_j^i - \alpha_i : M_i \to \bigoplus M_k$. Hence there is a unique map $\mu : \bigoplus_{i \leq j} M_i \to \bigoplus M_k$ such that $\alpha_j \varphi_j^i - \alpha_i = \mu \alpha_{ij}$ for all $i \leq j$. Let $\Psi : \bigoplus M_k \to \operatorname{coker} \mu$ be the natural map. We will show that the maps $\Psi \alpha_i : M_i \to \operatorname{coker} \mu$ make $\operatorname{coker} \mu$ the direct limit of the directed system (M_i, φ_j^i). Suppose that $\psi_i : M_i \to N$ are maps such that $\psi_i = \psi_j \varphi_j^i$ for all $i \leq j$. The universal property of $\bigoplus M_k$ ensures that there is a map $\rho : \bigoplus M_k \to N$ such that $\psi_i = \rho \alpha_i$ for all i. Since $\rho \circ (\alpha_j \varphi_j^i - \alpha_i) = \psi_j \varphi_j^i - \psi_i = 0$ we have $\rho \mu = 0$. Hence ρ factors through $\operatorname{coker} \mu$; more explicitly, there is a map $\tau : \operatorname{coker} \mu \to N$

such that $\tau\Psi = \rho$. Thus $\tau\Psi\alpha_i = \rho\alpha_i = \psi_i$. To see that τ is unique subject to this equality, suppose that $\tau' : \operatorname{coker}\mu \to N$ is such that $\tau'\Psi\alpha_i = \psi_i$ for all i; then $(\tau - \tau')\Psi\alpha_i = 0$ for all i, so $\tau\Psi = \tau'\Psi$, whence $\tau = \tau'$ because Ψ is epic. □

Corollary 4.9 *Let* I *be a directed category. If* A *is a cocomplete abelian category, then* $\varinjlim$ *extends to a functor* $\varinjlim : \mathsf{Fun}(\mathsf{I}, \mathsf{A}) \to \mathsf{A}$.

Proof. This follows from the fact that if (M_i, φ^i_j) and (N_i, ψ^i_j) are directed systems over the same index set, and $f_i : M_i \to N_i$ are morphisms compatible with the φ^i_js and ψ^i_js, then there is an induced morphism $\varinjlim M_i \to \varinjlim N_i$. We denote that morphism by $\varinjlim f_i$. □

Example 1.3.12 shows that a product of epimorphisms need not be an epimorphism. Taking that example in the opposite category shows that a direct sum of monics need not be monic. Thus a direct limit of monics need not be monic. In the notation of the previous proof, if each f_i is monic, then $\varinjlim f_i : \varinjlim M_i \to \varinjlim N_i$ need not be monic. On the other hand, it follows from Proposition 4.11 that $\varinjlim$ is a left adjoint, so is right exact, and hence a direct limit of epics is epic. More precisely, if each f_i is epic, then $(f_i)_i$ viewed as a morphism in $\mathsf{Fun}(\mathsf{I}, \mathsf{A})$ is epic by Corollary 3.4, so $\varinjlim f_i$ is epic.

Definition 4.10 Let I be a directed set viewed as a category, and let C be a category. An inverse system in C, with index set I, is a contravariant functor $I \to \mathsf{C}$; that is, it consists of objects M_i, indexed by the elements of I, and morphisms $\psi^j_i : M_j \to M_i$ whenever $i \le j$ satisfying:

- $\psi^i_i = \mathrm{id}_{M_i}$, and
- $\psi^j_i \circ \psi^k_j = \psi^k_i$ whenever $i \le j \le k$.

An inverse limit of this inverse system is an object $\varprojlim M_i$ in C and a set of morphisms

$$\psi_i : \varprojlim M_i \to M_i$$

such that

- $\psi_i = \psi^j_i \psi_j$ whenever $i \le j$, and
- if N is any object, and $\psi_i : N \to M_i$ are morphisms such that $\psi_i = \psi^j_i \psi_j$ whenever $i \le j$, then there exists a unique morphism $\rho : N \to \varprojlim M_i$ such that $\psi_i = \psi_i \circ \rho$ for all i.

If every inverse system in C has an inverse limit we say that inverse limits exist in C. ◊

Inverse systems in C are the same things as direct systems in C^{op}.

Since small limits exist in Set, so do small inverse limits. Explicitly, if (M_i, ψ^j_i) is an inverse system then

$$\varprojlim M_i = \{(x_i) \in \prod M_i \mid \psi^j_i(x_j) = x_i \text{ whenever } i \le j\}.$$

Proposition 4.11 *Let* A *be an abelian category and* I *a directed category. Define*

$$G : \mathsf{A} \to \mathsf{Fun}(\mathsf{I}, \mathsf{A})$$

as follows. If M *is an* A*-module, then* $G(M) = G_M$ *is the constant functor* $G_M(i) = M$ *and* $G_M(\alpha) = \mathrm{id}_M$ *for all objects* i *and morphisms* α *in* I*. If* $f : M \to N$ *is a morphism in* A*, then* $G(f) : G_M \to G_N$ *is the obvious natural transformation.*

1. *If* A *is complete, then* $\varprojlim : \mathsf{Fun}(\mathsf{I}, \mathsf{A}) \to \mathsf{A}$ *is a right adjoint to* G*.*

2. *If* A *is cocomplete, then* $\varinjlim : \mathsf{Fun}(\mathsf{I}, \mathsf{A}) \to \mathsf{A}$ *is a left adjoint to* G*.*

Proof. This follows from the definition of direct and inverse limits. □

Proposition 4.11 is a special case of an analogous result for limits and colimits.

Example **??** gives a sheaf-theoretic interpretation of inverse systems indexed by the natural numbers.

Direct and inverse limits are the representing objects for certain functors. If (M_i, φ^i_j) is a directed system, then for every module N there is an induced map

$$\psi^j_i = \mathrm{Hom}(\varphi^i_j, N) : \mathrm{Hom}(M_j, N) \to \mathrm{Hom}(M_i, N)$$

making $(\mathrm{Hom}(M_i, N), \psi^j_i)$ an inverse system in Ab. If the direct limit $\varinjlim M_i$ exists, then the maps $\varphi_i : M_i \to \varinjlim M_i$ induce maps $\mathrm{Hom}(\varinjlim M_i, N) \to \mathrm{Hom}(M_i, N)$, and it is easy to check that this makes $\mathrm{Hom}(\varinjlim M_i, N)$ an inverse limit of the inverse system. In other words

$$\mathrm{Hom}(\varinjlim M_i, N) = \varprojlim \mathrm{Hom}(M_i, N).$$

Similarly, if (L_i, ψ^j_i) is an inverse system having an inverse limit, then

$$\mathrm{Hom}(N, \varprojlim L_i) = \varprojlim \mathrm{Hom}(N, L_i).$$

Chapter 6 of [66] says some useful things about inverse limits.

EXERCISES

4.1 Show that there is a category Graph whose objects are the directed graphs and a morphism $f : (V_1, E_1) \to (V_2, E_2)$ is a set map sending vertices to vertices and paths to paths in such a way that $f(E^\alpha_\beta) \subset E^{f(\alpha)}_{f(\beta)}$.

4.2 Let X be a topological space and let $\mathsf{Open}(X)$ be the category of open subsets of X, as described in Example 1.13.

(a) Show that $\mathsf{Open}(X)$ forms a directed set if we define $V \leq U$ whenever $U \subset V$. [*Hint:* if U and V are open so is $U \cap V$.]

(b) If $x \in X$ show that the subcategory consisting of those open sets which contain x is a directed system.

4.3 (Compare with part (2) of Lemma 5.5.) Let (M_i, φ^i_j) be a directed system in $\mathsf{Mod}R$. Identify $\varinjlim M_i$ with $\coprod M_i/N$, as in Proposition 4.8. Show that

(a) every element in $\varinjlim M_i$ is an image of an element in some M_i, that is, of the form $x_i + N$ for some $x_i \in M_i$;

(b) $x_i + N = 0$ if and only if $\varphi^i_j x_i = 0$ for some $j \geq i$.

4.4 Let p be a prime number. For $m \leq n$ define $\varphi^n_m : \mathbb{Z}/(p^n) \to \mathbb{Z}/(p^m)$ to be the natural projection. Show this is an inverse system of rings and that its inverse limit exists in the category of rings. That inverse limit is called the ring of p-adic integers.

4.5 Let (M_i, φ^i_j) be a direct system in C, indexed by I. Show that if J is cofinal in I, then

$$\varinjlim{}_I M_i \cong \varinjlim{}_J M_i.$$

In particular, if there is a unique element $\omega \in I$ such that $i \leq \omega$ for all $i \in I$, then $\varinjlim M_i \cong M_\omega$.

4.6 Let (M_i, φ^i_j) be a direct system, indexed by an ordered set I. If there exists $k \in I$ such that φ^k_l is an isomorphism for all $k \leq l$, show that

$$\varinjlim M_i \cong M_k.$$

4.7 Do Exercise 2 in [117, Volume II, Section 2.5]. This gives an example where an inverse limit does not exist.

4.8 Inverse limits in Ring.

4.9 Let $f : Y \to X$ be a map of topological spaces, and let $\mathcal{F}$ be a sheaf of abelian groups on Y. Show that $f^{-1}\mathcal{F}$ defined by

$$(f^{-1}\mathcal{F})(U) = \varinjlim{}_{V \supset f(U)} \mathcal{F}(V),$$

where U is open in Y and the limit is taken over the open sets in X containing $f(U)$, is a sheaf on Y. Further, show that if $p \in Y$, then $(f^{-1}\mathcal{F})_p \cong \mathcal{F}_{f(p)}$.

2.5 Grothendieck's conditions Ab3, Ab4, and Ab5

Throughout this section A will denote an abelian category.

Definition 5.1 An abelian category is

- Ab3 if it has arbitrary direct sums (and hence arbitrary colimits);
- Ab4 if it satisfies Ab3 and direct sums are exact;
- Ab5, or has exact direct limits, if it satisfies Ab3, and whenever $\{L_i\}$, $\{M_i\}$, $\{N_i\}$ are directed systems over a common directed set, and there are exact sequences $L_i \to M_i \to N_i$ compatible with the maps in the directed systems, then the induced sequence $\varinjlim L_i \to \varinjlim M_i \to \varinjlim N_i$ is exact.

We say that A satisfies the dual conditions $\mathsf{Ab3}^*$, $\mathsf{Ab4}^*$, $\mathsf{Ab5}^*$ if A^{op} satisfies $\mathsf{Ab3}$, $\mathsf{Ab4}$, $\mathsf{Ab5}$ respectively. $\Diamond$

Remarks. 1. In an Ab3 category every small family of submodules of a given module has a sum, and in an Ab3* category every small family of submodules has an intersection (see section 2).

2. The opposite of the subcategory of Ab consisting of the torsion abelian groups does not satisfy the condition Ab4: Example 1.3.12 exhibits epimorphisms $f_m : \mathbb{Z}_{2^m} \to \mathbb{Z}_2$ such that the product $\prod f_m$ is not epic.

Proposition 5.2 *Suppose that A has products and coproducts. Then the following conditions are equivalent:*

1. *for any set of modules $\{M_i \mid i \in I\}$ the natural map*
$$\Psi : \bigoplus M_i \to \prod M_i$$
(see Proposition 1.4) is monic;
2. *for any set of modules $\{M_i \mid i \in I\}$ and any map $f : K \to \bigoplus M_j$, f is zero if and only if $\gamma_i f = 0$ for all i, where $\gamma_i : \bigoplus M_j \to M_i$ are the canonical projections.*

If either of these conditions holds, then A satisfies Ab4.

Proof. (1) $\Rightarrow$ (2) If $\rho_i : \prod M_j \to M_i$ are the canonical projections, then $\gamma_i = \rho_i \Psi$ by Proposition 1.4. Thus, if $f : K \to \bigoplus M_j$ is such that $\gamma_i f = 0$ for all i, then $\rho_i \Psi f = 0$ for all i, whence $\Psi f = 0$ by the universal property of $\prod M_j$. But Ψ is monic by hypothesis (1), so $f = 0$.

(2) $\Rightarrow$ (1) The inclusion $f : \ker \Psi \to \bigoplus M_j$ is such that $\Psi f = 0$, so $\rho_i \Psi f = 0$ for all i, whence $\gamma_i f = 0$ for all i. Thus hypothesis (2) implies that $f = 0$; that is, $\ker \Psi = 0$ as claimed.

Finally suppose that condition (1) holds. Let $f_i : M_i \to N_i$ be monics. There is a commutative diagram

$$\begin{array}{ccc} \coprod M_i & \xrightarrow{\coprod f_i} & \coprod N_i \\ {\scriptstyle \Psi_1}\downarrow & & \downarrow{\scriptstyle \Psi_2} \\ \prod M_i & \xrightarrow{\prod f_i} & \prod N_i \end{array}$$

By Lemma 1.3.10, $\prod f_i$ is monic, so $(\prod f_i) \circ \Psi_1$ is monic. Hence $\Psi_2 \circ (\coprod f_i)$ is monic. Condition (1) now implies that $\coprod f_i$ is monic. $\square$

Example 5.3 The conditions in Proposition 5.2 do not always hold. For example, take A to be the opposite of $\mathsf{Mod}k[x]$. The product in A coincides with the coproduct in $\mathsf{Mod}k[x]$. If the map $\bigoplus M_i \to \prod M_i$ were monic in A, then in $\mathsf{Mod}k[x]$ the map $\bigoplus M_i \to \prod M_i$ would be epic. In particular, the direct product of all simple $k[x]$-modules would be semisimple; but $k[x]$ is isomorphic to a submodule of this, so it too would be a semisimple $k[x]$-module. $\Diamond$

$\boxed{\text{Paul}}$ Does $(\mathsf{Mod}k[x])^{\mathrm{op}}$ satisfy Ab4? It does not satisfy Ab5 because inverse limits are not exact in $\mathsf{Mod}k[x]$.

Definition 5.4 A directed family of submodules of M is a collection of submodules $\{M_i \mid i \in I\}$ of M such that I becomes a directed set when we define $i \leq j$ if $M_i \subset M_j$. ◊

Lemma 5.5 *Let $\{M_i \mid i \in I\}$ with inclusions $\beta_i : M_i \to M$, be a directed family of submodules of a module M. Then*

1. $\varinjlim M/M_i \cong M/\sum_{i \in I} M_i$;
2. *if* A *is Ab5, then* $\varinjlim M_i = \sum M_i$;
3. *if* A *is Ab5, then* $\psi_j : \mathrm{Hom}(\sum M_i, N) \to \mathrm{Hom}(M_j, N)$ *where* $\psi_j(\tau) = \tau\beta_i$ *make* $\mathrm{Hom}(\sum M_i, N)$ *isomorphic to the inverse limit* $\varprojlim \mathrm{Hom}(M_i, N)$.

Proof. (1) Taking direct limits of the sequences $0 \to M_i \to M \to M/M_i \to 0$ yields an exact sequence

$$\varinjlim M_i \xrightarrow{\ h\ } M \longrightarrow \varinjlim M/M_i \longrightarrow 0.$$

Let $\theta_i : M_i \to \varinjlim M_i$ and $\alpha_i : M_i \to \bigoplus M_i$ be the maps occuring in the definition of the direct limit and direct sum. Then $h\theta_i = \beta_i$ for all i. There is a unique map $g : \bigoplus M_i \to M$ satisfying $g\alpha_i = \beta_i$. By Proposition 4.8, there is an epimorphism $f : \bigoplus M_i \to \varinjlim M_i$ such that $f\alpha_i = \theta_i$ for all i. Therefore

$$hf\alpha_i = h\theta_i = \beta_i$$

for all i. By the uniqueness of g, it follows that $hf = g$. Since f is epic, $\mathrm{im}\, g = \mathrm{im}\, h = \sum M_i$, and the result follows.

(2) If A satisfies Ab5, then h is monic, so is an isomorphism onto $\sum M_i$.

(3) By definition of $\varinjlim$, $\mathrm{Hom}(\varinjlim M_i, -) = \varprojlim \mathrm{Hom}(M_i, -)$ so, after (2), proving (3) is simply a matter of verifying that the maps are the correct ones. We leave that to the reader. □

Theorem 5.6 *The following conditions on an Ab3 category* A *are equivalent:*

1. A *satisfies Ab5;*
2. *for every morphism $f : M \to N$ and every directed family $\{N_j \mid j \in J\}$ of submodules of N,*

$$f^{-1}\left(\sum_{j \in J} N_j\right) = \sum_{j \in J} f^{-1}(N_j);$$

3. *for every directed system of submodules* $\{M_j \mid j \in J\}$ *of a module* M *and every submodule* L *of* M,

$$\Big(\sum M_j\Big) \cap L = \sum \Big(M_j \cap L\Big);$$

4. *for every directed system* $\{M_i \mid i \in I\}$, *we have*

$$\ker(M_i \to \varinjlim M_r) = \sum_{i \le j} \ker(M_i \to M_j).$$

Proof. (1) $\Rightarrow$ (2) The defining property of $f^{-1}(N_j)$ ensures that there is an exact sequence $0 \to f^{-1}(N_j) \to M \to N/N_j$. Taking the direct limit of these sequences yields the exact sequence in the top row of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim f^{-1}(N_j) \arrow[r] & M \arrow[r] \arrow[d, "="] & \varinjlim N/N_j \arrow[d, "\cong"] \\
0 \arrow[r] & f^{-1}(\sum N_j) \arrow[r] & M \arrow[r] & N/\sum N_j,
\end{tikzcd}$$

where the vertical isomorphism comes from part (1) of Lemma 5.5. It follows that $f^{-1}(\sum N_j)$ equals the image of $\varinjlim f^{-1}(N_j)$ in M; but that image is $\sum f^{-1}(N_j)$ by part (2) of Lemma 5.5.

(2) $\Rightarrow$ (3) Apply (2) to the inclusion $f : L \to M$.

(3) $\Rightarrow$ (2) The $f^{-1}(N_j)$ form a directed family of submodules of M, and fit into exact sequences.

$$0 \to f^{-1}(N_j) \to M \to f(M)/f(M) \cap N_j \to 0.$$

Therefore

$$\begin{aligned}
M/\sum f^{-1}(N_j) &= \varinjlim M/f^{-1}(N_j) \qquad \text{by Lemma 5.5(1)} \\
&\cong \varinjlim f(M)/f(M) \cap N_j \\
&= f(M)/\sum (f(M) \cap N_j) \qquad \text{by Lemma 5.5(1)} \\
&= f(M)/f(M) \cap (\sum N_j) \qquad \text{by hypothesis (3)} \\
&\cong M/f^{-1}(\sum N_j).
\end{aligned}$$

It follows that $f^{-1}(\sum N_j) = \sum f^{-1}(N_j)$ as required.

(2) $\Rightarrow$ (4) Let $\varphi_j^i : M_i \to M_j$ for $i \le j$ be the maps in the directed system, and $\varphi_i : M_i \to \varinjlim M_r$ the canonical maps. We must show that $\ker \varphi_i = \sum_{i \le j} \ker \varphi_j^i$. If $i \le j$, then $\varphi_i = \varphi_j \varphi_j^i$, so $\ker \varphi_j^i \subset \ker \varphi_i$.

If $\alpha_i : M_i \to \bigoplus M_r$ is the canonical inclusion, then by the proof of Proposition 4.8 there is an epimorphism $\Psi : \bigoplus M_r \to \varinjlim M_r$ such that $\varphi_i = \Psi \alpha_i$ for all i, and

$$\ker \Psi = \sum_S (\alpha_k \varphi_k^j - \alpha_j)(M_j)$$

where

$$S = \{(j,k) \in I \times I \mid j \leq k\}.$$

For each subset $F \subset S$, write

$$K_F := \sum_{(j,k)\in F} (\alpha_k \varphi_k^j - \alpha_j)(M_j).$$

Therefore

$$\begin{aligned}
\ker \varphi_i &= \alpha_i^{-1}(\ker \Psi) \\
&= \alpha_i^{-1}\Bigg(\sum_{\text{finite } F\subset S} K_F \Bigg) \\
&= \sum_{\text{finite } F\subset S} \alpha_i^{-1}(K_F)
\end{aligned}$$

where the last equality follows from hypothesis (2). We must therefore show that given a finite $F \subset S$, there is an $m \in I$ such that $\alpha_i^{-1}(K_F) \subset \ker \varphi_m^i$.

For a fixed finite $F \subset S$, choose $m \in I$ such that $m \geq i$ and $m \geq k$ for all $(j,k) \in F$. There is a unique map $\theta : \bigoplus M_r \to M_m$ such that

$$\theta\alpha_r = \begin{cases} \varphi_m^r & \text{if } r \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\theta \circ (\alpha_k \varphi_k^j - \alpha_j) = 0$ if $(j,k) \in F$. In other words, $K_F \subset \ker \theta$, whence

$$\alpha_i^{-1}(K_F) \subset \alpha_i^{-1}(\ker \theta) = \ker \theta\alpha_i = \ker \varphi_m^i.$$

This proves (4).

(4) $\Rightarrow$ (1) Let $\psi_j^i : L_i \to L_j$ and $\varphi_j^i : M_i \to M_j$ be directed systems indexed by a common set I. Suppose that $\theta_i : L_i \to M_i$ are monics such that $\varphi_j^i\theta_i = \theta_j\psi_j^i$ for all $i \leq j$. There are commutative diagrams

$$\begin{tikzcd}
L_i \arrow[r, "\theta_i"] \arrow[d, "\psi_i"'] & M_i \arrow[d, "\varphi_i"] \\
\varinjlim L_i \arrow[r, "\theta"'] & \varinjlim M_i.
\end{tikzcd}$$

Let $K = \ker \theta$. Since $\varinjlim L_i = \sum \psi_i(L_i)$,

$$K = K \cap \sum \psi_i(L_i) = \sum K \cap \psi_i(L_i)$$

by (3), so it suffices to prove that $K \cap \psi_i(L_i) = 0$ for all i.

Suppose to the contrary that $K \cap \psi_i(L_i) \neq 0$. Then $\psi_i^{-1}(K) \neq 0$, and

$$\psi_i^{-1}(K) = \ker \theta\psi_i = \ker \varphi_i\theta_i = \theta_i^{-1}(\ker \varphi_i).$$

By (4), this equals

$$\theta_i^{-1}(\sum_{i\leq j} \ker \varphi_j^i)$$

which equals

$$\sum_{i\leq j} \theta_i^{-1}(\ker \varphi_j^i) = \sum_{i\leq j} \ker \varphi_j^i \theta_i = \sum_{i\leq j} \ker \theta_j \psi_j^i = \sum_{i\leq j} \ker \psi_j^i = \ker \psi_i.$$

That is, $\ker \psi_i = \psi_i^{-1}(K)$. Therefore

$$K \cap \psi_i(L_i) = \psi_i(\psi_i^{-1}(K)) = \psi_i(\ker \psi_i) = 0,$$

contradicting the choice of i. $\square$

Corollary 5.7 *The category of modules over a ring is Ab5.*

Proof. By Proposition 4.8, $\mathsf{Mod}R$ has direct limits. It is easy to verify that condition (3) in Theorem 5.6 holds. An explicit proof that $\varinjlim$ is left exact can be found in [200, Theorem 2.18]. $\square$

The next example shows that $\mathsf{Mod}R$ is not an Ab5* category in general. Thus $(\mathsf{Mod}R)^{\mathrm{op}}$ is a typical abelian category in which direct limits fail to be exact.

Example 5.8 Inverse limits in $\mathsf{Mod}R$ need not be exact. If R is a discrete valuation ring with maximal ideal $\mathfrak{m}$ there are inverse systems (with obvious maps)

$$\begin{gathered} \cdots \to \mathfrak{m}^n \to \cdots\cdots \to \mathfrak{m}^2 \to \mathfrak{m}, \\ \cdots \to R \to \cdots\cdots \to R \to R, \\ \cdots \to R/\mathfrak{m}^n \to \cdots\cdots \to R/\mathfrak{m}^2 \to R/\mathfrak{m}. \end{gathered}$$

For each $n \geq 1$ there is an exact sequence

$$0 \to \mathfrak{m}^n \to R \to R/\mathfrak{m}^n \to 0. \tag{5-1}$$

Now $\varprojlim \mathfrak{m}^n = 0$ since $\bigcap \mathfrak{m}^n = 0$, and $\varprojlim R/\mathfrak{m}^n = \hat{R}$, the $\mathfrak{m}$-adic completion of R. Therefore taking inverse limits of the individual terms in the short exact sequences (5-1) gives a complex $0 \to 0 \to R \to \hat{R} \to 0$ which fails to be exact if R is not complete. $\lozenge$

Definition 5.9 An inverse system of abelian groups $(M_i, \varphi_i^j, i \in \mathbb{N})$, satisfies the Mittag-Leffler condition if, for each n, there exists $n_0 \geq n$ such that the image of $M_i \to M_n$ equals the image of $M_j \to M_n$ whenever $i, j \geq n_0$. $\lozenge$

Proposition 5.10 *[?, ???] Suppose that there are morphisms of inverse systems of abelian groups*

$$(L_i, \varphi_i^j) \to (M_i, \psi_i^j) \to (N_i, \phi_i^j),$$

where each system is indexed by $i \in \mathbb{N}$, and that the sequence $0 \to L_i \to M_i \to N_i \to 0$ is exact for all i. If (L_i) satisfies the Mittag-Leffler condition, then the sequence

$$0 \to \varprojlim L_i \to \varprojlim M_i \to \varprojlim N_i \to 0$$

is exact.

The next two results establish properties of Ab5 categories that are not shared by all abelian categories. With regard to the first of these, recall that Example 2.8 showed that in $(\mathsf{Mod}k[x])^{\mathrm{op}}$, which is not Ab5, a sum of simple modules need not be isomorphic to a direct sum of simple modules.

Proposition 5.11 *Suppose that $M = \bigoplus_{i \in I} S_i$ is a direct sum of simple modules in an Ab5 category. If L is a submodule of M, then there is a subset $J \subset I$ such that*

$$M = L \bigoplus (\bigoplus_{i \in J} S_i).$$

Proof. Give the set

$$\mathcal{S} := \{J \subset I \mid L \cap (\bigoplus_{j \in J} S_j) = 0\}.$$

a partial ordering by inclusion. Let $J_1 \subset J_2 \subset \dots$ be an ascending chain of elements in $\mathcal{S}$ and set $J = \cup_n J_n$. We may apply part 2 of Theorem 5.6 to the directed family of submodules $\bigoplus_{j \in J_n} S_j$ of M. It follows that

$$L \cap (\bigoplus_{j \in J} S_j) = L \cap (\sum_i \bigoplus_{j \in J_n} S_j) = \sum_i L \cap (\bigoplus_{j \in J_n} S_j) = 0.$$

Therefore J belongs to $\mathcal{S}$. Hence we can apply Zorn's lemma. Let J be a maximal member of $\mathcal{S}$. If $L + \sum_{j \in J} S_j$ is not equal to M, then some S_{n_0} is not a submodule of this sum. Therefore $J \cup \{n_0\}$ is in $\mathcal{S}$, contradicting the maximality of J. Thus $M = L + \sum_{j \in J} S_j$. □

Proposition 5.12 *Every sum of simple modules in and Ab5 category is isomorphic to a direct sum of simples.*

Proof. Every sum of simple modules is a quotient of a direct sum of simple modules. However, if M/L is such a quotient, then $M = L \oplus C$ where C is itself a direct sum of simples by Proposition 5.11. Since $M/L \cong C$ the result is true. □

Proposition 5.13 *If* A *satisfies Ab5, then the natural map*

$$\Psi : \bigoplus M_i \to \prod M_i$$

(see Proposition 1.4) is monic.

Proof. (Van den Bergh) By Lemma 4.4, $\bigoplus_{i \in I} M_i$ is the direct limit

$$\varinjlim{}_{F \subset I} \bigoplus_{i \in F} M_i$$

taken over all finite subsets $F \subset I$. We may view $\prod_{i \in I} M_i$ as the direct limit over such $F \subset I$ of the constant directed system. For each F, we have a monic

$$\bigoplus_{i \in F} M_i \cong \prod_{i \in F} M_i \to \prod_{i \in I} M_i.$$

By hypothesis, $\varinjlim$ is left exact, so this gives the desired result. □

EXERCISES

5.1 In Proposition 5.10 show that

(a) if (L_i) and (N_i) both satisfy the Mittag-Leffler condition, so does (M_i);

(b) if (M_i) satisfies the Mittag-Leffler condition, so does (N_i).

2.6 Finiteness conditions

Throughout this section A will denote an abelian category.

Several different finiteness conditions are important. First, there are conditions on individual A-modules: the familiar notions of noetherian, finitely generated, and finitely presented for modules over rings may be extended to objects in an arbitrary abelian category. Second, there are finiteness conditions on the whole category. The most important such condition is that A have a generator or, more generally, that it have a small set of generators (Definition 7.1 and Lemma 7.2). Third, one can combine these two ideas and require that A be generated by a small set of modules with each module having some prescribed finiteness property.

The next lemma sets the stage for the definition of a compact module which follows it.

Lemma 6.1 *Let* $\{N_i \mid i \in I\}$ *be a family of* A*-modules indexed by a small set* I*. Let* $\gamma_i : \bigoplus N_j \to N_i$ *be the natural projections obtained in Proposition 1.4.*

1. *There is a natural transformation* $\bigoplus \operatorname{Hom}(-, N_i) \to \operatorname{Hom}(-, \bigoplus N_i)$.

2. If M is an A-module, then the map

$$\Phi : \bigoplus_{i \in I} \operatorname{Hom}_{\mathsf{A}}(M, N_i) \to \operatorname{Hom}_{\mathsf{A}}(M, \bigoplus_{i \in I} N_i) \tag{6-1}$$

has the following properties:

(a) if θ_j denotes the j^{th} component of a map $\theta \in \oplus \operatorname{Hom}(M, N_i)$, then $\gamma_j \circ \Phi(\theta) = \theta_j$ for all j.

(b) Φ is injective;

(c) if A is an Ab5 category, then the image of Φ consists of those $f : M \to \bigoplus N_i$ such that $\{i \in I \mid \gamma_i f \neq 0\}$ is finite.

Proof. (1) Let $\alpha_j : N_j \to \bigoplus_{i \in I} N_i$ be the canonical injections. If, in the following diagram, the maps ε_j are the canonical injections, then the universal property of $\bigoplus \operatorname{Hom}(M, N_i)$ ensures the existence of a unique map Φ such that the diagram

$$\begin{tikzcd} \operatorname{Hom}(M, N_j) \arrow[r, "\varepsilon_j"] \arrow[d, "\operatorname{Hom}(M,\alpha_j)"'] & \bigoplus \operatorname{Hom}(M, N_i) \\ \operatorname{Hom}(M, \bigoplus N_i) & \end{tikzcd} \tag{6-2}$$

commutes. To prove the existence of the natural transformation, one takes two copies of the triangle (6-2), one for a module M and the other for a module M' and then uses a map $g : M' \to M$ to connect the two triangles in an obvious way, and checks appropriate commutativity conditions.

(2a) The previous diagram fits into the larger diagram

$$\begin{tikzcd} \operatorname{Hom}(M, N_j) \arrow[r, "\varepsilon_j"] \arrow[d, "\operatorname{Hom}(M,\alpha_j)"'] & \bigoplus \operatorname{Hom}(M, N_i) \arrow[d, "\pi_j"] \\ \operatorname{Hom}(M, \bigoplus N_i) \arrow[r, "\operatorname{Hom}(M,\gamma_j)"'] & \operatorname{Hom}(M, N_j), \end{tikzcd}$$

where π_j is the canonical projection. Since the j^{th} component of θ is $\pi_j(\theta)$, to prove (2a) it suffices to show that $\operatorname{Hom}(M, \gamma_j) \circ \Phi = \pi_j$ for all j.

Since this is a statement about maps in the category of abelian groups it suffices to check it on each component of $\bigoplus \operatorname{Hom}(M, N_i)$. That is, we must verify that $\operatorname{Hom}(M, \gamma_j) \circ \Phi \circ \varepsilon_i = \pi_j \varepsilon_i$ for all $i \in I$. The left-hand side of this is

$$\operatorname{Hom}(M, \gamma_j) \circ \operatorname{Hom}(M, \alpha_i) = \operatorname{Hom}(M, \gamma_j \alpha_i) = \operatorname{Hom}(M, \delta_j^i)$$

where $\delta_j^i : N_i \to N_j$ is id_{N_i} if $i = j$ and zero otherwise. The right-hand side, $\pi_j \varepsilon_i$, equals $\mathrm{id}_{\operatorname{Hom}(M,N_i)}$ if $i = j$ and zero otherwise, so we have the desired equality.

(2b) The map Φ in (6-1) is injective because if θ is non-zero, then some component of it, say θ_j, is non-zero, whence $\Phi(\theta)$ is non-zero.

(2c) Suppose that f is in the image of Φ. Then $f = \Phi(\theta)$. But θ has only finitely many non-zero components, and $\gamma_i f = \theta_i$. Hence $\gamma_i f$ is non-zero for only finitely many i. Conversely, if $\gamma_i f$ is non-zero for only finitely many i, then there is a map $\theta \in \bigoplus \mathrm{Hom}(M, N_i)$ such that $\theta_i = \gamma_i f$ for all i. Now $\gamma_i(\Phi(\theta) - f) = \theta_i - \gamma_i f = 0$ for all i. It follows from Propositions 5.13 and 5.2 that $\Phi(\theta) - f = 0$. □

Definition 6.2 An A-module M is

- compact if $\mathrm{Hom}_{\mathsf{A}}(M, -)$ commutes with direct sums—that is, the map Φ in (6-1) is always an isomorphism;
- finitely presented if $\mathrm{Hom}_{\mathsf{A}}(M, -)$ commutes with direct limits;
- finitely generated if whenever $M = \sum_{i \in I} M_i$ for some directed family of submodules $\{M_i \mid i \in I\}$, there is an index i_0 such that $M = M_{i_0}$;
- coherent if it is finitely presented and all its finitely generated submodules are finitely presented. ◇

Remarks. 1. Popescu [182, Section 3.5] uses the word *small* rather than *compact*. We do not adopt Popescu's usage because we wish to reserve the word *small* to indicate that certain sets belong to the universe in which we are working. Neeman [172] calls an object M in a triangulated category *compact* if $\mathrm{Hom}(M, -)$ commutes with direct sums.

2. Let (N_i, φ^i_j) be a directed system in A. Then $\mathrm{Hom}_{\mathsf{A}}(M, N_i)$ becomes a directed system of abelian groups in an obvious way. The maps $N_i \to \varinjlim N_i$ yield compatible homomorphisms $\mathrm{Hom}_{\mathsf{A}}(M, N_i) \to \mathrm{Hom}_{\mathsf{A}}(M, \varinjlim N_i)$ of abelian groups, so the universal property of the direct limit in Ab yields a group homomorphism

$$\Theta : \varinjlim \mathrm{Hom}_{\mathsf{A}}(M, N_i) \to \mathrm{Hom}_{\mathsf{A}}(M, \varinjlim N_i) \tag{6-3}$$

as follows: an element of the left hand side is the image of some $f : M \to N_j$, and $\Theta(f)$ is the composition $M \to N_j \to \varinjlim N_i$. The A-module M is finitely presented if Θ is an isomorphism for all directed systems.

3. In general the map Θ is neither injective nor surjective. To see that Θ is not always injective, consider a k-vector space with basis $\{e_1, e_2, \dots\}$ and quotients $N_i = M/ke_1 + \dots + ke_i$. It follows from part (1) of Lemma 5.5 that $\varinjlim N_i = 0$, so the right-hand side of (6-3) is zero. However, the natural surjections $\pi_i : M \to N_i$ are elements in $\mathrm{Hom}_{\mathsf{A}}(M, N_i)$ that are not zero in $\varinjlim \mathrm{Hom}_{\mathsf{A}}(M, N_i)$ (Proposition 4.8).

4. A simple module is finitely generated.

Theorem 6.3 *If A satisfies Ab5, then the following conditions on an A-module M are equivalent:*

1. *M is compact;*

2. *every map* $f : M \to \bigoplus N_i$ *has its image contained in a finite direct sum of the* N_i*s;*

3. $\mathrm{Hom}_{\mathsf{A}}(M, -)$ *commutes with countable direct sums;*

4. *if* M *is a sum of a countable set of submodules, say* $M = \sum_{i=1}^{\infty} M_i$*, then* $M = \sum_{i=1}^{n} M_i$ *for some* $n < \infty$*.*

Proof. Throughout this proof $\Phi : \bigoplus \mathrm{Hom}(M, N_i) \to \mathrm{Hom}(M, \bigoplus N_i)$ is the map in (6-1).

Before proceeding we observe that condition (4) is equivalent to the condition that if $M_1 \subset M_2 \subset \dots$ is a chain of submodules of M such that $M = \sum M_i$, then $M = M_n$ for some n.

(1) $\Leftrightarrow$ (2) By definition, compactness of M is equivalent to the condition that Φ is always an isomorphism. But Φ is always injective, so this is equivalent to Φ being surjective. By part (2c) of Lemma 6.1, this is equivalent to the condition that $\{i \in I \mid \gamma_i f \neq 0\}$ is finite for all $f : M \to \oplus N_i$. But such f are precisely the maps whose image is contained in a *finite* direct sum of the N_is.

(1) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (1) If (1) failed there would be a map $f : M \to \bigoplus_{i \in I} N_i$ such that $\gamma_i f$ is non-zero for infinitely many i. In particular, there would be an infinite countable set J such that $\gamma_j f \neq 0$ for all $j \in J$. But then, if $\pi : \oplus_{i \in I} N_i \to \bigoplus_{j \in J} N_j$ is the natural map, πf would be an element of $\mathrm{Hom}(M, \bigoplus_{j \in J} N_j)$ that is not in the image of the map $\bigoplus_{j \in J} \mathrm{Hom}(M, N_j) \to \mathrm{Hom}(M, \bigoplus_{j \in J} N_j)$. This would contradict hypothesis (3), so we conclude that (1) must hold.

(3) $\Rightarrow$ (4) Let $M_1 \subset M_2 \subset \dots \subset M$ be a chain of submodules of M such that $M = \sum M_i$. Let $\beta_j : M_j \to M$ and $\beta_k^j : M_j \to M_k$, $j \leq k$, be the inclusions. By Lemma 5.5, the inverse limit of the inverse system

$$\psi_j^k : \mathrm{Hom}(M_k, \bigoplus M/M_i) \to \mathrm{Hom}(M_j, \bigoplus M/M_i)$$

defined by $\psi_j^k(g) = g\beta_k^j$ for $j \leq k$ is $\mathrm{Hom}(M, \bigoplus M/M_i)$ together with the maps

$$\mathrm{Hom}(M, \bigoplus M/M_i) \xrightarrow{\psi_j} \mathrm{Hom}(M_j, \bigoplus M/M_i)$$

defined by $\psi_j(g) = g\beta_j$.

Since the image of M_k in M/M_i is zero for $i \geq k$, the map $f_k : M_k \to \bigoplus M/M_i$ that is the composition

$$M_k \to \bigoplus_{i=1}^{k} M/M_i \to \bigoplus_{i=1}^{\infty} M/M_i.$$

satisfies $\psi_j^k(f_k) = f_k\beta_k^j = f_j$ for $j \leq k$. Hence there is a map $f : M \to \bigoplus M/M_i$ such that $\psi_j(f) = f_j$ for all j. That is, $f\beta_j = f_j$ for all j. By hypothesis, f is in the image of the map

$$\Phi : \bigoplus \mathrm{Hom}(M, M/M_i) \to \mathrm{Hom}(M, \bigoplus M/M_i)$$

so the image of f is contained in a *finite* direct sum of various M/M_is. The only way for that to happen is if $M = M_n$ for some n. Thus (4) is true.

(4) $\Rightarrow$ (3) (cf. [190]) We must show that Φ is surjective when $I = \mathbb{N}$. Let $f : M \to \bigoplus_{i \in \mathbb{N}} N_i$. Then

$$M = f^{-1}\Big(\bigoplus_{i \in \mathbb{N}} N_i\Big) = f^{-1}\Big(\sum_{n=0}^{\infty}\sum_{i=1}^{n} N_i\Big).$$

By the Ab5 hypothesis and Theorem 5.6, this equals $\sum_{n=0}^{\infty} f^{-1}(\sum_{i=1}^{n} N_i)$. It follows from hypothesis (4) applied to the submodules $f^{-1}(\sum_{i=1}^{n} N_i)$ of M that $M = f^{-1}(\sum_{i=1}^{n} N_i)$ for some n. Thus $f(M) \subset N_1 \oplus \ldots \oplus N_n$. □

Corollary 6.4 *Every finitely generated module in an Ab5 category is compact.*

Proof. Let M be a finitely generated module. We must show that every homomorphism $f : M \to \bigoplus_{i \in I} N_i$ factors through a finite direct sum of the N_i.

Write $N = \bigoplus_{i \in I} N_i$ and for each finite subset $F \subset I$ write $N_F = \bigoplus_{i \in F} N_i$. If M_F denotes the kernel of the composition

$$M \xrightarrow{\ f\ } \textstyle\bigoplus_{i \in I} N_i \longrightarrow N/N_F, \tag{6-4}$$

then there is an exact sequence $0 \to M_F \to M \to N/N_F \to 0$. We now take the direct limit over all finite subsets $F \subset I$. The direct limit of the modules N/N_F is computed by taking the direct limit of the exact sequences $0 \to N_F \to N \to N/N_F \to 0$. By Lemma 4.4 and the Ab5 hypothesis, $\varinjlim N/N_F = N/\varinjlim N_F = 0$. Thus $\varinjlim M_F = M$. By the proof of Lemma 5.5, $\sum M_F = \varinjlim M_F = M$. But M is finitely generated so $M = M_F$ for some F. Hence the image of θ is contained in N_F for some finite $F \subset I$. □

The converse to Corollary 6.4 is false.

Example 6.5 (Rentschler [190, 5°]) A compact module need not be finitely generated. To show this we construct a valuation ring A with field of fractions K such that $\operatorname{Hom}_A(K, -)$ commutes with direct sums despite the fact that K is not a finitely generated A-module.

Let Ω denote the smallest uncountable ordinal (see [103] for details about ordinals). Thus Ω is an uncountable partially ordered set that is characterized by the following three properties:

1. Ω is well-ordered; i.e., every non-empty subset of Ω has a smallest element;
2. for every $a \in \Omega$, $\{x \in \Omega \mid x < a\}$ is countable;
3. if $a \in \Omega$, then $a = \{x \in \Omega \mid x < a\}$.

Let I be the set of ordinals $i < \Omega$ endowed with the opposite order to its natural one. Thus every countable subset of I has a lower bound. For each $i \in I$, set $\mathbb{Z}_i = \mathbb{Z}$ and define the abelian group

$$\Gamma = \bigoplus_{i \in I} \mathbb{Z}_i.$$

We endow Γ with the lexicographic ordering, thus making it a totally ordered group with the property that every decreasing sequence in Γ has a lower bound.

Define $\Gamma^+ = \{\gamma \in \Gamma \mid \gamma \geq 0\}$.

Let $k\Gamma$ be the group algebra over a base field k. Thus, $k\Gamma$ is the localization $k[x_i, x_i^{-1} \mid i \in I]$ of the polynomial ring on the indeterminates x_i, $i \in I$. Let K be the field of fractions of $k\Gamma$. There is a valuation $\nu : K \to \Gamma \cup \{\infty\}$ defined by declaring

$$\nu\left(\sum_{\gamma \in \Gamma} a_\gamma \gamma\right) = \inf\{\gamma \mid a_\gamma \neq 0\}.$$

Let A be the valuation ring associated to ν.

By [49, Ch. 6, Proposition 7, p. 104] every non-zero A-submodule of K is of the form

$$\{x \in K \mid \nu(x) \in \Gamma'\} \cup \{0\}$$

for a unique subset Γ' of Γ with the property that $\Gamma' + \Gamma^+ \subset \Gamma'$. We call such a Γ' a cone. For example, the submodule corresponding to the cone $\gamma + \Gamma^+$ is $A\gamma$.

We claim that if Γ' is a cone that is not equal to Γ, then it has a lower bound. If it did not, then given $\gamma \in \Gamma$, there would be some γ' in Γ' such that $\gamma' \not\geq \gamma$. But Γ is totally ordered, so then $\gamma > \gamma'$; this would imply that $\gamma = \gamma' + (\gamma - \gamma') \in \Gamma' + \Gamma^+ \subset \Gamma'$; it would follow that $\Gamma' = \Gamma$.

To prove that K is a compact A-module, it suffices to show that if $M_1 \subset M_2 \subset \dots$ is an ascending chain of proper A-submodules of K, then their sum is not equal to K. Let Γ_j be the cone corresponding to M_j. Since M_j is not equal to K, Γ_j is not equal to Γ; it therefore has a lower bound, say γ_j. The sequence $\{\gamma_1, \gamma_2, \dots\}$ has a lower bound, say γ, so $\Gamma_j \subset \gamma + \Gamma^+$ for all j. Hence $M_j \subset A\gamma$ for all j. It follows that the sum of the M_js is not K, so we conclude that K is compact.

It is clear that K is not a finitely generated A-module. $\Diamond$

Proposition 6.6 *1. A finite direct sum of compact modules is compact.*

2. A quotient of a compact module is compact.

Proof. (1) Let K and L be compact modules. If $f : K \oplus L \to \bigoplus N_i$, then each of $f(K)$ and $f(L)$ is contained in a finite direct sum of N_is, so $f(K) \oplus f(L)$ is too. It follows from criterion (2) of Theorem 6.3 that $K \oplus L$ is compact.

(2) Suppose that L is compact, and K is a submodule of L. Then any map $f : L/K \to \bigoplus N_i$ lifts to a map $L \to \bigoplus N_i$ having the same image. Since L is compact that image lies in a finite direct sum of the N_is. By Theorem 6.3, L/K is compact. $\square$

The next result gives a characterization of finitely generated modules that is similar in spirit to the definitions of compact and finitely presented modules.

Proposition 6.7 *A module M is finitely generated if and only if, for ever directed family (N_i, φ^i_j) in which all the φ^i_j are monic, the canonical map*

$$\varinjlim \operatorname{Hom}(M, N_i) \to \operatorname{Hom}(M, \sum N_i)$$

is an isomorphism.

Proof. □

Proposition 6.8 *Suppose that $0 \to L \to M \to N \to 0$ is an exact sequence in an Ab5 category.*

1. *If M is finitely generated, so is N.*
2. *If L and N are finitely generated, so is M.*

Proof. (1) If $\{N_i \mid i \in I\}$ is a directed family of submodules of N whose sum is N, then their pre-images form a directed family of submodules of M. By the Ab5 hypothesis, sums commute with pre-images (Theorem 5.6), so M is the sum of the pre-images of the N_i. Since M is finitely generated it is therefore equal to the pre-image of some N_i, whence N is equal to some N_i.

(2) Let $\{M_i \mid i \in I\}$ be a directed family of submodules of M whose sum is M. Set $L_i = L \cap M_i$ and $N_i = M_i + L/L$. Then L_i and N_i are a directed families of submodules of L and N respectively. By the Ab5 hypothesis, $L = \sum L_i$. Since the image of a sum is the sum of the images, $N = \sum N_i$. Since L and N are finitely generated, and since I is directed, there is a single $k \in I$ such that $L = L_k$ and $N = N_k$. It follows that $M = M_k$. □

Proposition 6.9 *A finitely presented module in an Ab5 category is finitely generated.*

Proof. Let M_i, $i \in I$, be a directed family of submodules of a finitely presented module M, and suppose that $M = \sum M_i$. By Lemma 5.5, the natural map $\varinjlim M_i \to \sum M_i$ is an isomorphism. Because M is finitely presented the natural map

$$\varinjlim \operatorname{Hom}(M, M_i) \to \operatorname{Hom}(M, \varinjlim M_i)$$

is an isomorphism. But $\varinjlim M_i = M$, so the identity map id_M is the image of an element in $\varinjlim \operatorname{Hom}(M, M_i)$. Hence there is an index i_0 such that id_M is in the image of the natural map $\operatorname{Hom}(M, M_{i_0}) \to \operatorname{Hom}(M, M)$. It follows that $M = M_{i_0}$. □

Proposition 6.10 *[182, Corollary 5.7, page 91]. Let M be a module over a ring R.*

1. *M is finitely generated in the sense of Definition 6.2 if and only if it is finitely generated in the usual sense;*

2. *M is finitely presented in the sense of Definition 6.2 if and only if it is finitely presented in the usual sense.*

Proof. (1) ($\Rightarrow$) The finitely generated (in the usual sense) submodules of M form a directed system. Their sum is M so, if M is finitely generated in the sense of Definition 6.2, M must equal one of them.

($\Leftarrow$) Suppose that M is finitely generated in the usual sense, and let M_i, $i \in I$, be a directed family of submodules whose sum is M. Let $\{m_1, \dots, m_n\}$ be a set of generators for M. Each m_j belongs to a *finite* sum of the M_is so, since I is directed, there is a single $i_0 \in I$ such that M_{i_0} contains every m_j. Hence $M = M_{i_0}$.

(2) ($\Leftarrow$) Suppose there is an exact sequence $0 \to K \to F \to M \to 0$ with F finitely generated free, and K finitely generated. Let N_i be a directed system in $\mathsf{Mod}R$. We must prove that the map $\Theta : \varinjlim \operatorname{Hom}_R(M, N_i) \to \operatorname{Hom}_R(M, \varinjlim N_i)$ in (6-3) is an isomorphism. Injectivity is easy to prove, so we only prove surjectivity.

Let $f : M \to \varinjlim N_i$. Since F is finitely generated and the system is directed, the composition $F \to M \to \varinjlim N_i$ factors through $F \to N_j$ for some j. Because the composition $K \to F \to N_j \to \varinjlim N_i$ is zero and K is finitely generated, $K \to F \to N_j \to N_k$ is zero for some $k \geq j$. This yields a factorization of the original map $M \to N_k \to \varinjlim N_i$. If $g : M \to N_k$ is this factor, then $\Theta(g) = f$.

($\Rightarrow$) By Proposition 6.9 and part (1), M is finitely generated in the usual sense, so there is an exact sequence $0 \to K \to F \to M \to 0$ with F a free R-module having a finite basis. The finitely generated submodules of K form a directed system, say $\{K_i \mid i \in I\}$, and $K = \sum K_i = \varinjlim K_i$. Taking direct limits of the following diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_i & \longrightarrow & F & \longrightarrow & F/K_i & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle \theta_i} & & \\
0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

gives a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varinjlim K_i & \longrightarrow & F & \longrightarrow & \varinjlim F/K_i & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \cong} & & \big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle \theta} & & \\
0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

In particular, θ is an isomorphism. Thus

$$\theta^{-1} \in \operatorname{Hom}(M, \varinjlim F/K_i) \cong \varinjlim \operatorname{Hom}(M, F/K_i).$$

This means that for some $j \in I$ there is a map $\psi : M \to F/K_j$ such that $\theta^{-1} = \rho_j \psi$ where $\rho_j : F/K_j \to \varinjlim F/K_i$ is the canonical map. Since $\theta \rho_j \psi = \mathrm{id}_M$,

$\theta\rho_j : F/K_j \to M$ is a split epic. Since F and K_j are both finitely generated it follows that M is finitely presented. □

$\boxed{\text{Paul}}$ If P is finitely presented, is there an exact sequence $F \to G \to P \to 0$ with F and G finitely generated? Do we need to assume that we have a set of generators? P itself will be finitely generated.

Definition 6.11 An A-module M

- is noetherian if any increasing sequence of subobjects $M_1 \subset M_2 \subset \ldots$ of M is eventually stationary;
- is artinian if any decreasing sequence of subobjects $M_1 \supset M_2 \supset \ldots$ of M is eventually stationary;
- has finite length if it is both artinian and noetherian.

If M has finite length a composition series for M is a finite sequence of subobjects $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ such that each M_i/M_{i-1} is simple. The quotients M_i/M_{i-1} are called the composition factors of M; they are determined up to isomorphism by M. ◊

Lemma 6.12 *A noetherian module is finitely presented, and hence finitely generated.*

Proof. [182, Exercise 1, page 370] □

EXERCISES

6.1 Show that if A is an abelian category with a set of finitely generated generators, then the natural map $\bigoplus M_i \to \prod M_i$ is monic. [Hint: apply the functor $\mathrm{Hom}_{\mathsf{A}}(N, -)$ with N a finitely generated submodule of the kernel.]

6.2 Give an example of a non-zero module M in an abelian category such that every non-zero submodule and every non-zero quotient module of M is not finitely generated.

6.3 Show that an A-module M is noetherian if and only if every submodule of it is finitely generated.

6.4 Let $0 \to L \to M \to N \to 0$ be an exact sequence of A-modules. Show that M is noetherian if and only if both L and N are noetherian.

6.5 Show that an A-module M is finitely presented if and only if it is finitely generated and the kernel of every epimorphism $N \to M$ from a finitely generated module N is finitely generated.

6.6 Show that a module is noetherian if and only if it is artinian as a module in the dual category.

6.7 Show that a module M is noetherian if and only if any set of submodules of M has a maximal member with respect to the ordering by inclusion.

2.7 Generators

Definition 7.1 An object L in a category C is a generator if the functor $\mathrm{Hom}_{\mathsf{C}}(L,-)$ is faithful. In other words, if $f_1, f_2 : M \to N$ are distinct morphisms in C, then $f_1 g \neq f_2 g$ for some $g : L \to M$.

We say that C has a set of generators if there is a *small* set of objects L_i, $i \in I$, such that if $f_1, f_2 : M \to N$ are distinct morphisms in C, then there exists and $i \in I$ and a $g \in \mathrm{Hom}_{\mathsf{C}}(L_i, M)$ such that $f_1 g \neq f_2 g$. $\Diamond$

The canonical example of a generator is R_R, the right regular representation of a ring R, in $\mathsf{Mod}R$.

The existence of a set of generators is a finiteness condition.

Lemma 7.2 *Let A be an abelian category having a set of generators. If M is an A-module, then the collection of submodules of M is a small set.*

Proof. [88, Proposition 5, p. 336] Let $\{P_\lambda \mid \lambda \in \Lambda\}$ be a set of generators for A. By the definition of a category, each $\mathrm{Hom}_{\mathsf{A}}(P_\lambda, M)$ is a small set. Hence so is their disjoint union, E. If N is a submodule of M define $E_N := \{f \in E \mid \mathrm{im}\, f \subset N\}$. If N' is a submodule of M that does not contain N, then the natural morphism $N \to N/N \cap N'$ is non-zero. Hence there is some $f \in E$ such that $f \in E_N$ but $f \notin E_{N \cap N'}$. It follows that the map $N \mapsto E_N$ is injective on the submodules of M. Hence the collection of submodules is a small set. $\square$

Proposition 7.3 *The following conditions on a family of modules M_α, $\alpha \in I$, in an $\mathsf{Ab3}$-category A are equivalent.*

1. *$\{M_\alpha \mid \alpha \in I\}$ is a set of generators;*
2. *$M = \bigoplus_\alpha M_\alpha$ is a generator;*
3. *every A-module is a quotient of a suitably large direct sum of copies of M.*

Proposition 7.4 *If the ring R has an identity, then $\mathsf{Mod}R$ has a set of noetherian generators if and only if R is right noetherian.*

Proof. Suppose that $\mathcal{S}$ is a set of noetherian generators for $\mathsf{Mod}R$. Let M be the direct sum all these generators. If J is a right ideal of R that is not equal to R, then the map $R \to R/J$ is non-zero, so there is a map $\varphi : M \to R$ such that $\varphi(M) \not\subset J$. Hence $R = \sum \varphi_i(M)$ where the sum is taken over all $\varphi_i \in \mathrm{Hom}_R(M, R)$. Hence there is a surjective map from some large direct sum of noetherian modules onto R. Therefore 1 is in the image of a *finite* direct sum of noetherian modules. But that finite sum is noetherian, so we conclude that R is right noetherian.

The converse is trivial. $\square$

If R does not have an identity this result fails. For example, suppose that R is the direct sum of an infinite number of fields. Let $\mathsf{Mod}R$ consist of those R-modules such that $MR = M$. Although R is not noetherian, $\mathsf{Mod}R$ is generated by the simple R-modules.

EXERCISES

7.1 ??

2.8 Grothendieck categories

Definition 8.1 An abelian category is

- a Grothendieck category if it has a set of generators and satisfies Ab5;
- locally noetherian if it is a Grothendieck category and has a set of noetherian generators;
- locally finitely presented if it is a Grothendieck category and has a set of finitely presented generators;
- locally finitely generated if it is a Grothendieck category and has a set of finitely generated generators;
- locally finite if it is a Grothendieck category and has a set of finite length generators. ◊

Proposition 8.2 *If* A *is locally noetherian, then the following conditions on an* A*-module are equivalent:*

1. *M is noetherian;*
2. *M is finitely generated;*
3. *M is finitely presented;*
4. *M is compact.*

Proof. (1) $\Rightarrow$ (2) It follows immediately from the definitions.

(2) $\Rightarrow$ (1) Since A is locally noetherian, M is a sum of noetherian submodules. That sum can be assumed to be a sum of a directed family of noetherian submodules, so if M is finitely generated we conclude that M is noetherian.

(2) $\Rightarrow$ (3)

(3) $\Rightarrow$ (2) Proposition 6.9. □

2.9 Projectives

Throughout this section A will denote an abelian category.

Definition 9.1 An A-module P is projective if $\operatorname{Hom}_{\mathsf{A}}(P,-)$ is exact. We say that A has enough projectives if every A-module is isomorphic to a quotient of an projective module. $\diamondsuit$

The only projective in $\mathsf{Mod}\mathbb{P}^1$ is the zero module. This is typical for projective schemes.

Lemma 9.2 *An R-module is projective if and only if it is a direct summand of a free module.*

Proof. First we show that a free module is projective. Let F be free with basis $\{f_\lambda \mid \lambda \in \Lambda\}$. Let $\alpha : M \to N$ be surjective, and let $\beta : F \to N$ be any map. For each $\lambda \in \Lambda$, choose $m_\lambda \in M$ such that $\alpha(m_\lambda) = \beta(f_\lambda)$. There is an R-module map $\gamma : F \to M$ such that $\gamma(f_\lambda) = m_\lambda$ for all λ. Since β and $\alpha\gamma$ agree on the generators f_λ, they are equal. Hence the map $\operatorname{Hom}_R(F,M) \to \operatorname{Hom}_R(F,N)$ is surjective.

If F is free and $F = P \oplus Q$, then $\operatorname{Hom}_R(F,-) = \operatorname{Hom}_R(P,-) \oplus \operatorname{Hom}_R(Q,-)$, so $\operatorname{Hom}_R(P,-)$ is exact. Hence a direct summand of a free module is projective.

Conversely, let P be a projective R-module. There is a surjective map $\theta : F \to P$ from some free module F. The map $\operatorname{Hom}(P,F) \to \operatorname{Hom}(P,P)$ is surjective, so there is a map $\beta : P \to F$ such that $\alpha\beta = \mathrm{id}_P$. Hence P is a direct summand of F. $\square$

Definition 9.3 A finitely generated projective generator in an abelian category is called a progenerator. $\diamondsuit$

Lemma 9.4 *An exact functor between abelian categories is faithful if and only if it sends non-zero objects to non-zero objects.*

Proof. Let $F : \mathsf{A} \to \mathsf{B}$ be the functor.

($\Rightarrow$) Suppose that F is faithful. If M is a non-zero A-module, then id_M is non-zero, so $0 \neq F(\mathrm{id}_M) = \mathrm{id}_{FM}$, whence FM is non-zero.

($\Leftarrow$) If f is a non-zero morphism in A, then its image is non-zero, whence $F(\operatorname{im} f)$ is non-zero. But F is exact so commutes with kernels and cokernels, and hence with images. Thus $\operatorname{im}(Ff) = F(\operatorname{im} f)$ is non-zero by hypothesis, whence Ff is non-zero. $\square$

Thus a projective A-module P is a generator if and only if $\operatorname{Hom}_{\mathsf{A}}(P,M)$ is non-zero for every non-zero A-module M.

Proposition 9.5 *A projective module in an Ab5 category is finitely generated if and only if it is compact.*

Proof. By Corollary 6.4, a finitely generated module in an Ab5 category is compact so we must prove that a projective compact module is finitely generated. Let P be the module in question. By the proof of Proposition 4.8, the direct limit of a directed system can be written as the cokernel of a map between direct sums taken over the modules in the directed system. That is, in the notation of Proposition 4.8, there is an exact sequence

$$\bigoplus_{i \leq j} M_i \xrightarrow{\ \mu\ } \bigoplus M_k \longrightarrow \varinjlim M_k \longrightarrow 0.$$

If we apply $\mathrm{Hom}(P,-)$ to this we obtain an exact sequence because P is projective, and the compactness hypothesis says that $\mathrm{Hom}(P,-)$ commutes with direct sums. Hence $\mathrm{Hom}(P, \varinjlim M_k) \cong \varinjlim \mathrm{Hom}(P, M_k)$. Thus $\mathrm{Hom}(P,-)$ commutes with direct limits. It follows that P is finitely presented and, in particular, finitely generated. $\square$

Theorem 9.6 *Let P be a progenerator in a cocomplete abelian category A, and define the ring $R := \mathrm{Hom}_{\mathsf{A}}(P,P)$. Then*

$$\mathrm{Hom}_{\mathsf{A}}(P,-) : \mathsf{A} \to \mathsf{Mod}R$$

is an equivalence of categories.

Proof. Write $F = \mathrm{Hom}_{\mathsf{A}}(P,-)$. Then F is exact because P is projective, and is faithful because P is a generator. By Theorem 1.6.21, it remains to show that every right R-module is isomorphic to one of the form FM, and that F is full, i.e., that the map

$$F : \mathrm{Hom}_{\mathsf{A}}(M,N) \to \mathrm{Hom}_R(FM,FN) \tag{9-1}$$

is surjective for every M and N in A. If $\alpha \in \mathrm{Hom}_{\mathsf{A}}(M,N)$ then $F(\alpha)$ is the right R-module map $F(\alpha)(f) = \alpha \circ f$ for $f \in FM = \mathrm{Hom}_{\mathsf{A}}(P,M)$.

Fix an A-module. We consider the two sides of (9-1) as contravariant functors

$$G := \mathrm{Hom}_{\mathsf{A}}(-,N) \ : M \mapsto \mathrm{Hom}_{\mathsf{A}}(M,N)$$

and

$$H := \mathrm{Hom}_R(-,FN) \circ F \ : M \mapsto \mathrm{Hom}_R(FM,FN),$$

whence F may be viewed as a natural transformation $t : G \to H$; thus, we must show that $t_M : GM \to HM$ is an isomorphism for all M (i.e., that t is a natural equivalence).

First t_P is an isomorphism since, for each $\alpha \in \mathrm{Hom}_{\mathsf{A}}(P,P)$, $t_P(\alpha)$ is left multiplication by α, and the map $R \to \mathrm{Hom}_R(R_R,R_R)$ sending $\alpha \in R$ to left multiplication by α is an isomorphism.

Now let M be arbitrary. Since P is a generator there is an exact sequence $Q_1 \to Q_0 \to M \to 0$ with Q_1 and Q_0 isomorphic to direct sums of copies of P. Applying G and H yields a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & GM & \longrightarrow & GQ_0 & \longrightarrow & GQ_1 \\ & & {\scriptstyle t_M}\big\downarrow & & {\scriptstyle t_{Q_0}}\big\downarrow & & {\scriptstyle t_{Q_1}}\big\downarrow \\ 0 & \longrightarrow & HM & \longrightarrow & HQ_0 & \longrightarrow & HQ_1 \end{array}$$

of abelian groups with exact rows. By the previous paragraph, and the fact that F commutes with direct sums, the second and third vertical maps are isomorphisms. We must show that the first vertical map is an isomorphism; we already know it is injective since F is faithful, so it remains to show it is surjective. This results from standard diagram chasing. Thus t_M is an isomorphism.

Finally, if M is an R-module there is an exact sequence

$$M_1 \xrightarrow{\ \psi\ } M_0 \xrightarrow{\ \varphi\ } M \longrightarrow 0 \tag{9-2}$$

in which M_1 and M_0 are free. There exist A-modules Q_1 and Q_0, both of which are direct sums of copies of P, such that $FQ_1 = M_1$ and $FQ_0 = M_0$. Since the map in (9-1) is surjective, $\psi = F\alpha$ for some $\alpha \in \mathrm{Hom}_{\mathsf{A}}(Q_1, Q_0)$. Applying F to the exact sequence $Q_1 \to Q_0 \to M \to 0$, where $M = \operatorname{coker}\alpha$, yields an exact sequence $FQ_1 \to FQ_0 \to FM \to 0$. Comparing this with (9-1), it follows that $M \cong FM$. □

Example 9.7 The field of complex numbers can be endowed with a $\mathbb{Z}_2$-grading as follows. Define $A_0 = \mathbb{R}$ and $A_1 = \mathbb{R}i$. The ring itself is a projective object in $\mathsf{GrMod}A$, the category of graded modules. It is clear that $A \cong A(1)$, and hence A is a generator. It is finitely generated, and therefore a progenerator. The endomorphism ring of A in $\mathsf{GrMod}A$ consists of the multiplications by elements of A_0. Hence $\mathsf{GrMod}A \cong \mathsf{Mod}\mathbb{R}$. ◊

The following result shows how close abelian categories are to module categories. However, its proof requires much of the material that will be developed later in this chapter.

Theorem 9.8 *[Mitchell's Theorem] Let A be a small abelian category (i.e., the objects of A form a set). Then there exists a ring R and a fully faithful exact covariant functor $\mathsf{A} \to \mathsf{Mod}R$.*

Proof. See [182, Theorem 11.6, Chapter 4] for the details. The basic idea is to embed A in a larger abelian category which has a projective generator and to apply the last result to the larger category. □

Mitchell's Theorem allows us to think of the objects in a small abelian category as modules over some ring, and then the morphisms correspond to module homomorphisms. In particular, objects may be thought of as having elements, and diagrams are therefore susceptible to 'diagram chasing' arguments. We will also abuse terminology by writing $M \subset N$ for a subobject M of an object N.

EXERCISES

9.1 Show that the dual of an abelian category is abelian.

9.2 Let A be an abelian category having coproducts. Show that $X \in \mathrm{Ob}(\mathsf{A})$ is a generator if and only if, for each $N \in \mathrm{Ob}(\mathsf{A})$, there exists an epimorphism $\coprod_I X \to N$ defined on some coproduct of copies of X.

9.3 If P is a projective object in an abelian category A then $\mathrm{Hom}_{\mathsf{A}}(P, -)$ need not commute with arbitary direct sums. In particular, if A is the category of k-vector spaces, and V is an infinite dimensional vector space with basis $\{e_\lambda\}$, show that $\mathrm{id}_V \notin \oplus \mathrm{Hom}_{\mathsf{A}}(V, ke_\lambda)$.

9.4 If $\{P_\alpha \mid \alpha \in I\}$ is a collection of projectives in A, show that $\coprod P_\alpha$ is projective.

9.5 Let (F, G) be an adjoint pair of functors between abelian categories. Show that F preserves projectives if G is exact.

9.6 Let A be an abelian category. Suppose that L and N are submodules of an A-module M. If F is a left exact functor on A, show that $FL \cap FN = F(L \cap N)$. If G is a right exact functor on A, show that $GL + GN = G(L + N)$.

9.7 Let M be an A-module. Show that $\mathrm{Hom}_{\mathsf{A}}(-, M)$ is a projective in $\mathsf{Fun}(\mathsf{A}^{\mathrm{op}}, \mathsf{Ab})$. Show that as M runs over all the A-modules this produces a set of generators for $\mathsf{Fun}(\mathsf{A}^{\mathrm{op}}, \mathsf{Ab})$.

9.8 [271, Proposition 2.3, page 16] Show that the opposite category to $\mathsf{Mod}R$ is not equivalent to $\mathsf{Mod}S$ for any ring S. (Hint: show that $(\mathsf{Mod}R)^{\mathrm{op}}$ does not have a progenerator.)

2.10 The functor $\mathrm{Hom}_R(-, R)$

Throughout this section R will denote a ring.

Let M be a right R-module. Composition of maps provides a map

$$\mathrm{Hom}_R(M, M) \times \mathrm{Hom}_R(R, M) \to \mathrm{Hom}(R, M)$$

that makes M into a left module over $\mathrm{End}_R(M)$, and hence into a $\mathrm{End}_R(M)$-R-bimodule.

We define $M^* = \mathrm{Hom}_R(M, R)$ and make this a *left* R-module via

$$(x.\mu)(m) = x.\mu(m). \tag{10-1}$$

Composition of maps gives a map

$$\mathrm{Hom}_R(M, R) \times \mathrm{Hom}_R(M, M) \to \mathrm{Hom}(M, R)$$

that makes M^* into an R-$\mathrm{End}_R(M)$-bimodule. More generally, if M is an S-R-bimodule, then M^* is an R-S-bimodule.

Let ${}_RM$ be a left R-module. The natural right R-module structure on $M^\vee := \mathrm{Hom}_R(M, R)$ is given by

$$(\rho.x)(m) = \rho(m)x. \tag{10-2}$$

In this way $M^\vee$ becomes an $\mathrm{End}_R M$-R-bimodule.

The trace ideal of a left R-module M is

$$T(M) := \sum_\rho \rho(M),$$

where the sum is over all $\rho \in \mathrm{Hom}_R(M,R)$. Since each $\rho(M)$ is a left ideal, $T(M)$ is a left ideal. It is also a right ideal because, $\rho(m)x = (\rho.x)(m)$ for all $x \in R$. If M is an R-S-bimodule, then the trace ideal is the image of the natural map

$$\Phi : M \otimes_S M^\vee \to R, \qquad m \otimes \rho \mapsto \rho(m). \tag{10-3}$$

There is also a natural map

$$\Psi : M^\vee \otimes_R M \to \mathrm{End}_R(M) \tag{10-4}$$

defined by

$$\Psi(\rho \otimes m)(m') = \rho(m')(m).$$

If M and $M^\vee$ are given their bimodule structures over R and $\mathrm{End}_R M$, then Ψ is a homomorphism of $\mathrm{End}_R(M)$ bimodules.

For a right module, the trace ideal is defined in a similar way, and it is the image of the natural map $M^* \otimes M \to R$.

Lemma 10.1 *An R-module M is a generator if and only $T(M) = R$.*

Proof. ($\Rightarrow$) If $T(M)$ were not equal to R, then the map $R \to R/T(M)$ would be non-zero, so there would be a map $M \to R$ for which the composition $M \to R \to R/T(M)$ would be non-zero. But then the image of the map $M \to R$ would not be contained in $T(M)$. This is contrary to the definition of $T(M)$, so we conclude that $T(M) = R$.

($\Leftarrow$) Let $f : K \to L$ be a non-zero map of right R-modules. Then there is a map $g : R \to K$ such that $fg \neq 0$. By hypothesis $R = \sum \varphi_\lambda(M)$ where the sum is over all $\varphi_\lambda \in \mathrm{Hom}_R(M,R)$, so because $fg(R) \neq 0$, $fg\varphi_\lambda \neq 0$ for some λ. Thus $g\varphi_\lambda : M \to K$ is the element that shows $\mathrm{Hom}_R(M,f) \neq 0$. $\square$

The rule $M \mapsto M^*$ is a contravariant functor from right modules to left modules. If R is a field, this is the duality functor sending a vector space to its dual. A finite dimensional vector space is naturally isomorphic to its double dual. If F is a finitely generated free module over any ring, then F^* is finitely generated free of the same rank, so F^{**} is isomorphic to F. Proposition 10.5 extends this duality result to finitely generated projective modules. First we need some notation.

For any right R-module M there is a natural right R-module homomorphism

$$\Lambda : M \to M^{**}$$

sending $m \in M$ to the map $\Lambda_m : M^* \to R$ defined by $\Lambda_m(f) = f(m)$. More formally, there is a natural transformation from the identity functor to the composition **.

Since $M \mapsto M^*$ is a contravariant left exact functor, if R is left noetherian, and M is finitely generated, then M^* is finitely generated. To see this, write M as a quotient of a finitely generated free module, F say, and observe that M^* is a submodule of F^*. Without the noetherian hypothesis it is possible for M to be finitely generated but M^* not finitely generated (Exercise 5).

Proposition 10.2 *Let M be a noetherian module over a noetherian ring R. Then the natural map $\Lambda : M \to M^{**}$ is injective if and only if M is a submodule of a free module.*

Proof. ($\Rightarrow$) There is a surjective map $\theta : F \to M^*$ for some finitely generated free module F. Since $\theta^* = \operatorname{Hom}(\theta, R)$ is injective so is the map $\theta^* \circ \Lambda : M \to F^*$, whence M is a first syzygy.

($\Leftarrow$) By hypothesis, there is an injective map $\varphi : M \to F$ with F finitely generated free. Since the diagram

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\Lambda"'] & F \arrow[d] \\ M^{**} \arrow[r, "\varphi^{**}"'] & F^{**} \end{tikzcd}$$

commutes, and $F \to F^{**}$ is an isomorphism, it follows that Λ is injective. $\square$

Corollary 10.3 *If M is a finitely generated R-module, then the natural map $M^* \to M^{***}$ is injective.*

Proof. The natural map $\Psi : M^* \to M^{***}$ is the map Λ for the module M^*; it sends $f \in M^*$ to Ψ_f, the map defined by $\Psi_f(\alpha) = \alpha(f)$ for $\alpha \in M^{**}$. There is a surjective map $F \to M$ with F finitely generated free, and hence an injective map $M^* \to F^*$. The result follows from the second half of the proof of Proposition 10.2. $\square$

Exercise 4 shows that the map $M^* \to M^{***}$ need not be not surjective.

Lemma 10.4 (Dual Basis Lemma) *Let P be a right R-module. Then P is projective if and only if there exist elements $p_\lambda \in P$ and $\varphi_\lambda \in \operatorname{Hom}_R(P, R)$ such that*

1. *for each $p \in P$, $\varphi_\lambda(p) = 0$ for all but finitely many λ, and*
2. *$p = \sum_\lambda p_\lambda \varphi_\lambda(p)$ for all $p \in P$.*

In this case, if the index set of λs is finite, then $\varphi = \sum \varphi(p_\lambda)\varphi_\lambda$ for all φ in $\operatorname{Hom}_R(P, R)$.

Proof. Suppose P is projective. Then there is a free module F, an epimorphism $\varphi : F \to P$, and a map $\psi : P \to F$ such that $\varphi\psi = \mathrm{id}_P$. Let $\{x_\lambda \mid \lambda \in \Lambda\}$ be a basis for F. Define $p_\lambda = \varphi(x_\lambda)$. If $p \in P$, then $\psi(p) = \sum x_\lambda r_\lambda$ for some elements $r_\lambda \in R$. Define $\varphi_\lambda : P \to R$ by $\varphi_\lambda(p) = r_\lambda$. This is a well-defined map because the x_λ are linearly independent. It is easy to check that φ_λ is a right R-module map. Now

$$p = \varphi\psi(p) = \sum p_\lambda \varphi_\lambda(p),$$

as required.

Conversely, suppose that such elements exist. Let F be free on a basis x_λ, and define $\varphi : F \to P$ by $\varphi(x_\lambda) = p_\lambda$. Define $\psi : P \to F$ by $\psi(p) = \sum x_\lambda \varphi_\lambda(p)$. Then

$$\varphi\psi(p) = \varphi\big(\sum x_\lambda \varphi_\lambda(p)\big) = \sum p_\lambda \varphi_\lambda(p) = p.$$

Therefore $\varphi\psi = \mathrm{id}_P$, showing that P is projective.

Finally, suppose the conditions are satisfied. Let $\varphi \in \mathrm{Hom}_R(P,R)$ and $p \in P$. Then

$$\varphi(p) = \varphi\Big(\sum p_\lambda \varphi_\lambda(p)\Big) = \Big(\sum \varphi(p_\lambda)\varphi_\lambda\Big)(p),$$

so $\varphi = \sum \varphi(p_\lambda)\varphi_\lambda$. □

Part (2) of the Dual Basis Lemma shows that the elements p_λ generate P. If P is finitely generated we may assume that there are only a finite number of p_λs and φ_λs. In this case, we say that the elements $\{p_\lambda\}$ and $\{\varphi_\lambda\}$ form a dual basis for P and P^*. We call P^* the dual of P. It is a left R-module. If P is finitely generated, so is P^*. The last statement in the Dual Basis Lemma shows that the φ_λ generate P^*. The symmetry in the Dual Basis Lemma shows that P^* is a projective left R-module.

The next result justifies the use of the word "dual".

Proposition 10.5 *If P is a finitely generated projective right R-module, then the natural map $\Lambda : P \to P^{**}$ is an isomorphism.*

Proof. The map is injective by Proposition 10.2. Take a dual basis as in the Dual Basis Lemma. The map Λ sends p_λ to the element of P^{**} that sends $\varphi \in P^*$ to $\varphi(p_\lambda)$. Since Λ is injective, we will think of p_λ as an element of P^{**} and write $\varphi(p_\lambda)$ as $p_\lambda(\varphi)$. By the last part of the Dual Basis Lemma,

$$\varphi = \sum p_\lambda(\varphi)\varphi_\lambda. \tag{10-5}$$

By part (2), applied to P^* and P^{**}, this says the p_λ are dual to the φ_λs. So, by the last part of the Dual Basis Lemma applied to P^* and P^{**}, the p_λs generate P^{**}. □

Corollary 10.6 *The rule $P \mapsto P^*$ is a duality between the category of finitely generated projective right R-modules and the category of finitely generated projective left R-modules.*

Proposition 10.7 *If P is a finitely generated projective right R-module, then the natural map $\Psi : P \otimes_R P^* \to \operatorname{End}_R P$ is an isomorphism.*

Proof. The map Ψ is defined by $\Psi(q \otimes \alpha)(p) = q\alpha(p)$ (this is the analogue of (10-4) for right modules). Let $p_1, \dots, p_n \in P$ and $\varphi_1, \dots, \varphi_n \in P^*$ be dual bases.

Since Ψ is a homomorphism of $\operatorname{End}_R(P)$-bimodules, to prove it is surjective it suffices to show that its image contains the identity. However,

$$\Psi\left(\sum_{i=1}^{n} p_i \otimes \varphi_i\right)(p) = \sum_{i=1}^{n} p_i\varphi_i(p) = p,$$

so

$$\Psi\left(\sum_{i=1}^{n} p_i \otimes \varphi_i\right) = \operatorname{id}_P .$$

On the other hand, suppose that $\sum_j q_j \otimes \alpha_j \in \ker \Psi$. Then

$$\sum_j q_j\alpha_j(p_i) = \Psi\left(\sum_j q_j \otimes \alpha_j\right)(p_i) = 0.$$

But $\sum_i \alpha_j(p_i)\varphi_i$ equals α_j by (10-5), so

$$\sum_j q_j \otimes \alpha_j = \sum_j q_j \otimes \sum_i \alpha_j(p_i)\varphi_i = \sum_{i,j} q_j\alpha_j(p_i) \otimes \varphi_i = 0.$$

Therefore Ψ is injective. □

Corollary 10.8 *Let P be a finitely generated projective right R-module. Let T be the trace ideal of P. Then $T^2 = T$.*

Proof. Let $S = \operatorname{End}_R P$. Then T^2 is the image of the natural map

$$P^* \otimes_S P \otimes_R P^* \otimes_S P \to R.$$

By Proposition 10.7, the middle term, $P \otimes_R P^*$, is isomorphic to S, so the result follows. □

EXERCISES

10.1 Check that the rules in (10-1) and (10-2) make M^* and $M^\vee$ R-modules.

10.2 Let $M = mR$ be a cyclic R-module. Let N be any right R-module. Show that there is a bijection between the elements of $\operatorname{Hom}_R(M, N)$ and the elements $n \in N$ such that $\operatorname{Ann}(m) \subset \operatorname{Ann}(n)$; a homomorphism θ corresponds to $\theta(n)$.

10.3 Let e be an idempotent in R. Show that $(Re)^\vee \cong eR$.

10.4 Let $k[x, y]$ be the polynomial ring, define $R = k[x, y]/(x, y)^2$ and let $M = R/(x, y)$. Show that

(a) $M^* \cong M \oplus M$;

(b) no two $M^{**\cdots**}$ are isomorphic;

(c) the image of $\Lambda : M \to M^{**}$ is not an essential submodule of M^{**}.

10.5 [Musson] When R is not noetherian, the hypothesis that M is finitely generated does not imply that M^* is finitely generated. Let R be the ring of $\mathbb{Z} \times \mathbb{Z}$-matrices over the field k such that each column has only finitely many non-zero entries. Let V be the subspace of R consisting of those matrices whose entries outside the 0^{th}-column are zero. Then V is a simple left R-module. For exach $n \in \mathbb{Z}$, let $f_n : V \to R$ be the map embedding V as the n^{th} column of R. By considering the elements f_n, show that V^* is not a finitely generated right R-module.

$\boxed{\text{Paul}}$ Check this exercise.

2.11 Morita equivalence

Definition 11.1 Two rings R and S are Morita equivalent if the categories $\mathsf{Mod}R$ and $\mathsf{Mod}S$ are equivalent. $\Diamond$

Morita equivalence is an equivalence relation on rings.

The standard example is that all matrix algebras over the same field are Morita equivalent. The simplest way to see this is to observe that both $M_n(k)$ and k have a unique simple right module, k^n and k respectively, and that the endomorphism ring of this simple is isomorphic to k in each case, and every other module is a direct sum of copies of this simple module.

In the language of Chapter 3, two rings are Morita equivalent if they are coordinate rings of the same non-commutative space.

Lemma 11.2 (Watt's Theorem) *Let R and S be rings, and $f^* : \mathsf{Mod}R \to \mathsf{Mod}S$ a right exact additive functor commuting with direct sums. Then f^*R has the structure of an R-S-bimodule and f^* is naturally equivalent to $- \otimes_R f^*R$.*

Proof. Write B for the right S-module f^*R. The map

$$x \mapsto \lambda_x = \text{left multiplication by } x$$

is a ring isomorphism $R \to \operatorname{Hom}_R(R_R, R_R)$. Since f^* is additive the map $f^* : \operatorname{Hom}_R(R_R, R_R) \to \operatorname{Hom}_S(B, B)$ is a ring homomorphism, thus making B an R-S-bimodule with the action of $x \in R$ defined by $x.b = (f^*\lambda_x)(b)$ for $b \in B$.

Now fix a right R-module M. For each $m \in M$ define $\varphi_m \in \operatorname{Hom}_R(R, M)$ by $\varphi_m(x) = mx$; thus $f^*\varphi_m \in \operatorname{Hom}_S(B, f^*M)$. If $b \in B$, $m \in M$ and $x \in R$ then

$$(f^*\varphi_{mx})(b) = f^*(\varphi_m \circ \lambda_x)(b) = (f^*\varphi_m \circ f^*\lambda_x)(b) = (f^*\varphi_m)(xb).$$

Hence the rule $t_M(m \otimes b) := (f^*\varphi_m)(b)$ gives a well-defined map

$$t_M : M \otimes_R B \to f^*M;$$

it is a right S-module map since $f^*\varphi_m$ is. It is routine to show that $t : -\otimes_R B \to f^*$ is a natural transformation, so it remains to show that t_M is an isomorphism for all M.

Since f^* commutes with direct sums, if $Q = \oplus_I R$, then the map $t_Q : Q \otimes_R B \to f^*Q$ is an isomorphism. Now, for an arbitary M, there is an exact sequence $Q_1 \to Q_0 \to M \to 0$ with each Q_i a free R-module. In the commutative diagram

$$\begin{array}{ccccccc}
Q_1 \otimes_R B & \longrightarrow & Q \otimes_R B & \longrightarrow & M \otimes_R B & \longrightarrow & 0 \\
{\scriptstyle t_{Q_1}}\downarrow & & {\scriptstyle t_{Q_0}}\downarrow & & {\scriptstyle t_M}\downarrow & & \\
f^*Q_1 & \longrightarrow & f^*Q_0 & \longrightarrow & f^*M & \longrightarrow & 0
\end{array}$$

the first two vertical maps are isomorphisms, and the rows are exact, so a diagram chase shows that t_M is an isomorphism. □

Theorem 11.3 *If $f^* : \mathsf{Mod}R \to \mathsf{Mod}S$ is an equivalence of categories then there is an R-S-bimodule B such that f^* is naturally equivalent to $- \otimes_R B$. Moreover,*

1. *B_S is a progenerator,*
2. $\mathrm{End}_S B \cong R$*;*
3. *any two rings having a bimodule satisfying these two conditions are Morita equivalent.*

Proof. Lemma 11.2 shows that f^* is naturally equivalent to $- \otimes_R B$ where $B = f^*R$ with its natural bimodule structure. The functor $\mathrm{Hom}_S(B, -)$ is a right adjoint to f^*. Because $\mathrm{Hom}_S(B, -)$ is an equivalence of categories, B_S is a progenerator. Lemma 11.2 implies that $\mathrm{Hom}_S(B, -)$ is equivalent to $- \otimes_S B^\vee$, where $B^\vee = \mathrm{Hom}_S(B, S)$. Furthermore, because f^* sends R_R to B_S, $\mathrm{Hom}_S(B, -)$ sends B_S to R_R, and sends $\mathrm{Hom}_S(B, B)$ isomorphically to $\mathrm{Hom}_R(R_R, R_R)$, which is isomorphic to R.

Because $- \otimes_R B$ and $\mathrm{Hom}_S(B, -)$ are quasi-inverses, $- \otimes_R B$ is right adjoint to $- \otimes_S B^\vee$. Therefore $- \otimes_R B$ is naturally equivalent to $\mathrm{Hom}_R(B^\vee, -)$.

Because $- \otimes_R B$ and $- \otimes_S B^\vee$ are quasi-inverses to each other, there are bimodule isomorphisms $B \otimes_S B^\vee \cong R$ and $B^\vee \otimes_R B \cong S$.

The symmetry of the situation shows that $\mathrm{Hom}_R(B^\vee, R) \cong B$ as R-S-bimodules, and the same arguments show that B is a progenerator as an R-module, and that $\mathrm{Hom}_R(B, B) \cong S$. □

If P is a finitely generated projective right R-module, then $\mathrm{Hom}_R(P, -)$ is equivalent to $- \otimes_R P^\vee$.

Proposition 11.4 *The ring of natural transformations of the identity functor on $\mathsf{Mod}R$ is isomorphic to the center of R.*

Proof. Each central element $z \in R$ induces a natural transformation $\tau : \mathrm{id}_{\mathsf{Mod}R} \to \mathrm{id}_{\mathsf{Mod}R}$ defined by requiring $\tau_M : M \to M$ to be the map $m \mapsto mz$. Since z is central, τ_M is a module homomorphism. If $f : M \to N$ is a module homomorphism, then $\tau_N \circ f = f \circ \tau_M$, so τ is a natural transformation of the identity functor. Hence there is a set map from the center of R to the ring of natural transformations of the identity functor. It is easy to check that this is a ring homomorphism.

On the other hand , if μ is a natural transformation of the identity functor, then $\mu_R : R \to R$ is a map of right R-modules, so is given by left multiplication by some element $z \in R$. Now fix $r \in R$, and let $f : R \to R$ be left multiplication by r. Since μ is a natural transformation of the identity functor, $\mu_R \circ f = f \circ \mu_R$. Thus $zrx = rzx$ for all $x \in R$. It follows that z is central. Now, if M is any R-module, and $m \in M$, there is a map $g : R \to M$ given by $g(x) = mx$. Since $\mu_M \circ g = g \circ \mu_R$, we have in particular that

$$\mu_M(m) = \mu_M(g(1)) = g(\mu_R(1)) = g(z) = mz.$$

Thus $\mu_M = \tau_M$. Therefore the ring homomorphism from the center of R to $\mathrm{End}(\mathrm{id}_{\mathsf{Mod}R})$ is surjective. It is also injective because z can be recovered from τ as $z = \tau_R(1)$. □

Corollary 11.5 *Two commutative rings are Morita equivalent if and only if they are isomorphic.*

EXERCISES

11.1 Let R be any ring, and n and d any integers ≥ 1. Show that $M_n(R)$ is Morita equivalent to $M_d(R)$.

11.2 Let R be a ring, and P a projective left R-module. Suppose further that P is a generator. Set $P^\vee = \mathrm{Hom}_R(P, R)$ and $S = \mathrm{Hom}_R(P, P)$. Show that the ring

$$\begin{pmatrix} R & P \\ P^\vee & S \end{pmatrix}$$

is Morita equivalent to R.

2.12 Injectives

All categories in this section are abelian.

Definition 12.1 An A-module E is injective if $\mathrm{Hom}_{\mathsf{A}}(-, E)$ is exact. We say that A has enough injectives if every A-module is isomorphic to a submodule of an injective module. An injective envelope of an A-module M is a monomorphism $\psi : M \to E$ where E is an injective such that if $\psi' : M \to I$ is a monic to any injective I, then there is a unique morphism $\rho : E \to I$ such that $\psi' = \rho\psi$. If every A-module has an injective envelope we say that A has injective envelopes. ♢

The fact that an A-module is injective if and only if it is projective when considered as an object in the opposite category A^{op} leads to parallels between the theories of projectives and injectives. However, the abelian categories that turn up most often are Grothendieck categories, and a Grothendieck category has enough injectives but need not have enough projectives. The category of quasi-coherent sheaves on a projective algebraic variety and the category of co-modules over a coalgebra are typical examples. Thus it is better to use injectives than projectives. The fact that there is not a complete parallel between the two is because although the opposite of an abelian category is abelian, the only Grothendieck category whose opposite is a Grothendieck category is the zero category.

Definition 12.2 A monic $f : L \to M$ is essential, and M is called an essential extension of L, if every non-zero submodule of M has non-zero intersection with the image of f. We also say that $f(L)$ is an essential submodule of M. $\Diamond$

Lemma 12.3 *If L is a submodule of an injective module E, then E is an injective envelope of of L if and only if L is an essential submodule of E.*

Thus an injective envelope is a maximal essential extension.

The proof that $\mathsf{Mod}R$ has enough injectives can be found in several places such as [59, Chapter 1, Section 3] and ?? The proof below uses Proposition 12.8 which will be frequently used later on.

Lemma 12.4 *A direct product of injectives is injective.*

Proof. This follows from the fact that in any category C, if $M \in \mathsf{C}$ and some objects E_α have a product, then $\prod \mathrm{Hom}_{\mathsf{C}}(M, E_\alpha) \cong \mathrm{Hom}_{\mathsf{C}}(M, \prod E_\alpha)$. $\Box$

Lemma 12.4 is equivalent to the result that a direct sum of projectives is projective because passing to the opposite category interchanges projectives with injectives and direct sums with direct products.

Proposition 12.5 *An inverse limit of injectives is injective.*

Proof. $\Box$

Cartan and Eilenberg [59, Theorem 3.2, page 8] give a a proof of Baer's criterion that an R-module E is injective if and only if the restriction map $\mathrm{Hom}_R(R, E) \to \mathrm{Hom}_R(I, E)$ is surjective for all right ideals I of R. This is a special case of the following result which appears in Grothendieck's Tohoku paper.

Lemma 12.6 *[94, Lemme 1, pg. 136] Suppose that A satisfies $\mathsf{Ab5}$ and has a generator G. Then an A-module E is injective if and only if for every submodule M of G the restriction map $\mathrm{Hom}_{\mathsf{A}}(G, E) \to \mathrm{Hom}_{\mathsf{A}}(M, E)$ is surjective.*

An R-module E is divisible if for every regular element x in R, every element of E is a multiple of x; that is, $Ex = E$.

If x is a right regular element in a ring R, then left multiplication by x is an injective map from R to itself so, if E is an injective R-module the induced map $\mathrm{Hom}_R(R, E) \to \mathrm{Hom}_R(R, E)$ is surjective. But $\mathrm{Hom}_R(R, E) \cong E$, and the induced action is $e \mapsto ex$, so $Ex = E$. Therefore every injective module over a domain is divisible. Baer's criterion implies that the converse also holds for a ring in which every right idea is principal: a module is injective if and only if it is divisible (see [**?**, ?] for example).

We now apply this to $\mathbb{Z}$.

Example 12.7 The category of abelian groups has enough injectives. First, $\mathbb{Q}$ is injective because it is divisible. Every $\mathbb{Z}$-module is a quotient of a suitably large product of copies of $\mathbb{Z}$, and is therefore a submodule of a quotient of a product of copies of $\mathbb{Q}$. But that product is injective by Lemma 12.4, and hence divisible, and a quotient of a divisible module is divisible, so every $\mathbb{Z}$-module embeds in an injective module. $\diamondsuit$

Proposition 12.8 *A right adjoint to an exact functor sends injectives to injectives.*

Proof. Let (F, G) be an adjoint pair with $F : \mathsf{A} \to \mathsf{B}$. Suppose that I is an injective in B. Then $\mathrm{Hom}_{\mathsf{A}}(-, GI)$ is isomorphic to $\mathrm{Hom}_{\mathsf{B}}(-, I) \circ F$ which is a composition of exact functors, so is exact. Hence GI is injective. $\square$

Theorem 12.9 *The category of modules over a ring has enough injectives.*

Proof. Let R be a ring. Since R has an identity, there is a ring homomorphism $\mathbb{Z} \to R$. The associated direct image functor $f_* = \mathrm{Hom}_R(R, -) : \mathsf{Mod}R \to \mathsf{Mod}\mathbb{Z}$ has a right adjoint $f^! = \mathrm{Hom}_{\mathbb{Z}}(R, -)$ (1.6.4). Since f_* is restriction of scalars it is exact. Therefore $f^!$ preserves injectives.

Now let M be an R-module. As a $\mathbb{Z}$-module, it embeds in an injective abelian group, say I. There are inclusions of right R-modules,

$$M \cong \mathrm{Hom}_R(R, M) \subset \mathrm{Hom}_{\mathbb{Z}}(R, M) \subset \mathrm{Hom}_{\mathbb{Z}}(R, I) = f^! I.$$

Therefore M is a submodule of the injective R-module $f^! I$. $\square$

Example 12.10 An abelian category can have enough injectives but not have injective envelopes. If A is the opposite category to $\mathsf{Mod}k[x]$, then A has enough injectives because $\mathsf{Mod}k[x]$ has enough projectives. However, $\mathsf{Mod}k[x]$ does not have projective covers, so A does not have injective envelopes. See Exercises 5-7.

Definition 12.11 An A-module N is a cogenerator if the functor $\mathrm{Hom}_{\mathsf{A}}(-, N)$ is faithful. A set of A-modules $\{N_\lambda \mid \lambda \in \Lambda\}$ cogenerates A if whenever $f : U \to V$ is a non-zero morphism in A, there exists a λ and a morphism $g : V \to N_\lambda$ such that $gf \neq 0$. $\diamondsuit$

By analogy with Lemma 9.4, or by that result interpreted in the opposite category, an injective E is a cogenerator if and only if $\operatorname{Hom}_{\mathsf{A}}(M, E)$ is non-zero whenever M is a non-zero A-module.

Lemma 12.12 *Suppose that A has injective envelopes. If A has a noetherian set of generators, then the set of injective envelopes of the simple A-modules cogenerates A.*

Proof. Let $f : U \to V$ be a non-zero map in A. Then $f(U)$ has a non-zero noetherian submodule, say M, and there is a non-zero map from M to some simple module, say S. Hence there is a non-zero map from M to the injective envelope of S. This extends to a map g from V to that injective. Clearly $gf \neq 0$. $\square$

Lemma 12.13 *Let y be a normal regular element of R. Define $\sigma \in \operatorname{Aut} R$ by $yr = r^\sigma y$ for all $r \in R$, and let σ^* and σ_* be the inverse and direct image functors associated to σ. Let E be an injective R-module, and define $\beta : E \to E$ by $\beta(e) = ey$. Then there is an exact sequence*

$$0 \longrightarrow \{e \in E \mid ey = 0\} \longrightarrow E \xrightarrow{\beta} \sigma^* E \longrightarrow 0.$$

Furthermore, $\{e \in E \mid ey = 0\}$ is an injective $R/(y)$-module.

Proof. By Lemma 1.6.5, β is a homomorphism. It is surjective because E is divisible. The direct image functor $i_* : \mathsf{Mod} R/(y) \to \mathsf{Mod} R$ has a right adjoint $i^! = \operatorname{Hom}_R(R/yR, -)$. Since i_* is exact, it follows from Proposition 12.8 that $i^! E$ is an injective $R/(y)$-module. But $i^! E$ is the submodule of E consisting of elements annihilated by y, which is exactly $\ker \beta$. $\square$

Let x be a normal element in a ring R. Thus $xR = Rx$. An element m in a module M is x-torsion if $mx^n = 0$ for $n \gg 0$. Because x is normal, the x-torsion elements form a submodule. We say that M itself is x-torsion if every element in it is. The x-torsion modules form a localizing subcategory of $\mathsf{Mod} R$.

If x is also regular (i.e., not a zero-divisor), then for each $r \in R$ there is a unique r' in R such that $xr' = rx$. We therefore write $r' = x^{-1} r x$. In particular, for all integers n, $x^{-n} r x^n$ is a uniquely determined element of R.

Proposition 12.14 *[98, Theorem 4] Let R be a right noetherian ring, and $x \in R$ a regular normal element. If M is x-torsion, then so is its injective envelope.*

Proof. Let E be an essential extension of M and suppose, to the contrary, that E is not x-torsion. Choose a non-zero $e \in E$ and $n \geq 0$, subject to the conditions that e is not x-torsion, and the right ideal

$$x^n (\operatorname{Ann} e) x^{-n}$$

is as large as possible. Since M is essential in E, there is an element r in R such that er is a non-zero element of M. Hence $erx^d = 0$ for $d \gg 0$. Certainly ex^d is not x-torsion. However,

$$\operatorname{Ann}(ex^d) \supset x^{-d}(\operatorname{Ann} e)x^d$$

and

$$x^{n+d}\operatorname{Ann}(ex^d)x^{-(n+d)} \supset x^n(\operatorname{Ann} e)x^{-n}$$

so, by the choice of e, we conclude that there is equality. This implies that $\operatorname{Ann}(ex^d) = x^{-d}(\operatorname{Ann} e)x^d$. But $x^{-d}rx^d$ annihilates ex^d, so r annihilates e. This contradicts the choice of r. We conclude that E is x-torsion. $\square$

Theorem 12.15 (Matlis) *[241, Prop. 4.5, Ch. V, pg. 124] In* A *has a set of noetherian generators, then every injective* A*-module is isomorphic to a direct sum of indecomposable injectives.*

EXERCISES

12.1 Show that an injective envelope is unique up to isomorphism.

12.2 Let A be an abelian category having products. Show that an A-module N is a cogenerator if and only if every A-module embeds in some product of copies of N.

12.3 Let $0 \to M \to E^0 \to E^1 \to E^2 \to \dots$ be a minimal injective resolution. If S is a simple module show that the differential in the complex $0 \to \operatorname{Hom}(S, E^0) \to \operatorname{Hom}(S, E^1) \to \dots$ is zero, and hence that $\operatorname{Ext}^n(S, M) \cong \operatorname{Hom}(S, E^n)$.

12.4 Show that an injective I is indecomposable if and only if $\operatorname{Hom}_{\mathsf{A}}(I, I)$ has a unique maximal two-sided ideal and the quotient by it is a division ring.

12.5 An epimorphism $f : M \to N$ in an abelian category A is superfluous if the only submodule L of M such that $L + \ker f = M$ is $L = M$. Show that f is superfluous if and only if it is an essential monomorphism in A^{op}.

12.6 Let A be an abelian category. A projective cover of a module M is a pair (P, f) consisting of a projective P and a superfluous epimorphism $f : P \to M$. Show that (P, f) is a projective cover of M if and only if (P, f) is an injective envelope of M in A^{op}.

12.7 Show that the simple modules over $k[x]$ do not have projective covers. Hence deduce that the opposite category to $\mathsf{Mod}k[x]$ does not have enough injectives. Thus, it is not a Grothendieck category. This can also be seen by observing that inverse limits fail to be exact in $\mathsf{Mod}k[x]$, so direct limits are not exact in the opposite category.

2.13 Quotient categories

Throughout this section A will denote an abelian category having a set of generators.

Two good references for quotient categories and localization of categories are Gabriel's thesis [88] and Stenstrom's book [241]. We have drawn heavily on both, and details that we omit can often be found in both those accounts.

Definition 13.1 A non-empty full subcategory T of an abelian category A is a Serre subcategory if, for all short exact sequences $0 \to M' \to M \to M'' \to 0$ in A, M belongs to T if and only if both M' and M'' do. In particular, the zero module is in T. The A-modules in T are called torsion modules. An A-module is torsion-free if its only submodule belonging to T is the zero submodule. Thus the zero module is the only A-module which is both torsion and torsion-free. $\Diamond$

For the rest of this section T will denote a Serre subcategory of A.

Since T is closed under submodules and quotients the the inclusion $\mathsf{T} \to \mathsf{A}$ preserves kernels and cokernels; in other words kernels and cokernels in T agree with those in A. Since T is closed under extensions it is closed under finite direct sums and products, and these agree with those in A. It follows that T is an abelian category, and the inclusion functor is exact.

If M_1 and M_2 are submodules of M that are torsion, then so is their sum since it is a quotient of $M_1 \oplus M_2$, which occurs in an exact sequence $0 \to M_1 \to M_1 \oplus M_2 \to M_2 \to 0$.

Example 13.2 If $\mathcal{S}$ is a multiplicatively closed subset of a commutative ring R, then the $\mathcal{S}$-torsion modules form a Serre subcategory of $\mathsf{Mod}R$. A module is $\mathcal{S}$-torsion if every element of it is annihilated by some element of $\mathcal{S}$. More generally, suppose that R is a ring having ring of fractions $\operatorname{Fract} R$ and S is an intermediate ring, $R \subset S \subset \operatorname{Fract} R$. If ${}_RS$ is flat, then $\{M \in \mathsf{Mod}R \mid M \otimes_R S = 0\}$ is a Serre subcategory of $\mathsf{Mod}R$. $\Diamond$

The general principle behind this example is that if $F : \mathsf{A} \to \mathsf{B}$ is an exact functor on an abelian category A, then the full subcategory of A consisting of those M such that $FM = 0$ is a Serre subcategory.

Definition 13.3 Let A be an abelian category and T a Serre subcategory. The quotient category A/T is defined as follows:

- its objects are the objects of A;
- if M and N are A-modules then
$$\operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(M, N) := \varinjlim \operatorname{Hom}_{\mathsf{A}}(M', N/N'),$$
where the direct limit is taken over all submodules M' of M and all submodules N' of N with the property that M/M' and N' are torsion;
- the composition of morphisms in A/T is induced by that in A. $\Diamond$

Proposition 13.4 *Definition 13.3 makes sense.*

Proof. First, the direct limit makes sense. Fix two A-modules M and N. Because A has a set of generators, the collection of all pairs (M', N') of submodules

$M' \subset M$ and $N' \subset N$, such that M/M' and N' are torsion is a small set. Let I denote that set. We define

$$(M', N') \le (M'', N'')$$

if $M'' \subset M'$ and $N' \subset N''$. Thus I is a quasi-ordered set. If $(M', N') \le (M'', N'')$, the natural morphisms $M'' \to M'$ and $N/N' \to N/N''$ induce maps

$$\mathrm{Hom}_{\mathsf{A}}(M', N/N') \to \mathrm{Hom}_{\mathsf{A}}(M'', N/N') \to \mathrm{Hom}_{\mathsf{A}}(M'', N/N'').$$

Thus $\mathrm{Hom}(M', N/N')$ is a direct system indexed by I. Since direct limits exist in the category of abelian groups (1.4.8), the definition of $\mathrm{Hom}_{\mathsf{A/T}}$ makes sense.

Since $(M, 0) \in I$, $\mathrm{Hom}_{\mathsf{A}}(M, N)$ appears in the direct system. The set I is directed because if (M_1', N_1') and (M_2', N_2') are in I, then so is $(M_1' \cap M_2', N_1' + N_2')$, and

$$(M_i', N_i') \le (M_1' \cap M_2', N_1' + N_2').$$

Hence every morphism in $\mathrm{Hom}_{\mathsf{A/T}}(M, N)$ is the image of a morphism in $\mathrm{Hom}_{\mathsf{A}}(M', N/N')$ for some $(M', N') \in I$ (cf. Exercise 3).

Second, there is a well-defined composition of morphisms in $\mathsf{A/T}$: we will state the main steps required to verify this, leaving the details to the reader. The composition

$$\mathrm{Hom}_{\mathsf{A/T}}(N, Z) \times \mathrm{Hom}_{\mathsf{A/T}}(M, N) \to \mathrm{Hom}_{\mathsf{A/T}}(M, Z)$$

is defined as follows. Let $\bar{f} \in \mathrm{Hom}_{\mathsf{A/T}}(N, Z)$ and $\bar{g} \in \mathrm{Hom}_{\mathsf{A,T}}(M, N)$. By the previous paragraph, $\bar{f}$ and $\bar{g}$ are images of morphisms $g : M' \to N/N'$ and $f : N'' \to Z/Z'$ in A where M/M', N', N/N'', and Z' belong to T. Define $M'' := g^{-1}(N' + N''/N')$, check that M/M'' is torsion, and define

$$g' : M'' \to N' + N''/N'$$

to be the restriction of g to M''. Both $f(N' \cap N'')$ and $Z'' := Z' + f(N' \cap N'')$ are torsion. Now define

$$f' : N''/N' \cap N'' \to Z/Z''$$

to be the map induced by f. Define h to be the composition

$$M'' \xrightarrow{g'} (N' + N''/N') \xrightarrow{\sim} (N''/N' \cap N'') \xrightarrow{f'} Z/Z'',$$

where the middle map is the natural isomorphism. Finally, one checks that $\bar{h}$, the image of h in $\mathrm{Hom}_{\mathsf{A/T}}(M, Z)$, depends only on $\bar{f}$ and $\bar{g}$ and not on a choice of representatives f and g.

Third, $\mathrm{Hom}_{\mathsf{A/T}}(M, M)$ contains an identity morphism, namely the image of id_M in the direct limit. $\square$

Definition 13.5 Let T be a Serre subcategory of A. The quotient functor

$$\pi : \mathsf{A} \to \mathsf{A}/\mathsf{T}$$

is defined by $\pi M = M$ on modules, and $\pi f =$ the image of f in the direct limit, on morphisms. ◊

It follows from the definitions that A/T is an additive category and that $\pi : \mathsf{A} \to \mathsf{A}/\mathsf{T}$ is an additive functor.

Lemma 13.6 *Let M be an A-module. Then $\pi M \cong 0$ if and only if M is torsion.*

Proof. Write I for the directed set used in defining $\mathrm{Hom}_{\mathsf{A}/\mathsf{T}}(\pi M, \pi M)$.

($\Rightarrow$) Since $\pi M \cong 0$, $\mathrm{Hom}_{\mathsf{A}/\mathsf{T}}(\pi M, \pi M) = 0$. In particular, the image in the direct limit of the identity morphism id_M is zero. Thus, since the set I in the proof of Proposition 13.4 is directed, there exists $(M', N') \in I$ such that the map $M' \to M/N'$ induced by id_M is the zero map. But the image of this map is $M' + N'/N'$, so $M' \subset N'$. Thus, since T is Serre, M' is torsion whence M is torsion.

($\Leftarrow$) It suffices to show that $\mathrm{Hom}_{\mathsf{A}/\mathsf{T}}(\pi M, \pi M) = 0$. Every morphism in this set is of the form πf for some $f : M' \to M/N'$ and some $(M', N') \in I$. But $(M', N') \leq (0, N') \in I$ since M is torsion, and the image of f in $\mathrm{Hom}_{\mathsf{A}}(0, N')$ is zero, so $\pi f = 0$. □

Proposition 13.7 *Let $f : M \to N$ be a morphism in A. Then*

1. *the kernel and cokernel of πf are $\pi(\ker f)$ and $\pi(\operatorname{coker} f)$ respectively;*
2. *πf is zero if and only if $\operatorname{im} f$ is torsion;*
3. *πf is monic if and only if $\ker f$ is torsion;*
4. *πf is epic if and only if $\operatorname{coker} f$ is torsion;*
5. *πf is an isomorphism if and only if both $\ker f$ and $\operatorname{coker} f$ are torsion.*

Theorem 13.8 *Let T be a Serre subcategory of A. Then A/T is abelian and the quotient functor $\pi : \mathsf{A} \to \mathsf{A}/\mathsf{T}$ is exact.*

Theorem 13.9 *Let A be an abelian category, T a Serre subcategory and D another abelian category.*

1. *Let $F : \mathsf{A} \to \mathsf{D}$ be an exact functor such that $FM = 0$ for all torsion modules M. Then there is a functor $G : \mathsf{A}/\mathsf{T} \to \mathsf{D}$, unique up to natural isomorphism, such that $F = G\pi$; that is, the diagram*

$$\begin{tikzcd} \mathsf{A} \arrow[r, "F"] \arrow[d, "\pi"'] & \mathsf{D} \arrow[d, "="] \\ \mathsf{A}/\mathsf{T} \arrow[r, "G"'] & \mathsf{D} \end{tikzcd}$$

commutes up to natural equivalence.

2. *If* $G : \mathsf{A}/\mathsf{T} \to \mathsf{D}$ *is a functor, then* G *is exact if and only if* $G\pi$ *is exact.*

Corollary 13.10 *Let* T *be a Serre subcategory of an abelian category* A. *Let* B *be a full subcategory of* A *that is closed under quotients and submodules. Then*

1. *the inclusion functor* $i_* : \mathsf{B} \to \mathsf{A}$ *is exact;*

2. $\mathsf{B} \cap \mathsf{T}$ *is a Serre subcategory of* B*;*

3. *there is an exact functor* $f_* : \mathsf{B}/\mathsf{B} \cap \mathsf{T} \to \mathsf{A}/\mathsf{T}$ *such that the diagram*

$$\begin{tikzcd} \mathsf{B} \arrow[r, "i_*"] \arrow[d, "\pi_1"'] & \mathsf{A} \arrow[d, "\pi_2"] \\ \mathsf{B}/\mathsf{B} \cap \mathsf{T} \arrow[r, "f_*"'] & \mathsf{A}/\mathsf{T} \end{tikzcd}$$

commutes up to natural equivalence.

Proof. (1) This is clear: the kernel and cokernel in A of a morphism in B both belong to B, so i_* commutes with kernels and cokernels.

(2) This holds because $\mathsf{B} \cap \mathsf{T}$ consists of precisely those modules on which the exact functor $\pi_2 i_*$ vanishes.

(3) It is clear that $\pi_2 i_*$ vanishes on $\mathsf{B} \cap \mathsf{T}$, so the existence of f_* follows from Theorem 13.9. The right-hand side of the equivalence $f_*\pi_1 \cong \pi_2 i_*$ is exact so, by Theorem 13.9, f_* is exact. □

$\boxed{\text{Paul}}$ When is f_* full and faithful?

Proposition 13.11 *[182, Exercise 6, page 174] Let* $\mathsf{S} \subset \mathsf{T} \subset \mathsf{A}$ *be two Serre subcategories of* A. *Then* T/S *is a Serre subcategory of* A/S, *and*

$$\mathsf{A}/\mathsf{T} \cong (\mathsf{A}/\mathsf{S})/(\mathsf{T}/\mathsf{S}).$$

Proposition 13.12 *[88, Corollaire 1, page 368] If* $0 \to L' \to M' \to N' \to 0$ *is an exact sequence in* A/T, *then there is an exact sequence* $0 \to L \to M \to N \to 0$ *in* A, *and a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & L' \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \pi L \arrow[r] & \pi M \arrow[r] & \pi N \arrow[r] & 0 \end{tikzcd}$$

such that α, β, γ *are isomorphisms in* A/T.

Let $\varphi \in \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\pi M, \pi N)$. As remarked in the proof of Proposition 13.4, φ is the image of a map $f \in \operatorname{Hom}_{\mathsf{A}}(M', N/N')$ for some $(M', N') \in I$. Write $s_1 : M' \to M$ for the inclusion, and $s_2 : N \to N/N'$ for the surjection. Both πs_1 and πs_2 are isomorphisms, so we may write

$$\varphi = (\pi s_2)^{-1} \circ (\pi f) \circ (\pi s_1)^{-1}.$$

This point of view may be used as the starting point for the definition of a quotient category. That is, rather than starting with a class of modules, the Serre subcategory, one begins with a class of morphisms which are to be inverted. This latter point of view is more general, and leads to the notion of a category of fractions (see [8] for details). This point of view is required to develop the notion of universal localization (see [63, Chapter 7] and [203, Chapter 4]).

The quotient functor preserves direct sums.

Example 13.13 The quotient functor $\mathsf{A} \to \mathsf{A}/\mathsf{T}$ need not preserve products. For example, let R be a k-algebra with an infinite dimensional simple module S. Let e_λ, $\lambda \in \Lambda$, be a k-basis for S, and set $I_\lambda = \operatorname{Ann} e_\lambda$. There is a diagonal injection $R/\operatorname{Ann} S \to \prod_\lambda R/I_\lambda$. Now take $\mathsf{A} = \mathsf{Mod} R$, and let T be all finite length R-modules having all composition factors isomorphic to S. We may choose an R such that $R/\operatorname{Ann} S$ is not of finite length, in which case $\prod_\lambda R/I_\lambda$ is not in T, so the quotient functor does not send this to zero, although it sends each component to zero. $\Diamond$

2.14 Localizing subcategories

In this section A denotes an abelian category and T denotes a Serre subcategory of A.

Definition 14.1 A Serre subcategory T of A is called a localizing subcategory if $\pi : \mathsf{A} \to \mathsf{A}/\mathsf{T}$ has a right adjoint. We write ω for the right adjoint and call it the section functor. $\Diamond$

The importance of localizing subcategories will be apparent when we discuss open subspaces in Section 3.7.

The key result is that when A has injective envelopes T is a localizing category if and only if it is closed under direct sums (Theorems 14.8 and 14.12).

A direct sum of torsion modules need not be a torsion module. For example, consider the category of k-vector spaces and declare a vector space to be torsion if it has finite dimension. This is a Serre subcategory that is not localizing.

Lemma 14.2 *Let T be a Serre subcategory of A. The following are equivalent:*

- *every A-module has a largest torsion submodule;*
- *the inclusion functor $\mathsf{T} \to \mathsf{A}$ has a right adjoint;*

- *every direct sum of torsion modules is torsion.*

Proof. Let $i_* : \mathsf{T} \to \mathsf{A}$ denote the inclusion functor.

(1) $\Rightarrow$ (2) We construct a right adjoint τ to i_* as follows. If M is an A-module, then τM is defined to be the largest torsion submodule of M. If $f : M \to N$ is a map of A-modules, then $f(\tau M)$ is a quotient of τM so is torsion, and therefore contained in τN. We define $\tau f : \tau M \to \tau N$ to be the restriction of f. It is easy to check that τ is a functor $\mathsf{A} \to \mathsf{T}$. It is a right adjoint to i_* because if M is a torsion module and N is an A-module, then the image of any map $f : M \to N$ is contained in τN. In other words, the natural map $\operatorname{Hom}_{\mathsf{A}}(M, \tau N) \to \operatorname{Hom}_{\mathsf{A}}(M, N)$ is an isomorphism, so

$$\operatorname{Hom}_{\mathsf{A}}(i_*M, N) = \operatorname{Hom}_{\mathsf{A}}(M, N) \cong \operatorname{Hom}_{\mathsf{A}}(M, \tau N) = \operatorname{Hom}_{\mathsf{T}}(M, \tau N).$$

(2) $\Leftarrow$ (1) Let $i^! : \mathsf{A} \to \mathsf{T}$ be a right adjoint to i_*. Let N be an A-module. By part (2b) of Theorem 1.6.18, the map $\varepsilon_N : i_*i^!N \to N$ is monic so we can view $i^!N$ as a submodule of N. It is of course torsion. To see that it is the largest torsion submodule, suppose that M is a torsion submodule of N. The inclusion of M in N can be viewed as an element of $\operatorname{Hom}_{\mathsf{A}}(i_*M, N)$. However, the adjunction isomorphism $\nu : \operatorname{Hom}_{\mathsf{T}}(M, i^!N) \to \operatorname{Hom}_{\mathsf{A}}(i_*M, N)$ satisfies $\nu(\alpha) = \varepsilon_N \circ i_*(\alpha)$, so every map $i_*M \to N$ factors as a composition

$$i_*M \longrightarrow i_*i^!N \xrightarrow{\varepsilon_N} N.$$

In particular, the inclusion of M in N factors in this way. Therefore M is contained in the submodule $i^!N$ of N. Thus $i^!N$ is the largest torsion submodule of N.

(1) $\Leftarrow$ (3) If M_i, $i \in I$, are torsion modules, then $\oplus M_i$ is the sum of the submodules M_i, so is torsion.

(3) $\Leftarrow$ (1) If M_i, $i \in I$, is the set of all torsion submodules of a module M, then their sum is a quotient of their direct sum, so is torsion. Ths sum must, of course, be the largest torsion submodule of M. □

Definition 14.3 Let T be a Serre subcategory of A. If the inclusion functor $\mathsf{T} \to \mathsf{A}$ has a right adjoint, then that adjoint is called the torsion functor and is denoted by τ. ◊

Definition 14.4 If an A-module M has a largest torsion submodule, that submodule is denoted by τM and is called the torsion submodule of M. We will often indicate the existence of a largest torsion submodule by saying 'suppose that τM exists'. ◊

Lemma 14.5 *If τN exists, then* $\operatorname{Hom}_{\mathsf{A}}(M, N/\tau N) = 0$ *for all torsion modules M. In particular, $N/\tau N$ is torsion-free.*

Proof. Suppose that M is in T and that $f : M \to N/\tau N$. Write N' for the kernel of the composition $N \to N/\tau N \to \operatorname{coker} f$. Then there is an exact sequence $0 \to \tau N \to N' \to N'/\tau N \cong \operatorname{im} f \to 0$. Since M is torsion so is $\operatorname{im} f$, and hence so is N' as T is Serre. Since τN is the largest torsion submodule of N, $N' \subset \tau N$. Therefore $\operatorname{im} f = 0$, whence $f = 0$ as required. □

Lemma 14.6 *Let M and N be A-modules. If τN exists, then*

$$\operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(M, N) = \varinjlim \operatorname{Hom}_{\mathsf{A}}(M', N/\tau N) \tag{14-1}$$

where the direct limit is taken over

$$J := \{(M', \tau N) \mid M' \subset M \text{ and } M/M' \text{ is torsion}\}.$$

Proof. It is easy to see that J is cofinal in the set I defined in the proof of Proposition 13.4. □

Proposition 14.7 *Suppose that T is a localizing subcategory of A. Let π and ω denote the quotient and section functors. Let $\mathcal{F}$ be an A/T-module. Then*

1. *$\omega\mathcal{F}$ is torsion-free;*
2. *if $f \in \operatorname{Hom}_{\mathsf{A}}(M, N)$ and πf is an isomorphism, then the map*

   $$\operatorname{Hom}(f, \omega\mathcal{F}) : \operatorname{Hom}_{\mathsf{A}}(N, \omega\mathcal{F}) \to \operatorname{Hom}_{\mathsf{A}}(M, \omega\mathcal{F})$$

   is an isomorphism;
3. *the map $\pi : \operatorname{Hom}_{\mathsf{A}}(M, \omega\mathcal{F}) \to \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\pi M, \pi\omega\mathcal{F})$ is an isomorphism for all A-modules M;*
4. *if Z is a torsion module, then every exact sequence of the form $0 \to \omega\mathcal{F} \to N \to Z \to 0$ splits;*
5. *$\pi\omega \cong \mathrm{id}_{\mathsf{A}/\mathsf{T}}$;*
6. *ω is fully faithful.*

Proof. (1) If M is torsion, then $\operatorname{Hom}_{\mathsf{A}}(M, \omega\mathcal{F}) \cong \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\pi M, \mathcal{F}) = 0$ so $\omega\mathcal{F}$ is torsion-free.

(2) By the adjoint property there is a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathsf{A}}(N, \omega\mathcal{F}) \arrow[r] \arrow[d, "\operatorname{Hom}(f,\omega\mathcal{F})"'] & \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\pi N, \mathcal{F}) \arrow[d, "\operatorname{Hom}(\pi f,\mathcal{F})"] \\
\operatorname{Hom}_{\mathsf{A}}(M, \omega\mathcal{F}) \arrow[r] & \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\pi M, \mathcal{F})
\end{tikzcd}$$

in which the horizontal maps are isomorphisms. By Proposition 13.7(4), πf is an isomorphism, so the right-hand vertical map is an isomorphism; hence the left-hand map is an isomorphism.

(3) Since $\omega\mathcal{F}$ is torsion-free, $\tau(\omega\mathcal{F})$ exists—it is zero. Thus, by (14-1), the map $f \mapsto \pi f$ is the natural map

$$\mathrm{Hom}_{\mathsf{A}}(M, \omega\mathcal{F}) \to \varinjlim \mathrm{Hom}_{\mathsf{A}}(M', \omega\mathcal{F}) \tag{14-2}$$

where the direct limit is taken over the $M' \subset M$ such that M/M' is torsion. By (2), all the maps $\mathrm{Hom}_{\mathsf{A}}(M', \omega\mathcal{F}) \to \mathrm{Hom}_{\mathsf{A}}(M'', \omega\mathcal{F})$ in the direct system are isomorphisms, whence so is (14-2).

(4) Let $f : \omega\mathcal{F} \to N$ be the map in the exact sequence. Then the map $\mathrm{Hom}(f, \omega\mathcal{F}) : \mathrm{Hom}_{\mathsf{A}}(N, \omega\mathcal{F}) \to \mathrm{Hom}_{\mathsf{A}}(\omega\mathcal{F}, \omega\mathcal{F})$ is an isomorphism by (2), so there exists $g : N \to \omega\mathcal{F}$ such that $f \circ g = \mathrm{id}_N$.

(5) Let $\varepsilon : \pi\omega \to \mathrm{id}_{\mathsf{A}/\mathsf{T}}$ be the counit. We must show that $\varepsilon_{\mathcal{F}} : \pi\omega\mathcal{F} \to \mathcal{F}$ is an isomorphism for each $\mathcal{F}$ in A/T. By Yoneda's Lemma, it suffices to prove that

$$\mathrm{Hom}(\mathcal{G}, \varepsilon_{\mathcal{F}}) : \mathrm{Hom}_{\mathsf{A}/\mathsf{T}}(\mathcal{G}, \pi\omega\mathcal{F}) \to \mathrm{Hom}_{\mathsf{A}/\mathsf{T}}(\mathcal{G}, \mathcal{F})$$

is an isomorphism for all $\mathcal{G}$ in A/T. Such a $\mathcal{G}$ is equal to πM for some A-module M, so we must show show that the bottom map in the following diagram is an isomorphism:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{A}}(M, \omega\mathcal{F}) & \xrightarrow{\nu} & \mathrm{Hom}_{\mathsf{A}/\mathsf{T}}(\pi M, \mathcal{F}) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle =} \\ \mathrm{Hom}_{\mathsf{A}/\mathsf{T}}(\mathcal{G}, \pi\omega\mathcal{F}) & \longrightarrow & \mathrm{Hom}_{\mathsf{A}/\mathsf{T}}(\mathcal{G}, \mathcal{F}) \end{array} \tag{14-3}$$

This diagram commutes by (6-7) in Proposition 1.6.12, and the left-hand vertical map is an isomorphism by (3), so the bottom map is an isomorphism too.

(6) This follows from (5) and Theorem 1.6.15. □

Theorem 14.8 *If T is a localizing subcategory of A, then there is a torsion functor $\tau : \mathsf{A} \to \mathsf{T}$. Furthermore, if π and ω denote the quotient and section functors, then for each M in A there is an exact sequence*

$$0 \longrightarrow \tau N \longrightarrow N \xrightarrow{\eta_N} \omega\pi N \longrightarrow \mathrm{coker}\,\eta_N \longrightarrow 0$$

in which η_N is an essential map, and $\mathrm{coker}\,\eta_N$ a torsion module.

Proof. If $W = \ker\eta_M$ and $Z = \mathrm{coker}\,\eta_M$, then there is an exact sequence

$$0 \longrightarrow \pi W \longrightarrow \pi M \xrightarrow{\pi(\eta_M)} \pi\omega\pi M \longrightarrow \pi Z \longrightarrow 0$$

in A/T. By Proposition 1.6.12, $\varepsilon_{\pi M} \circ \pi(\eta_M) = \mathrm{id}_{\pi M}$. However, part (5) of the previous result shows that $\varepsilon_{\pi M}$ is an isomorphism. Hence $\pi(\eta_M)$ is an isomorphism. Therefore $\pi W = \pi Z = 0$, showing that both W and Z are torsion (Proposition 13.7).

By Proposition 14.7(1), $\omega\pi M$ is torsion-free, so W contains every torsion submodule of M. Thus W is the largest torsion submodule of M.

If T is a submodule of $\omega\pi M$ such that $T \cap \eta_M(M) = 0$, then T embeds in Z, so is torsion. But $\omega\pi M$ is torsion-free, so $T = 0$. Thus $\eta_M(M)$ is essential in $\omega\pi M$. □

Lemma 14.9 *An essential extension of a torsion-free module is torsion-free.*

Proof. Let Q be an essential extension of a torsion-free module N. If $M \subset Q$ is a torsion module, so is $M \cap N$. Therefore $M \cap N = 0$, whence $M = 0$. $\square$

Example 14.10 An essential extension of a torsion module need not be torsion. Let R be a ring having a non-split extension $0 \to S \to M \to S' \to 0$ of two non-isomorphic simples (2×2 triangular matrices is such a ring). If T consists of all direct limits of finite length R-modules all of whose composition factors are isomorphic to S, then T is a localizing subcategory. Although S is torsion its essential extension M is not. $\Diamond$

The example also shows that applying π to an essential monic need not produce an essential monic.

Lemma 14.11 *Applying ω to an essential monic produces an essential monic.*

Proof. Because it is a right adjoint ω preserves monics. Let $\mathcal{L} \to \mathcal{M}$ be an essential monic in A/T. Suppose there is a direct sum $\omega\mathcal{L} \oplus N \subset \omega\mathcal{M}$. Applying π to this produces a direct sum $\mathcal{L} \oplus \pi N \subset \mathcal{M}$, so $\pi N = 0$. But N is torsion-free because $\omega\mathcal{M}$ is, so we deduce that $N = 0$. $\square$

Theorem 14.12 *Let T be a Serre subcategory of A. Suppose that a torsion functor $\tau : \mathsf{A} \to \mathsf{T}$ exists. If A has injective envelopes, then*

1. *T is a localizing subcategory of A;*
2. *for each N in A, $\omega\pi N$ is isomorphic to the largest submodule of the injective envelope of $N/\tau N$ which extends $N/\tau N$ by a torsion module.*

Proof. To show that the quotient functor $\pi : \mathsf{A} \to \mathsf{A}/\mathsf{T}$ has a right adjoint it suffices, by Proposition 1.6.20, to show that the functor

$$M \mapsto \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\pi M, \pi N)$$

is representable for each N in A. The representing object will be the module H we define next.

Fix N in A. Write $\bar{N} = N/\tau N$. Let H be the largest essential extension of a torsion module by $\bar{N}$. Explicitly, if $\alpha : \bar{N} \to E$ is the inclusion of N in an injective envelope, H is the kernel of the composition

$$E \to \operatorname{coker} \alpha \to \operatorname{coker} \alpha / \tau(\operatorname{coker} \alpha).$$

This gives rise to an exact sequence

$$0 \longrightarrow \tau N \longrightarrow N \xrightarrow{\ f\ } H \longrightarrow \operatorname{coker} f \longrightarrow 0$$

in which $\ker f$ and $\operatorname{coker} f$ are both torsion. In particular, $\pi f : \pi N \to \pi H$ is an isomorphism in A/T. By Lemma 14.5, $\bar{N}$ is torsion-free, hence so is H

by Lemma 14.9. Moreover, $E/H \cong \operatorname{coker}\alpha/\tau(\operatorname{coker}\alpha)$ is also torsion-free by Lemma 14.5.

Since πf is an isomorphism so is the map

$$\operatorname{Hom}(\pi M, \pi f) : \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\pi M, \pi N) \to \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\pi M, \pi H).$$

Thus, it suffices to show that H is a representing object for the functor

$$M \mapsto \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\pi M, \pi H).$$

We will do this by showing that $\pi : \operatorname{Hom}_{\mathsf{A}}(M, H) \to \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\pi M, \pi H)$ is an isomorphism.

Since H is torsion-free,

$$\operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\pi M, \pi H) = \varinjlim \operatorname{Hom}_{\mathsf{A}}(M', H)$$

where the direct limit is taken over those $M' \subset M$ for which M/M' is torsion. We will show for such an M' that the natural map $\operatorname{Hom}_{\mathsf{A}}(M, H) \to \operatorname{Hom}_{\mathsf{A}}(M', H)$ is an isomorphism. Since $\operatorname{Hom}_{\mathsf{A}}(-, H)$ is left exact and M/M' is torsion whereas H is torsion-free, it follows from Lemma 14.5 that this map is injective, so it remains to prove it is surjective. To see this, let $f' \in \operatorname{Hom}_{\mathsf{A}}(M', H)$ and consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "f'"'] & M \arrow[r] & M/M' \arrow[r] & 0 \\
0 \arrow[r] & H \arrow[r] & E \arrow[r] & E/H \arrow[r] & 0.
\end{tikzcd}$$

Since E is injective there is a morphism $f : M \to E$ extending the composition $M' \to H \to E$. It follows that there exists a morphism $g : M/M' \to E/H$ making the diagram commute. But E/H is torsion-free and M/M' is torsion, so $g = 0$ by Lemma 14.5. Therefore the image of f is contained in H and f' is the restriction of f. Hence the map $\operatorname{Hom}_{\mathsf{A}}(M, H) \to \operatorname{Hom}_{\mathsf{A}}(M', H)$ is surjective, and hence an isomorphism. □

Corollary 14.13 *Let* T *be a Serre subcategory of an abelian category* A. *Suppose that* A *has direct sums (i.e.,* A *is* $\mathsf{Ab3}$*), and injective envelopes. Then* T *is localizing if and only if* T *is closed under arbitrary direct sums.*

Proof. ($\Rightarrow$) By hypothesis, π has a right adjoint so it commutes with direct sums (Corollary 1.7.14). Therefore, if N_α are torsion modules, then $\pi(\oplus N_\alpha) \cong \oplus \pi N_\alpha = 0$, whence $\oplus N_\alpha$ is in T.

($\Leftarrow$) The direct sum of all the torsion submodules of a given module is torsion. But the sum of those submodules is a quotient of their direct sum, so is also torsion. Hence every module has a largest torsion submodule. It follows from Lemma 14.2 and Theorem 14.12 that T is localizing. □

A comparison of homological issues in A and A/T requires an understanding of the relation between injectives in A and A/T.

Theorem 14.14 *Suppose that* T *is a localizing subcategory of* A*, and that* A *has injective envelopes.*

1. ω *sends injectives to injectives, and injective envelopes to injective envelopes.*
2. *The injectives in* A/T *are* $\{\pi Q \mid Q$ *is a torsion-free injective in* $\mathsf{A}\}$.
3. A/T *has enough injectives.*
4. *If* Q *is a torsion-free injective* A*-module, then* $Q \cong \omega\pi Q$.

Proof. (1) Because it is right adjoint to an exact functor ω preserves injectives (Proposition 12.8). Then by Lemma 14.11 it preserves injective envelopes.

(2) If Q is a torsion-free injective, then $\operatorname{Hom}_{\mathsf{A}}(-,Q)$ is an exact functor vanishing on T so, by Theorem 13.9, the rule

$$\pi M \mapsto \operatorname{Hom}_{\mathsf{A}}(M,Q) \tag{14-4}$$

defines an exact functor on A/T. By Theorem 14.12(2), $Q \cong \omega\pi Q$ so, by Proposition 14.7(3),

$$\operatorname{Hom}_{\mathsf{A}}(M,Q) \cong \operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(\pi M, \pi Q).$$

Therefore the functor defined by (14-4) is equivalent to $\operatorname{Hom}_{\mathsf{A}/\mathsf{T}}(-,\pi Q)$. But (14-4) is an exact functor, so πQ is injective.

Let $\mathcal{Q}$ be an injective in A/T. Then $\omega\mathcal{Q}$ is injective by (1), and is torsion-free by Proposition 14.7(1). Moreover, $\pi\omega\mathcal{Q} \cong \mathcal{Q}$ by Proposition 14.7(5), so every injective in A/T is of the form πQ for some injective A-module Q.

(3) Let $\mathcal{F}$ be an A/T-module, and let $f : \omega\mathcal{F} \to Q$ be the inclusion of $\omega\mathcal{F}$ in its injective envelope. Since $\omega\mathcal{F}$ is torsion-free, so is Q (Lemma 14.9). But πf is monic, so πQ is an injective containing $\pi\omega\mathcal{F} \cong \mathcal{F}$. Thus A/T has enough injectives. $\square$

Theorem 16.3 shows that if A is a Grothendieck category and T a localizing subcategory, then A/T is a Grothendieck category.

Next we show how that the right derived functors of τ and ω are closely related when T is a localizing subcategory that is closed under injective envelopes.

Clearly, T is closed under injective envelopes if and only if every essential extension of a torsion module is torsion. This condition is sometimes described in the literature as a *stable torsion theory* (see [58, p. 46] and [240, p. 20] for example). In ???, we show that this condition is equivalent to an appropriate formulation of the Artin-Rees property.

Theorem 14.15 *Let* T *be a localizing subcategory of* A. *Suppose that* A *has enough injectives and that* T *is closed under injective envelopes. Then*

1. *every injective in* A *is a direct sum of a torsion injective and a torsion-free injective;*

2. *for $i \geq 1$, the right-derived functors of τ and ω satisfy*

$$R^{i+1}\tau M \cong R^i\omega(\pi M)$$

for all A*-modules M;*

3. *there is an exact sequence* $0 \to \tau M \to M \to \omega\pi M \to R^1\tau M \to 0$.

Proof. (1) Let E be an injective in A. Since E contains a copy of the injective envelope of τE, and since that injective is torsion by hypothesis, τE is injective. Therefore it is a direct summand of E, say $E = \tau E \oplus Q$. Clearly Q is a torsion-free injective.

(2) Let $M \to E^\bullet$ be an injective resolution of M. For each j, write I^j for the torsion submodule of E^j, and set $Q^j = E^j/I^j$. Then there is an exact sequence of complexes

$$0 \to I^\bullet \to E^\bullet \to Q^\bullet \to 0$$

which gives a long exact sequence

$$\cdots \to h^{i-1}(Q^\bullet) \to h^i(I^\bullet) \to h^i(E^\bullet) \to h^i(Q^\bullet) \to h^{i+1}(I^\bullet) \to \cdots$$

in homology. However, $h^i(I^\bullet) = R^i\tau M$, and $h^i(E^\bullet) = 0$ for $i \geq 1$. Therefore, for $i \geq 1$, $R^{i+1}\tau M \cong h^i(Q^\bullet)$.

By Theorem 14.14, πQ^j is injective in A/T, and $\omega\pi Q^j \cong Q^j$. Since π is exact, $\pi M \to \pi E^\bullet$ is an injective resolution in A/T. However, the complexes $\pi Q^\bullet$ and $\pi E^\bullet$ are isomorphic. Therefore, $\pi M \to \pi Q^\bullet$ is an injective resolution in A/T, so

$$R^i\omega(\pi M) \cong h^i(\omega\pi Q^\bullet) \cong h^i(Q^\bullet).$$

This completes the proof of (2), and (3) is given by the left-hand segment of the long homology sequence. □

Example 14.16 Let R be a commutative ring, and $\mathfrak{m}$ a maximal ideal in R. A module M is supported at $\mathfrak{m}$ if each element of M is killed by a power of $\mathfrak{m}$. Such modules form a Serre subcategory of $\mathsf{Mod}R$. This is a localizing subcategory, and the torsion functor τ is

$$\tau = \varinjlim \operatorname{Hom}_R(R/\mathfrak{m}^n, -).$$

The right derived functors of τ are therefore

$$R^i\tau = \varinjlim \operatorname{Ext}^i_R(R/\mathfrak{m}^n, -).$$

We write $H^i_{\mathfrak{m}}(M)$ for $R^i\tau M$, and call this the i^{th} local cohomology module of M with respect to τ.

The corresponding quotient category of $\mathsf{Mod}R$ is the category of quasi-coherent modules on the open complement in $\operatorname{Spec} R$ of $\mathfrak{m}$. This is called the punctured spectrum [79, Chapter 6] of R. If we write X for $\operatorname{Spec} R$, U for the punctured spectrum, and $j : U \to X$ for the inclusion map, then $j^* = \pi$ and

$j_* = \omega$. Therefore, if $M \in \mathsf{Mod}R$, and $\mathcal{M} = j^*M$ is its restriction to U, then $R^i j_* \mathcal{M} \cong H^{i+1}_{\mathfrak{m}}(M)$ for $i \geq 1$.

For example, when $X = \mathbb{A}^2$, and $U = \mathbb{A}^2 \backslash \{0\}$, one sees that $R^1 j_* \mathcal{O}_U \neq 0$ because $H^2_{\mathfrak{m}}(R)$ is isomorphic to the injective envelope of $R/\mathfrak{m}$ (ref???). $\lozenge$

The next result identifies an important source of Serre subcategories that are closed under injective envelopes. Its importance will be apparent when we examine the open complement to a hypersurface in a non-commutative space (3.3.7).

Proposition 14.17 *Let R be a right noetherian ring, and $x \in R$ a regular normal element. The x-torsion modules form a localizing category that is closed under injective envelopes.*

Proof. This is a consequence of Proposition 12.14. $\square$

Lemma 14.18 *Let T be a localizing subcategory of A. Suppose that M and N are simple A-modules, and that neither belongs to T. Then $M \cong N$ if and only if $\pi M \cong \pi N$.*

Proof. Let $f : \pi M \to \pi N$ be an isomorphism. This leads to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \omega\pi M & \longrightarrow & M' & \longrightarrow & 0 \\ & & & & {\scriptstyle\cong}\big\downarrow{\scriptstyle \omega f} & & & & \\ 0 & \longrightarrow & N & \longrightarrow & \omega\pi N & \longrightarrow & N' & \longrightarrow & 0 \end{array}$$

Since M is torsion-free, the image of M in N' is zero, so ωf sends M to N. Similarly, $\omega(f^{-1})$ sends N to M. It follows that $M \cong N$. $\square$

Lemma 14.19 *Let T be a localizing subcategory of A.*

1. *If M is noetherian, so is πM.*
2. *Suppose that every A-module is the union of its noetherian submodules. If $\mathcal{M}$ is a noetherian A/T-module, then there is a noetherian A-module M such that $\mathcal{M} \cong \pi M$.*

Proof. (1) Replacing M by $M/\tau M$, we may assume that M is torsion-free. Let $\mathcal{N}_1 \subset \mathcal{N}_2 \subset \dots$ be an ascending chain of submodules of πM. Because ω is left exact, $\omega\mathcal{N}_1 \subset \omega\mathcal{N}_2 \subset \dots$ is an ascending chain of submodules of $\omega\pi M$. Thus $\omega\mathcal{N}_1 \cap M \subset \omega\mathcal{N}_2 \cap M \subset \dots$ is an ascending chain of submodules of M. Since M is noetherian, it follows that this chain stabilizes. However, since π is left exact, it commutes with intersection. Thus, for large i,

$$\mathcal{N}_i = \pi\omega\mathcal{N}_i \cap \pi M = \pi(\omega\mathcal{N}_i \cap M) = \pi(\omega\mathcal{N}_{i+1} \cap M) = \mathcal{N}_{i+1}.$$

Hence the original chain stabilizes, and we conclude that πM is noetherian.

(2) By hypothesis, $\omega\mathcal{M}$ is the union of its noetherian submodules, say $\omega\mathcal{M} = \varinjlim M_i$, where each M_i is noetherian. Because π has a right adjoint, it commutes with direct limits, so $\mathcal{M} \cong \pi\omega\mathcal{M} = \varinjlim \pi M_i$. Each πM_i is a submodule of $\mathcal{M}$. By hypothesis, $\mathcal{M}$ is noetherian, so for some i, $\mathcal{M} = \pi M_i$. □

EXERCISES

14.1 Let $f : R \to S$ be a homomorphism of rings. Show that the full subcategory of $\mathsf{Mod}R$ consisting of those M for which $M \otimes_R S = 0$ is a Serre subcategory if and only if ${}_RS$ is flat.

14.2 Fill in the details required to show that the composition of morphisms in A/T is well-defined.

14.3 Show that if T is a Serre subcategory of A, then T^{op} is a Serre subcategory of A^{op}. Is $\mathsf{A}^{\mathrm{op}}/\mathsf{T}^{\mathrm{op}}$ equivalent to $(\mathsf{A}/\mathsf{T})^{\mathrm{op}}$? Give an example to show that T can be localizing, but T^{op} not localizing.

14.4 [91] Let P be an A-module. Define the full subcategory

$$P^{\perp} := \{M \mid \operatorname{Hom}(M, P) = 0\}.$$

(a) Show that $P^{\perp}$ need not be a Serre subcategory.

(b) Find conditions on P which ensure that $P^{\perp}$ is a Serre subcategory.

14.5 Let A denote the category of k-vector spaces, and let T be the full subcategory consisting of the finite dimensional vector spaces. Show that T is a Serre subcategory, but not a localizing subcategory.

14.6 Show that the artinian (respectively, noetherian) modules in an abelian category form a Serre subcategory.

14.7 Suppose that π has a right adjoint ω. Show that

(a) M indecomposable does not imply πM indecomposable (Hint: consider $M = k[x,y]/(xy)$ and its image in $\mathsf{Mod}\mathbb{P}^1$);

(b) $\omega\pi M$ indecomposable implies πM indecomposable;

(c) M indecomposable does not imply $\omega\pi M$ indecomposable.

14.8 Give conditions which ensure that a localization of $\mathsf{Mod}R$ is of the form $\mathsf{Mod}S$ for some ring S.

2.15 Left exact functors

Definition 15.1 Let A and C be abelian categories. The full subcategory of $\mathsf{Fun}(\mathsf{C},\mathsf{A})$ consisting of the left exact functors is denoted by $\mathsf{Lex}(\mathsf{C},\mathsf{A})$. ◊

We will show that $\mathsf{Lex}(\mathsf{C},\mathsf{A})$ is abelian when C has enough injectives.

Proposition 15.2 *Let $\tau : F \to G$ be a natural transformation of left exact functors. The kernel of τ in $\mathsf{Fun}(\mathsf{C},\mathsf{A})$ is left exact. However,* $\operatorname{coker}\tau$ *is not usually the same as its cokernel in* $\mathsf{Fun}(\mathsf{C},\mathsf{A})$.

Corollary 15.3 *The inclusion* $\mathsf{Lex}(\mathsf{C},\mathsf{A}) \to \mathsf{Fun}(\mathsf{C},\mathsf{A})$ *is left exact. In particular, kernels in* $\mathsf{Lex}(\mathsf{C},\mathsf{A})$ *coincide with kernels in* $\mathsf{Fun}(\mathsf{C},\mathsf{A})$.

Lemma 15.4 *Let* A *and* C *be abelian categories. If* C *has enough injectives, then a left exact functor* $F : \mathsf{C} \to \mathsf{A}$ *is determined by its value on injectives.*

Proof. Let M be an arbitrary C-module. Let $0 \to M \to E \to E'$ be the start of an injective resolution of M. Then $FM = \ker(FE \to FE')$.

$\boxed{\text{Paul}}$ show it doesn't depend on the inj res, and how is $F(f)$ defined. □

Proposition 15.5 *Suppose that* C *is an abelian category with enough injectives. Let* $\mathsf{Inj}(\mathsf{C})$ *denote the full subcategory of* C *consisting of injective* C*-modules. Then*

1. $\mathsf{Lex}(\mathsf{C},\mathsf{A}) \cong \mathsf{Fun}(\mathsf{Inj}(\mathsf{C}),\mathsf{A})$.
2. $\mathsf{Lex}(\mathsf{C},\mathsf{A})$ *is an abelian category.*

Proof. For brevity write $\mathsf{I} = \mathsf{Inj}(\mathsf{C})$. Restriction yields a natural functor $\mathsf{Lex}(\mathsf{C},\mathsf{A}) \to \mathsf{Fun}(\mathsf{I},\mathsf{A})$.

Let $F : \mathsf{I} \to \mathsf{A}$ be any functor. We extend F to C as follows. Let M be a C-module and take an exact sequence $0 \to M \to I_0 \to I_1$ with I_0 and I_1 injective; define $F'M := \ker(FI_0 \to FI_1)$. It is proved in [88, Chapitre I, Section 9] that F' is well-defined, and left exact.

$\boxed{\text{MORE}}$

□

The composition

$$\mathsf{Fun}(\mathsf{C},\mathsf{A}) \to \mathsf{Fun}(\mathsf{I},\mathsf{A}) \to \mathsf{Lex}(\mathsf{C},\mathsf{A}) \tag{15-1}$$

sends a functor F to its 0^{th} right derived functor R^0F. Thus, if $\tau : F \to G$ is a natural transformation of left exact functors, its cokernel in $\mathsf{Lex}(\mathsf{C},\mathsf{A})$ is the 0^{th} right derived functor of its cokernel in $\mathsf{Fun}(\mathsf{C},\mathsf{A})$. Is (15-1) left adjoint to the inclusion $\mathsf{Lex}(\mathsf{C},\mathsf{A}) \to \mathsf{Fun}(\mathsf{C},\mathsf{A})$?

Lemma 15.6 *Suppose that* C *is an abelian category with enough injectives. The embedding* $\mathsf{Lex}(\mathsf{C},\mathsf{A}) \to \mathsf{Fun}(\mathsf{C},\mathsf{A})$ *preserves monics.*

Proof. Let $0 \to F \to G$ be an exact sequence of functors in $\mathsf{Lex}(\mathsf{C},\mathsf{A})$. Since $\mathsf{Lex}(\mathsf{C},\mathsf{A})$ is equivalent to $\mathsf{Fun}(\mathsf{Inj}(\mathsf{C}),\mathsf{A})$, the sequence $0 \to F(I) \to G(I)$ is exact for all injectives I in C. Now, let M be an arbitrary C-module. By hypothesis, there is a monic $M \to I$ for some injective I. Since F and G are left exact, the vertical maps in the commutative diagram

$$\begin{tikzcd} F(M) \arrow[r] \arrow[d] & G(M) \arrow[d] \\ F(I) \arrow[r] & G(I) \end{tikzcd}$$

are monic. But the bottom map is monic, so the top one is too. Hence $F \to G$ is monic in $\mathsf{Fun}(\mathsf{C}, \mathsf{A})$. □

Give example to show this can't be improved.

EXERCISES

15.1 Let R be a ring, M and M' non-isomorphic simple left R-modules, and suppose that M is not projective. For each left R-module N define

$$FN := \{f \in \mathrm{Hom}_R(N, M) \mid f \text{ is not a split epimorphism}\}.$$

Show that F extends to a functor $\mathsf{Mod}R \to \mathsf{Ab}$, and hence that F is a proper subfunctor of $\mathrm{Hom}_R(-, M)$. [*Hint:* check that $F(M' \oplus M) = 0$ but $\mathrm{Hom}_R(M' \oplus M, M) \neq 0$, and that $FM \neq 0$ for some M.] This exercise shows that the functor $\mathsf{Mod}R \to \mathsf{Fun}(\mathsf{Mod}R, \mathsf{Ab})$ does not in general send irreducible modules to irreducible modules.

2.16 Examples and properties of Grothendieck categories

The first source of Grothendieck categories is module categories (16.2), and more are obtained as quotients of Grothendieck categories (16.3). These two observations culminate in the Gabriel-Popescu theorem characterizing Grothendieck categories as quotients of module categories (16.5).

Proposition 16.1 *If* A *is an abelian category, then* $\mathsf{Fun}(\mathsf{C}, \mathsf{A})$ *has the properties Ab3, Ab4, or Ab5 according to whether* A *does.*

Corollary 16.2 *Let G be a group and R a G-graded ring. Then* $\mathsf{GrMod}_G R$*, the category of G-graded R-modules with degree zero homomorphisms, is a Grothendieck category.*

Proof. Since $\mathsf{GrMod}_G R$ is of the form $\mathsf{Fun}(\mathsf{C}, \mathsf{Ab})$ it satsifies the condition Ab5. The module $R(i)$, $i \in G$, defined by $R(i)_j = R_{i+j}$, constitute a set of generators for $\mathsf{GrMod}_G R$. □

The basic properties of $\mathsf{GrMod}_G R$ are treated in more detail, and in a more straightforward way, in Chapter **??**.

Theorem 16.3 *Let* A *be a Grothendieck category, and* T *a localizing subcategory. Then* T *and* A/T *are Grothendieck categories.*

Proof. [182, Proposition 9, p. 378] By hypothesis, T is closed under direct limits, so satisfies $\mathsf{Ab3}$. Since $\mathsf{Ab5}$ holds for A it holds for T. So, to show T is Grothendieck, it remains to show that it has a set of generators.

Let $\{P_\lambda \mid \lambda \in \Lambda\}$ be a set of generators for A. The submodules of each P_λ form a set by Lemma 7.2, so the quotients of each P_λ also form a set. Hence the collection of all quotients of all P_λ is a set. Those quotients that belong to T provide a set of generators for T.

To show that $\{\pi P_\lambda \mid \lambda \in \Lambda\}$ is a set of generators for A/T consider a non-zero map $f : M \to N$ in A/T. Because $\pi \circ \omega \cong \mathrm{id}_{\mathsf{A}/\mathsf{T}}$, $\omega(f)$ is non-zero. Hence there is a morphism $g : P_\lambda \to \omega M$ such that $\omega(f) \circ g \neq 0$. The image of $\omega(f) \circ g$ is not torsion because it is a non-zero submodule of ωN so, by Proposition 13.7, $\pi(\omega(f) \circ g)$ is non-zero. Hence $f \circ \pi(g)$ is non-zero, and we conclude that $\{\pi P_\lambda \mid \lambda \in \Lambda\}$ is a set of generators for A/T.

Since π is a left adjoint it commutes with direct limits. So, if we are given a directed system in A/T, apply ω to it and take the direct limit of that in A, then apply π.

$\boxed{\text{Paul}}$ MORE TO DO □

Lemma 16.4 *Let* A *be a locally noetherian category and* T *a localizing subcategory. Let* $\pi : \mathsf{A} \to \mathsf{A}/\mathsf{T}$ *be the quotient functor and* $\omega : \mathsf{A}/\mathsf{T} \to \mathsf{A}$ *its right adjoint. Then* ω *commutes with direct sums.*

Proof. Let $\mathcal{M}_i$ be objects in A/T. We must show that the natural map $\oplus_i \omega\mathcal{M}_i \to \omega(\oplus_i \mathcal{M}_i)$ is an isomorphism. Define $M_i = \omega\mathcal{M}_i$. Thus $\pi M_i \cong \mathcal{M}_i$. Since π commutes with direct sums, it suffices to show that the natural map $\oplus_i M_i \to \omega\pi(\oplus_i M_i)$ is an isomorphism. Since each M_i is torsion-free, so is $\oplus_i M_i$; hence this map is monic. To show it is epic we must show that $\mathrm{Ext}^1_{\mathsf{A}}(N, \oplus_i M_i)$ is zero for all torsion modules N. However, since A is locally noetherian N is a direct limit of noetherian modules, and it therefore suffices to prove the vanishing of this Ext-group when N is noetherian. Since A is a Grothendieck category, direct sums are exact; since A is locally noetherian a direct sum of injectives is injective. Hence the direct sum of injective resolutions for each M_i is an injective resolution of $\oplus_i M_i$. Choose an injective resolution $M_i \to E_i^\bullet$ for each i. We must show that

$$0 \to \mathrm{Hom}_{\mathsf{A}}(N, \oplus E_i^0) \to \mathrm{Hom}_{\mathsf{A}}(N, \oplus E_i^1) \to \dots$$

is exact. Since N is noetherian, $\mathrm{Hom}_{\mathsf{A}}(N, -)$ commutes with direct sums; it follows that $\mathrm{Ext}^1_{\mathsf{A}}(N, \oplus_i M_i) = 0$ because $\mathrm{Ext}^1_{\mathsf{A}}(N, M_i) = 0$ for each i. □

Theorem 16.5 (Gabriel-Popescu) *Let* A *be a Grothendieck category. If* U *is a generator for* A*, and* R *is its endomorphism ring, then* $\mathrm{Hom}_{\mathsf{A}}(U, -) : \mathsf{A} \to \mathsf{Mod}R$ *is an equivalence between* A *and a quotient category of* $\mathsf{Mod}R$ *by a localizing subcategory.*

Proof. [241, page 220] The functor $\mathrm{Hom}_{\mathsf{A}}(U, -)$ is faithful because U is a generator. It commutes with products by definition of products, so has a left adjoint by Proposition 16.7. We denote that left adjoint by $- \otimes_R U$. The plan of the proof is to show that $\mathrm{Hom}_{\mathsf{A}}(U, -)$ is full, that $- \otimes_R U$ is exact, and then appeal to the proof of Theorem 3.7.2.

To show that $\mathrm{Hom}_{\mathsf{A}}(U, -)$ is full, we must show that every R-module map $\mathrm{Hom}_{\mathsf{A}}(U, M) \to \mathrm{Hom}_{\mathsf{A}}(U, N)$ is induced by a map $f : M \to N$ of A-modules. This is certainly true if $M = U$.

Because $\mathrm{Hom}_{\mathsf{A}}(U,-)$ is fully faithful, Theorem 1.6.15 implies that the composition $\mathrm{Hom}_{\mathsf{A}}(U,-) \otimes_R U$ is naturally equivalent to the identity functor id_{A}, so Theorem 3.7.2 applies, and says that A is equivalent to $\mathsf{Mod}R$ modulo a localizing subcategory. □

The Gabriel-Popescu theorem can be viewed as an enhancement of Mitchell's theorem 9.8. For Grothendieck categories this subcategory may be chosen so that the inclusion functor has an exact left adjoint.

Corollary 16.6 *A Grothendieck category is complete.*

Proof. This follows from the Gabriel-Popescu theorem and the following observation. Let A be an abelian category having products, and let $\pi : \mathsf{A} \to \mathsf{A}/\mathsf{T}$ be a quotient functor with right adjoint ω. Then A/T has products: if M_α are objects in A/T, then $\pi(\prod(\omega M_\alpha))$ is their product in A/T. □

Thus, in a Grothendieck category, one may form both sums and intersections of arbitrary sets of submodules of a given module (see page **??**).

A full subcategory of an abelian category is a Giraud subcategory if the inclusion functor has an exact left adjoint. Thus the Gabriel-Popescu theorem says that a Grothendieck category is equivalent to a Giraud subcategory of $\mathsf{Mod}R$ for some ring R. A left exact functor preserves limits, so commutes with products. Thus an inclusion functor having a left adjoint will commute with products; equivalently, if A is a subcategory of D and products in A differ from products in D, then A is not a Giraud subcategory.

Proposition 16.7 *Let C and D be Grothendieck categories.*

1. *A right exact functor $F : \mathsf{C} \to \mathsf{D}$ has a right adjoint if and only if it commutes with direct sums.*

2. *A left exact functor $F : \mathsf{C} \to \mathsf{D}$ has a left adjoint if and only if it commutes with direct products.*

Proof. □

Let X be a scheme. It is pointed out in [250, Appendix B] that the category $\mathsf{Mod}\mathcal{O}_X$ of all $\mathcal{O}_X$-modules is a Grothendieck category. However, on [250, page 409] they say that "It seems to be unknown whether, for general schemes X, $\mathsf{Qcoh}X$ has a set of generators, enough injectives, or even all limits." There is a forgetful functor $F : \mathsf{Qcoh}X \to \mathsf{Mod}\mathcal{O}_X$ that is exact and commutes with direct sums (so has a right adjoint), and reflects exactness. It follows that $\mathsf{Qcoh}X$ is cocomplete. On the positive side, they say that if X is quasi-separated and quasi-compact, then $\mathsf{Qcoh}X$ is a Grothendieck category. A topological space is quasi-compact if every open cover has a finite subcover, and a scheme is said to be quasi-compact if its underlying topological space is quasi-compact.

EXERCISES

16.1 Let C be a Giraud subcategory of an abelian category D. Let $i_* : \mathsf{C} \to \mathsf{D}$ be the inclusion, and write i^* for its left adjoint.

(a) Show that C has products if D does. Explicitly, given C-modules M_α, show that $i^*(\prod i_* M_\alpha)$ is a product of the M_α's in C.

(b) Define T to be the full subcategory of D consisting of modules M such that $i^*M = 0$. Show that T is a Serre subcategory of D.

(c) Let $\pi : \mathsf{D} \to \mathsf{D}/\mathsf{T}$ be the quotient functor. Show that $\pi i_* : \mathsf{C} \to \mathsf{D}/\mathsf{T}$ is an equivalence of categories.

16.2 In a Grothendieck category show that the canonical map $\oplus M_i \to \prod M_i$ is monic.

$\boxed{\text{Paul}}$ Is the category of complexes of R-modules equivalent to the category of graded modules over $R[\partial]/(\partial^2)$ with $\deg \partial = 1$? If so, then it is easy to see that complexes have injective resolutions.

16.3 [107, Exercise II 2.13] A topological space is noetherian if and only if every open subset is quasi-compact. Every affine scheme is quasi-compact.

16.4 Show that in a locally noetherian category, if M/N is a noetherian module, then there is a noetherian submodule L of M such that $L + N = M$. Where was exactness of direct limits used in the argument?

2.17 Injectives in a Grothendieck category

Theorem 17.1 *[94, Théorème 1.10.1] A Grothendieck category* A *has enough injectives. Indeed, there is a functor* E*, and a monic natural transformation* $\tau : \mathrm{id}_{\mathsf{A}} \to E$ *such that* $E(M)$ *is injective for all* M*.*

Proof. The category of modules over a ring has enough injectives (Theorem 12.9) so the result follows from Theorem 14.14 and the Gabriel-Popescu Theorem. $\square$

A Grothendieck category even has injective envelopes [89].

The next result is a natural extension of Baer's criterion that an R-module E is injective if and only if every homomorphism $f : I \to E$ from a right ideal extends to a homomorphism $g : R \to E$. It is proved in a similar way.

Proposition 17.2 *[94, Lemma 1, page 136] Suppose that* A *is a Grothendieck category. Then an* A*-module* E *is injective if and only if for every submodule* L *of a generator* M*, every morphism* $f : L \to E$ *extends to a morphism* $g : M \to E$*.*

Theorem 17.3 *In a locally noetherian category a direct sum of injectives is injective.*

Proof. See [241, Proposition 4.3, page 123] or [181, Theorem 3, page 207]. $\square$

Theorem 17.4 (Matlis) *An injective module in a locally noetherian category is isomorphic to a direct sum of injectives in an essentially unique way.*

Proof. See [241, Proposition 4.5, page 124]. □

Proposition 17.5 *The following conditions on a module M in a locally neotherian category are equivalent:*

1. *M is noetherian;*
2. *M is finitely generated;*
3. *M is finitely presented;*
4. $\mathrm{Hom}(M, -)$ *commutes with direct sums.*

Proof.

□

2.18 Presheaves and Sheaves

Example 18.1 Make the natural numbers a topological space, denoted by $\mathbf{N}$, by declaring the open sets to be the empty set, $\mathbb{N}$ itself, and the intervals $[0, n] = \{0, 1, \ldots, n\}$. Suppose now that C is a category in which inverse limits exist. Then the category of inverse systems in C is equivalent to the category of C-valued sheaves on $\mathbf{N}$; the sheaf $\tilde{M}$ associated to the inverse system $(M_n)_{n\in\mathbb{N}}$ is defined by $\tilde{M}([0, n]) = M_n$ and $\tilde{M}(\mathbf{N}) = \varprojlim M_n$. Thus taking the inverse limit coincides with taking the global sections, and left exactness of inverse limits is a special case of the fact that the global section functor is left exact. The right derived functors of $\varprojlim$ are therefore the same things as the higher sheaf cohomology functors $H^i(\mathbf{N}, -)$. These derived functors may be computed via flasque resolutions, and using these J.-E. Roos [?] has shown that $R^2 \varprojlim = 0$ for inverse systems indexed by $\mathbb{N}$. ◊

Theorem 18.2 *The sheaves of abelian groups on a topological space form a Grothendieck category.*

Proof. Let P and S respectively denote the category of presheaves and sheaves of abelian groups on X. By ???, P is a Grothendieck category. We will show that S is equivalent to the quotient category P/T where T is the localizing subcategory of P consisting of those presheaves $\mathcal{F}$ such that the stalks $\mathcal{F}_x$ are zero for all points x. The result will then follow from Theorem 16.3.

We consider S as a full subcategory of P, and denote by $\omega : \mathsf{S} \to \mathsf{P}$ the inclusion functor. We will show that ω has an exact left adjoint, namely the sheafification functor. If $\mathcal{F}$ is a presheaf, define the presheaf $\mathcal{F}^+$ by

$$\mathcal{F}^+(U) := \{(s_x) \in \prod_{x\in U} \mathcal{F}_x \mid \text{there exists an open covering } U_\lambda \text{ of } U$$
$$\text{elements } s_\lambda \in \mathcal{F}(U_\lambda) \text{ such that } s_\lambda|_x = s_x \text{ for all } x \in U_\lambda\}$$

where $s_\lambda|x = \rho_x^{U_\lambda}(s_\lambda)$ and $\rho_x^{U_\lambda} : \mathcal{F}(U_\lambda) \to \mathcal{F}_x$ is the map defined by the universal property of $\mathcal{F}_x$ as a direct limit. It is straightforward to verify that $\mathcal{F}^+$ is a sheaf. The rule $\mathcal{F} \to \mathcal{F}^+$ extends in an obvious way to a functor $\pi : \mathsf{P} \to \mathsf{S}$. We will show that π is a left adjoint to ω and exact.

$\boxed{\text{MORE}}$

□

The existence of π in the previous proof is another illustration of the adage that left adjoints to forgetful functors solve universal problems.

Definition 18.3 A sheaf $\mathcal{F}$ is flasque if the restristion map $\mathcal{F}(U) \to \mathcal{F}(V)$ is epic whenever $V \subset U$. ◊

Flasques are acyclic for $H^0(X, -)$, so one can compute $H^q(X, -)$ via flasque resolutions.

Chapter 3

Non-commutative spaces

Our philosophy is that non-commutative spaces are made manifest by the modules that live on them in the same way that the properties of a commutative scheme X are manifested by the category $\mathsf{Qcoh}X$ of quasi-coherent $\mathcal{O}_X$-modules. The modules over a non-commutative space form, by definition, an abelian category. That category is the basic object of study in non-commutative geometry. In short, a non-commutative space is an abelian category.

This chapter expands on this philosophy. It begins with the notions of affine spaces and affine maps, the geometric versions of rings and ring homomorphisms. The new language allows one to re-express some elementary ring-theoretic results in a more geometric way. Watt's Theorem in section 3.3 is a good example of this. Section four introduces a family of well-understood affine non-commutative spaces having finite dimensional coordinate rings, namely the path algebras of quivers. Section five introduces the notion of a map between non-commutative spaces, and illustrates the idea with examples drawn from the previous section. A map is defined as an adjoint pair of functors, the direct, and inverse, image functors between the module categories over the spaces. In section four we introduce the notion of an open subspace. Closed points are defined and discussed in section four. Open subspaces, and open complements, are intorduced in sections six and seven. Some examples related to projective modules are treated in section eight. For example, if e is an idempotent in a ring R, then the open complement in $\mathsf{Mod}R$ to the zero locus of e is isomorphic to $\mathsf{Mod}(eRe)$. This applies in particular to a skew group ring $R = A * G$ when G is a finite group whose order is a unit in A. In the final section of this chapter we draw pictures of some non-commutative spaces, and add to the supply of examples that has already been accumulated. The pictures suggest the existence of various maps between different spaces. Almost all the examples in this chapter are *affine* spaces. An affine space is defined as one of the form $\mathsf{Mod}R$, the category of right modules over a (possibly non-commutative) ring R.

A principal goal of this chapter is to provide a rapid introduction to a broad range of examples, and to introduce some of the basic definitions and ideas. The examples will provide motivation and context for the more systematic development of the subject in subsequent chapters. The definitions and results will give the reader some idea of where we are headed, and what the flavor of the subject is. This approach results some unevenness in the level of discussion. Although most of this chapter is accessible to someone who has had a basic graduate course covering some non-commutative and commutative algebra, and some affine algebraic geometry, there are several places where we assume more of the reader. However, the reader can always return to parts of this chapter after reading some of the subsequent chapters.

The commutative background. Our definitions of spaces, maps, closed subspaces, open complements, closed points, and so on are motivated by a mixture of commutative algebraic geometry and non-commutative algebra. To prepare for what follows we first review some of this material.

The usual geometric object associated to a commutative ring R is the affine scheme $\operatorname{Spec} R$, the spectrum of R. As a set $\operatorname{Spec} R$ consists of the prime ideals of R. This set is then endowed with the Zariski topology, the closed sets of which are the sets

$$\mathcal{V}(I) := \{\mathfrak{p} \mid \mathfrak{p} \supset I\},$$

where I ranges over all ideals of R. As a scheme $\operatorname{Spec} R$ is a pair consisting of the set of prime ideals of R endowed with the Zariski topology, together with the structure sheaf $\mathcal{O}_{\operatorname{Spec} R}$ of rings on it. One recovers R from this data as the global sections of $\mathcal{O}_{\operatorname{Spec} R}$. The ring R can also be recovered from $\mathsf{Mod}R$. It is the only *commutative* endomorphism ring of a progenerator. This is because Morita equivalent commutative rings are isomorphic. More generally, the center of a ring is a Morita invariant because it is the ring of natural transformations of the identity functor on the module category (Corollary 2.11.5).

If R is noetherian, then $\operatorname{Spec} R$ can be recovered directly from $\mathsf{Mod}R$ as follows. The isomorphism classes of the indecomposable injectives in $\mathsf{Mod}R$ are in bijection with the prime ideals [158, Theorem 18.4, p. 145]. The injective corresponding to the prime $\mathfrak{p}$ is the injective envelope of $R/\mathfrak{p}$, which we denote by $E(R/\mathfrak{p})$. This gives $\operatorname{Spec} R$ as a set. The Zariski topology can be recovered from the fact that $\mathfrak{p} \subset \mathfrak{q}$ if and only if there is a non-zero map $E(R/\mathfrak{q}) \to E(R/\mathfrak{p})$. The closed subschemes can also be described directly in terms of $\mathsf{Mod}R$ through Theorem 3.14 below. The open complements of the closed subschemes can be recovered through localizations of the module category (see section 3.7). The local rings $R_\mathfrak{p}$ can be recovered as the unique commutative endomorphism rings of progenerators in various quotient categories of $\mathsf{Mod}R$. Thus, one sees that the commutativity of R is not needed for the basic constructions of algebraic geometry. Indeed, the ring itself is not needed. All that is needed is the category $\mathsf{Mod}R$.

Once one realizes that the ring can be dispensed with it is natural when working with non-affine schemes to emphasize the properties of the category of quasi-coherent modules, and to de-emphasize the fact that one has sheaves of modules over a sheaf of rings on some underlying topological space.

The closed points of $\operatorname{Spec} R$ are precisely the maximal ideals. The set of maximal ideals is in bijection with the set of isomorphism classes of simple modules. Up to isomorphism the simple modules are the quotients $R/\mathfrak{m}$ as $\mathfrak{m}$ ranges over the set of maximal ideals. Thus, over a non-commutative ring, the set of isomorphism classes of the simple modules would appear to be a reasonable substitute for the set of closed points of any geometric space associated to the ring. However, this is too naive. Non-commutative algebras over a field k can have both finite dimensional and infinite dimensional simple modules. Although the finite dimensional ones behave rather like the simple modules over a commutative ring, the infinite dimensional ones do not. For example, the infinite dimensional ones usually defy classification. Thus our first definition of closed points will be in terms of the simple modules that behave like the simples over a commutative ring (see Definition 4.1 and Theorem 4.13).

Consider a homomorphism

$$\varphi : R \to S$$

of commutative rings. Since an ideal $\mathfrak{p}$ in S is prime exactly when $S/\mathfrak{p}$ is a domain, it follows that $\varphi^{-1}(\mathfrak{p})$ is prime. Therefore φ determines a map

$$f : \operatorname{Spec} S \to \operatorname{Spec} R$$

defined by $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. It is immediate that f is a continuous map. Therefore the rule $R \mapsto \operatorname{Spec} R$ is a contravariant functor from commutative rings to topological spaces. The inclusion $\mathbb{Z} \to \mathbb{Q}$ shows that the map f need not send maximal ideals to maximal ideals. This is why one works with the space of prime ideals rather than just the maximal ideals.

Each S-module N can be made into an R-module by defining

$$x.r = x\varphi(r)$$

for $x \in N$ and $r \in R$. This rule provides a functor

$$f_* : \mathsf{Mod}S \to \mathsf{Mod}R$$

because a homomorphism of S-modules is automatically an R-module homomorphism. It is obvious that f_* is exact.

The functor f_* is related to images under f in the following way. Suppose that p is a closed point of $\operatorname{Spec} S$ such that $f(p)$ is a closed point of $\operatorname{Spec} R$. Write $\mathfrak{n}$ and $\mathfrak{m}$ for the maximal ideals of S and R corresponding to p and $f(p)$. Then φ induces an inclusion $R/\mathfrak{m} \to S/\mathfrak{n}$. Therefore, as an R-module, $S/\mathfrak{n}$ is isomorphic to a direct sum of copies of $R/\mathfrak{m}$. In other words, if $\mathcal{O}_p$ and $\mathcal{O}_{f(p)}$ denote the simple modules $S/\mathfrak{n}$ and $R/\mathfrak{m}$ respectively, then $f_*(\mathcal{O}_p)$ is a direct sum of copies of $\mathcal{O}_{f(p)}$. If we define $\mathsf{Mod}p$ and $\mathsf{Mod}f(p)$ to consist of all direct sums of the simple modules $\mathcal{O}_p$ and $\mathcal{O}_{f(p)}$ respectively, f_* sends $\mathsf{Mod}p$ to $\mathsf{Mod}f(p)$.

The functor f_* has a left adjoint, namely $- \otimes_R S$, which we denote by

$$f^* : \mathsf{Mod}R \to \mathsf{Mod}S.$$

The action of f^* is related to the fibers of f in the following way. Let q be a closed point of $\operatorname{Spec} R$, and p a point of $\operatorname{Spec} S$. Let $\mathfrak{m}$ be the maximal ideal of R corresponding to q, and let $\mathfrak{p}$ be the prime ideal of S that corresponds to p. Then p is in the fiber $f^{-1}(q)$ if and only if $\varphi^{-1}(\mathfrak{p}) = \mathfrak{m}$; since $\mathfrak{m}$ is maximal, this is equivalent to the condition that $\varphi(\mathfrak{m}) \subset \mathfrak{p}$, or that $\mathfrak{p}$ is a prime ideal of $S/\varphi(\mathfrak{m})S = R/\mathfrak{m} \otimes_R S = f^*(\mathcal{O}_q)$. Therefore $f^{-1}(q) = \operatorname{Spec} f^*\mathcal{O}_q$.

The functors f_* and f^* are determined by φ, but one can also recover φ from these functors. For example, if $r \in R$, let $\rho : R \to R$ be multiplication by r. Then $f^*(\rho)$ is an S-module homomorphism from $S = f^*R$ to itself. Such a homomorphism is multiplication by an element of S. That element is $\varphi(r)$. Since the data φ is equivalent to the data (f^*, f_*), and since non-commutative spaces are categories, it is natural to define maps directly in terms of an adjoint pair of functors. This is what we do in Definition 3.1.

3.1 Spaces and subspaces

Definition 1.1 A non-commutative space X is a Grothendieck category $\mathsf{Mod}X$. Thus $X = \mathsf{Mod}X$. The objects in $\mathsf{Mod}X$ are called X-modules. $\lozenge$

A non-commutative space is denoted by a single letter X when we wish to think of it as a geometric object, and by $\mathsf{Mod}X$ when we wish to emphasize that it is a category. Thus a non-commutative space is an imaginary geometric object which manifests itself through its category of modules. The space is the category.

Definition 1.2 A space X is noetherian if every X-module is a direct limit of noetherian X-modules (i.e., if $\mathsf{Mod}X$ is locally noetherian). If X is noetherian, we denote by $\mathsf{mod}X$ the full subcategory of $\mathsf{Mod}X$ consisting of the noetherian modules. $\lozenge$

Example 1.3 Let X be a commutative noetherian scheme. Then $\mathsf{Mod}X$ denotes the category of quasi-coherent $\mathcal{O}_X$-modules, and $\mathsf{mod}X$ consists of the coherent X-modules. When we refer to an X-module, we always mean a quasi-coherent $\mathcal{O}_X$-module.

If X is quasi-compact and quasi-separated, then $\mathsf{Mod}X$ is a Grothendieck category [250, ??]. Most everyday schemes are quasi-compact and quasi-separated. For example, an open subscheme of a closed subscheme of a projective space $\mathbb{P}^n$ over a field is. Thus, $\mathsf{Mod}X$ is a Grothendieck category for such schemes. $\lozenge$

Definition 1.4 A space X is affine if $\mathsf{Mod}X$ has a progenerator. Equivalently, $\mathsf{Mod}X$ is equivalent to $\mathsf{Mod}R$ for some ring R. We call R a coordinate ring of X. $\lozenge$

An affine space can have many different coordinate rings.

The ur-example of a non-commutative ring is the ring $M_n(k)$ of $n \times n$ matrices over a field k. However, from the module-theoretic point of view this ring is commutative because $\mathsf{Mod}M_n(k)$ is equivalent to the category of k-vector spaces. From the module theoretic point of view, the difference between k and $M_n(k)$ is that one takes not just the category $\mathsf{Mod}X$, but also a distinguished object in it. If V denotes the unique simple object, and one takes as the distinguished object $V^{\oplus n}$, then one can recover $M_n(k)$ as the endomorphism ring of this distinguished object.

Definition 1.5 An enriched non-commutative space as a pair $(X, \mathcal{O}_X)$ consisting of a Grothendieck category $X = \mathsf{Mod}X$ together with a distinguished object $\mathcal{O}_X$ in it, which we call the structure module. $\Diamond$

The structure module $\mathcal{O}_X$ determines a ring, $R = \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{O}_X)$, and a functor

$$\operatorname{Hom}_X(\mathcal{O}_X, -) : \mathsf{Mod}X \to \mathsf{Mod}R.$$

If $\mathcal{O}_X$ is a progenerator in $\mathsf{Mod}X$ this is an equivalence of categories. Different coordinate rings can arise from different progenerators.

The ring T of $n \times n$ lower triangular matrices over k is a good first example of a non-commutative ring. There are n simple right T-modules up to isomorphism so, by analogy with the commutative case, we think of the non-commutative space $\mathsf{Mod}T$ as having n closed points. Of course, this is the same as the spectrum of the commutative ring $k \times \cdots \times k = k^{\times n}$. The ring $k^{\times n}$ is isomorphic to T/I where I is its nilpotent radical, the strictly lower triangular matrices. The difference between T and $k^{\times n}$ lies in the nilpotent structure. However, the role of the nilpotent structure in a non-commutative space is a little different from its role in the commutative case.

The closed points of an affine scheme $\operatorname{Spec} R$ are in bijection with the full subcategories of $\mathsf{Mod}R$ of the form $\mathsf{Mod}R/\mathfrak{m}$, where $\mathfrak{m}$ runs over the maximal ideals. The closed subschemes of $\operatorname{Spec} R$ are in bijection with the full subcategories $\mathsf{Mod}R/I$ for the various ideals I in R. However, the subcategories of $\mathsf{Mod}R$ of the form $\mathsf{Mod}R/I$ can be characterized without reference to the ring structures as the full subcategories that are closed under subquotients and such that the inclusion functor has both a right and a left adjoint (Theorem 3.14). That result motivates the definition of a closed subspace (Definition 2.1). But first we simply define a subspace.

Definition 1.6 A subspace Y of a space X is a full Grothendieck subcategory $\mathsf{Mod}Y$ of $\mathsf{Mod}X$ that is closed under direct sums and isomorphisms, and such that the inclusion functor $i_* : \mathsf{Mod}Y \to \mathsf{Mod}X$ is left exact. We indicate that Y is a subspace of X by saying *let $i : Y \to X$ be the inclusion of a subspace.* $\Diamond$

A subcategory that is closed under isomorphisms is said to be replete [83, p. 75]. Whenever we take a full subcategory of $\mathsf{Mod}X$ we will insist that it is closed under isomorphisms.

If Y is a subspace of X, and M is a Y-module, then every Y-submodule of M is an X-module because i_* is left exact, but a quotient of M in $\mathsf{Mod}Y$ need not be a quotient in $\mathsf{Mod}X$.

Example 1.7 The inclusion of a full subcategory in an abelian category need not be an exact functor. For example, the section functor $\omega : \mathsf{A}/\mathsf{T} \to \mathsf{A}$ associated to a localization is not generally exact. A simple example of this arises from the inclusion

$$j : U = \mathbb{A}^2 \backslash \{0\} \to \mathbb{A}^2 = \operatorname{Spec} k[X, Y]$$

of the punctured affine plane in the affine plane. Write $A = k[X, Y]$ and let $\mathfrak{m}$ be the ideal vanishing at the origin. The direct image functor $j_* : \mathsf{Mod}U \to \mathsf{Mod}\mathbb{A}^2$ realizes $\mathsf{Mod}U$ as the full subcategory of $\mathsf{Mod}\mathbb{A}^2$ consisting of those modules M such that $\operatorname{Hom}_A(A/\mathfrak{m}, M) = \operatorname{Ext}^1_A(A/\mathfrak{m}, M) = 0$. It is clear that $\mathsf{Mod}U$ is closed under submodules in $\mathsf{Mod}\mathbb{A}^2$, but is not closed under quotient modules. The inclusion j_* is not exact. $\boxed{\text{Paul}}$ give an example. ♢

Lemma 1.8 *Let $i_* : \mathsf{B} \to \mathsf{A}$ be the inclusion of a full subcategory of an abelian category A. Suppose that B is abelian. Then i_* is left exact if and only if B is closed under kernels in A.*

Proof. ($\Rightarrow$) Let $\alpha : M \to N$ be a morphism in B, and let K be its kernel in B. Then $0 \to K \to M \to N$ is exact in B, and hence exact in A. Therefore K is also the kernel of α in A. In other words, the kernel in A of a morphism in B is in B.

($\Leftarrow$) Let $0 \to K \to M \to N \to 0$ be an exact sequence in B. Then K is the kernel in B of $M \to N$, so is also the kernel in A. Therefore $0 \to K \to M \to N$ is exact in A. This says that i_* is left exact. □

If Y is a subspace of X, the inclusion $\mathsf{Mod}Y \to \mathsf{Mod}X$ preserves monics. I do not know if it is reasonable to ask that this preserve epics. I don't even know if it does this in the commutative case.

Lemma 1.9 *Let B be a full subcategory of an abelian category A. The following two statements are equivalent.*

1. *The category B is abelian, and the inclusion functor $i_* : \mathsf{B} \to \mathsf{A}$ is exact.*
2. *If $\alpha : M \to N$ is a morphism in B, then the kernel and cokernel of α taken in A belong to B.*

If either condition holds, then B is also closed under images.

Proof. Let $\alpha : M \to N$ be a morphism in B. Let (K, β) and (C, γ) denote the kernel and cokernel of α in A.

(1) $\Rightarrow$ (2) By Lemma 1.8, K is in B. By the dual version of Lemma 1.8, C is also in B.

(2) $\Rightarrow$ (1) By hypothesis, $\beta : K \to M$ is a morphism in B. To see it has the appropriate universal property in B, suppose that $\gamma : L \to M$ is a morphism in B such that $\alpha\gamma = 0$. Using the universal property of $\beta : K \to M$ in A, there is a unique morphism $\delta : L \to K$ in A such that $\gamma = \beta\delta$. Because B is full, and

because L and K are in B, δ is in B. Its uniqueness is clear. Thus (K,β) is a kernel of α in B. The result for cokernels is proved in the same way.

Thus kernels and cokernels of morphisms in B are the same as those taken in A. It follows that $\operatorname{coker}\ker\alpha$ is unambiguous—it belongs to B, and is the same whether computed in A or B. The same applies to $\ker\operatorname{coker}\alpha$. The canonical morphism $\operatorname{coker}\ker\alpha \to \ker\operatorname{coker}\alpha$, and its inverse, are in B because B is full. Therefore that morphism is an isomorphism in B. Hence B is abelian. Now we may apply Lemma 1.8, and its dual version, to conclude that i_* is exact, thus proving (1).

The final remark about images is immediate because an image is a kernel of a cokernel, and these are the same whether taken in A or in B. □

We can view a category as its collection of morphisms by identifying each object with the identity morphism on it. Hence if Y and Z are subspaces of X, we say that Z is contained in Y, or lies on Y, and write $Z \subset Y$, if $\mathsf{Mod}Z \subset \mathsf{Mod}Y$. If Z is a subspace of X that is contained in another subspace Y of X, then Z is a subspace of Y because Lemma 1.8 ensures that the inclusion $\mathsf{Mod}Z \to \mathsf{Mod}Y$ is left exact. Conversely, if Y is a subspace of X, and Z is a subspace of Y, then Z is a subspace of X.

The empty space is defined by declaring $\mathsf{Mod}\phi$ to be the abelian category with one object, the zero module. The empty subspace of X consists of just the zero module. It is contained in every subspace of X.

Definition 1.10 If X and Y are spaces, their disjoint union $X \coprod Y$ is defined by

$$\mathsf{Mod}(X \coprod Y) := (\mathsf{Mod}X) \times (\mathsf{Mod}Y),$$

the product of the categories. ◊

$\boxed{\text{Paul}}$ Check if this is a Grothendieck category.

Definition 1.11 If Y and Z are subspaces of X, their intersection $Y \cap Z$ is defined by

$$\mathsf{Mod}(Y \cap Z) := \mathsf{Mod}Y \bigcap \mathsf{Mod}Z$$

provided that this category is a Grothendieck category. ◊

I do not know conditions that ensure that an intersection of Grothendieck categories is again Grothendieck, so this notion of intersection is at present rather useless. However, when the intersections do exist as subspaces one has

$$\begin{aligned} &Y \cap Z = Z \cap Y, \quad && Y \cap \phi = \phi, \\ &Y \cap X = Y, && W \cap (Y \cap Z) = (W \cap Y) \cap Z. \end{aligned}$$

3.2 Closed subspaces

Definition 2.1 [88, page ??] [258] A subspace Y of a non-commutative space X is closed if $\mathsf{Mod}Y$ is closed under subquotients, and the inclusion functor $i_* : \mathsf{Mod}Y \to \mathsf{Mod}X$ has both a left and a right adjoint. We denote the left adjoint by i^* and the right adjoint by $i^!$. We write $i : Y \to X$ for the "inclusion map" (see Definition 3.1). ◊

For example, the empty subspace and X itself are closed subspaces of X.

Lemma 2.2 *A subspace Y of a space X is closed if and only if the inclusion $i_* : \mathsf{Mod}Y \to \mathsf{Mod}X$ is exact and $\mathsf{Mod}Y$ is closed under subquotients, products, and coproducts, in $\mathsf{Mod}X$.*

Proof. This follows from Proposition 2.16.7. □

Example 2.3 (The zero locus of an ideal.) Let I be a two sided ideal of a ring R. Let X and Y be the affine spaces with coordinate rings R and R/I respectively. We can, and will, identify $\mathsf{Mod}Y$ with the full subcategory of $\mathsf{Mod}X$ consisting of those R-modules annihilated by I. It is clear that the inclusion of $\mathsf{Mod}Y$ in $\mathsf{Mod}X$ is exact, and that $\mathsf{Mod}Y$ is closed under subquotients. Since products and direct sums of modules annihilated by I are annihilated by I, Y is a closed subspace of X. Theorem 3.14 will show that every closed subspace of X is of this form. We often write $\mathcal{Z}(I)$ for Y and call it the zero locus of I. If $x_1, \ldots, x_n \in R$ we write $\mathcal{Z}(x_1, \ldots, x_n)$ for the zero locus of the two-sided ideal generated by those elements. ◊

A non-commutative ring may have few two-sided ideals, so the notion of closed subspace is not very effective in non-commutative algebraic geometry. For that reason the less restrictive notion of a weakly closed subspace is important (see section 3.8).

Proposition 2.4 *Let $i : Z \to X$ be a closed subspace of a space X. If M is an X-module, then the natural maps $M \to i_*i^*M$ and $i_*i^!M \to M$ are epic and monic respectively.*

Proof. The cokernel C in the sequence $M \to i_*i^*M \to C \to 0$ is a Z-module because $\mathsf{Mod}Z$ is closed under quotients. The induced sequence $i^*M \to i^*i_*i^*M \to i^*C \to 0$ is exact because i^* is a left adjoint. But the natural transformation $\mathrm{id}_Z \to i^*i_*$ is an isomorphism by Theorem 2.6.15, so $i^*C = 0$. However, $C \cong i^*i_*C = i^*C$, so we conclude that $C = 0$. Hence the map $M \to i_*i^*M$ is epic.

The other case is similar. □

Proposition 2.5 *Let $Z \subset Y \subset X$ be subspaces of a non-commutative space X. Suppose that Z is closed in X.*

1. *If* $\mathsf{Mod}Z$ *is closed under quotient modules in* $\mathsf{Mod}Y$*, then* Z *is a closed subspace of* Y*.*

2. *If* Y *is closed in* X*, then* Z *is a closed subspace of* Y*.*

Proof. Let $\beta_* : \mathsf{Mod}Z \to \mathsf{Mod}Y$ and $\alpha_* : \mathsf{Mod}Y \to \mathsf{Mod}X$ are the inclusion functors. Then $i_* := \alpha_*\beta_*$ is the inclusion $\mathsf{Mod}Z \to \mathsf{Mod}X$. By hypothesis, i_* has a left adjoint i^* and a right adjoint $i^!$. Define

$$\beta^! := i^!\alpha_* \quad \text{and} \quad \beta^* = i^*\alpha_*.$$

Thus $\beta^!$ and β^* are functors from $\mathsf{Mod}Y$ to $\mathsf{Mod}Z$.

(1)

(2) By hypothesis, α_* has a left adjoint α^*, and a right adjoint $\alpha^!$. Let $M \in \mathsf{Mod}Z$ and $N \in \mathsf{Mod}Y$. Then

$$\begin{aligned}\mathrm{Hom}_Z(\beta^*N, M) &= \mathrm{Hom}_Z(i^*\alpha_*N, M)\\ &\cong \mathrm{Hom}_X(\alpha_*N, i_*M)\\ &\cong \mathrm{Hom}_Y(N, \alpha^!\alpha_*\beta_*M)\\ &\cong \mathrm{Hom}_Y(N, \beta_*M),\end{aligned}$$

so β^* is left adjoint to β_*. Also

$$\begin{aligned}\mathrm{Hom}_Z(M, \beta^!N) &= \mathrm{Hom}_Z(M, i^!\alpha_*N)\\ &\cong \mathrm{Hom}_X(i_*M, \alpha_*N)\\ &\cong \mathrm{Hom}_Y(\alpha^*\alpha_*\beta_*M, N)\\ &\cong \mathrm{Hom}_Y(\beta_*M, N),\end{aligned}$$

so $\beta^!$ is right adjoint to β_*. It is clear that $\mathsf{Mod}Z$ is closed in $\mathsf{Mod}Y$ under submodules and quotients, so Z is a closed subspace of Y. $\square$

Proposition 2.6 *An intersection of closed subspaces is a closed subspace.*

Proof. Suppose that Y and Z are closed subspaces of X. Let f be a morphism in $\mathsf{Mod}Y \cap \mathsf{Mod}Z$. Since $\mathsf{Mod}Y$ and $\mathsf{Mod}Z$ are closed under kernels and cokernels, the kernel and cokernel of f in $\mathsf{Mod}X$ belong to $\mathsf{Mod}Y \cap \mathsf{Mod}Z$. Hence the inclusion of $\mathsf{Mod}Y \cap \mathsf{Mod}Z$ in $\mathsf{Mod}X$ is exact. Both $\mathsf{Mod}Y$ and $\mathsf{Mod}Z$ are closed under products and coproducts in $\mathsf{Mod}X$, hence so is $\mathsf{Mod}Y \cap \mathsf{Mod}Z$. Thus, the inclusion of $\mathsf{Mod}Y \cap \mathsf{Mod}Z$ in $\mathsf{Mod}X$ has both a right and a left adjoint. $\square$

If Y and Z are closed in X, is $Y \cap Z$ closed in Y?

EXERCISES

2.1 Let T denote the ring of $n \times n$ lower triangular matrices over the field k. Let $y = \sum_{i=1}^{n-1} e_{i+1,i}$.

(a) Show that $yT = Ty$, and that this consists of all strictly lower triangular matrices. Thus $T/yT \cong k^{\times n}$.

(b) Show that the simple module corresponding to e_{ii}, that is, the simple module on which e_{ii} acts as the identity, is $T/yT + (1 - e_{ii})T$.

2.2 Retain the notation in Exercise 1. Let U denote the ring of $n \times n$ upper triangular matrices.

(a) Show there is an algebra isomorphism $T \to U$ given by the k-linear map $e_{ij} \mapsto e_{n-i,n-j}$.

(b) Show there is an algebra anti-isomorphism $T \to U$ given by the k-linear map $e_{ij} \mapsto e_{ji}$.

2.3 Let R be a commutative ring. Prove that there is a bijection between the indecomposable injectives and the prime ideals.

2.4 Show that the natural transformations from the identity functor on $\mathsf{Mod}\, R$ to itself form a ring that is isomorphic to the center of R.

2.5 Let R be a commutative ring, and $\mathfrak{p}$ a prime ideal. Show that every element of the injective envelope of $R/\mathfrak{p}$ is killed by a power of $\mathfrak{p}$. Show this is false for non-commutative rings by examining the lower triangular matrix algebra.

2.6 Let R be the ring $k\langle x, y\rangle/(xy + yx)$. Suppose that $\operatorname{char} k \neq 2$. Show that the center, say Z, of R is $k[x^2, y^2]$, and that this is isomorphic to the polynomial ring in two variables. By finding a k-vector-space basis for R, show R is free of rank four as a Z-module. Find all simple R-modules when k is algebraically closed.

2.7 This is an open-ended exercise about the ring R in the previous exercise. Try to think of a sensible definition of the undefined terms (no one else has defined them so you are free to do as you wish). Consider the correspondence (coming from Hilbert's Nullstellensatz) between curves and points in the affine plane and the modules over the commutative polynomial ring on two variables. Using the analogy with this, draw a picture of a non-commutative space which has "coordinate ring R". What is a "line" and a "parabola" in your space? What does it mean to say "two lines intersect"? Does it make sense to speak of three "lines" having a "common point of intersection"? What does it mean to say a line is "tangent" to a parabola in this space?

3.3 Affine spaces and affine maps

The category of affine schemes is the opposite of the category of commutative rings and ring homomorphisms. As a direct generalization of this one might be tempted to define the category of affine spaces to be the opposite of the category of rings and ring homomorphisms. However, this does not seem to be appropriate (see the remark after Theorem 3.6). We prefer to define the morphisms in this category, the affine maps, in a more abstract way that will make the definition of maps between arbitrary spaces seem more natural.

Affine maps are special kinds of maps. In this section we examine affine maps, and delay a study of maps until section 3.6. Recall that a morphism

$f : Y \to X$ between commutative schemes gives rise to an adjoint pair of functors f^* and f_*, the inverse, and direct image functors.

Definition 3.1 A map $f : Y \to X$ of spaces is an adjoint pair of functors (f^*, f_*) such that $f_* : \mathsf{Mod}Y \to \mathsf{Mod}X$. We call f^* the inverse image functor and f_* the direct image functor. We identify two maps having isomorphic direct image functors. Thus a map is a natural equivalence class of adjoint pairs (f^*, f_*). $\Diamond$

If $g : Z \to Y$ and $f : Y \to X$ are maps their composition fg is defined by $(fg)_* = f_* g_*$. It is clear that a composition of maps is a map. Thus there is a 2-category of spaces and maps.

Definition 3.2 An affine map $f : Y \to X$ of spaces is an adjoint triple $(f^*, f_*, f^!)$ such that $f_* : \mathsf{Mod}Y \to \mathsf{Mod}X$ is faithful. We identify two affine maps having isomorphic direct image functors. $\Diamond$

A composition of affine maps is an affine map. The inclusion of a closed subspace is an affine map. If f is an affine map, then f_* is exact because it has both adjoints.

Example 3.3 A ring homomorphism $\varphi : R \to S$ induces an affine map of affine spaces $f : \mathsf{Mod}S \to \mathsf{Mod}R$ defined by

$$f^* := - \otimes_R S \quad \text{and} \quad f_* := \operatorname{Hom}_S({}_R S, -).$$

Since f_* is "view an S-module as an R-module", it is faithful. Since f_* is naturally equivalent to $- \otimes_S S_R$, it has a right adjoint, namely

$$f^! := \operatorname{Hom}_R({}_S S, -).$$

Thus f is an affine map. $\Diamond$

Proposition 3.4 *Let $f : Y \to X$ be an affine map. If X is affine, so is Y.*

Proof. Let P be a progenerator in $\mathsf{Mod}X$. Since f^* is left adjoint to an exact functor it sends projectives to projectives. Since $\operatorname{Hom}_Y(f^*P, -) \cong \operatorname{Hom}_X(P, -) \circ f_*$ is a composition of faithful functors it is faithful. Thus f^*P is a projective generator. Since f_* has a right adjoint it commutes with direct sums, hence so does $\operatorname{Hom}_Y(f^*P, -)$. Thus f^*P is finitely generated, and therefore a progenerator. In other words, Y is affine.

If R is a coordinate ring of X, then $S = \operatorname{Hom}_Y(f^*R, f^*R)$ is a coordinate ring of Y, and $\operatorname{Hom}_Y(f^*R, -) : \mathsf{Mod}Y \to \mathsf{Mod}S$ is an equivalence of categories. $\square$

Theorem 3.5 (Watt's Theorem) *Let $f : Y \to X$ be a map of affine spaces. Suppose that $X = \mathsf{Mod}R$ and $Y = \mathsf{Mod}S$. Then there exists an R-S-bimodule B such that*

$$f^* = - \otimes_R B, \quad \textit{and} \quad f_* = \operatorname{Hom}_S(B, -).$$

Proof. If M is a right S-module, then $f_*M \cong \operatorname{Hom}_R(R, f_*M) \cong \operatorname{Hom}_S(f^*R, M)$. The first isomorphism is as right R-modules, and the second is as abelian groups. We must show that this is actually an isomorphism of R-modules. Write B for the right S-module f^*R. If $r \in R$, then $\lambda_r : R \to R$, $\lambda_r(a) = ra$ is a homomorphism of right R-modules, so $f^*(\lambda_r) : B \to B$ is a right S-module map. Thus B becomes an R-S-bimodule, so $\operatorname{Hom}_S(B, -)$ is a functor from $\mathsf{Mod}R$.

If $r \in R$, then the adjoint isomorphism gives a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_R(R, f_*M) & \longrightarrow & \operatorname{Hom}_S(f^*R, M) \\ \Big\downarrow{\scriptstyle -\circ\lambda_r} & & {\scriptstyle -\circ f^*(\lambda_r)}\Big\downarrow \\ \operatorname{Hom}_R(R, f_*M) & \longrightarrow & \operatorname{Hom}_S(f^*R, M) \end{array}$$

When $\operatorname{Hom}_R(R, f_*M)$ is identified with f_*M in the obvious way, the left-hand vertical map is $x \mapsto xr$. The right-hand vertical map is $y \mapsto yr$, where this the natural action of R on $\operatorname{Hom}_S(B, M)$. In other words, this commutative diagram says that the map $\operatorname{Hom}_R(R, f_*M) \to \operatorname{Hom}_S(f^*R, M)$ is a right R-module map. It follows that f_* is naturally equivalent to $\operatorname{Hom}_S(B, -)$. Therefore f^* is naturally equivalent to $- \otimes_R B$. $\square$

The next result is a partial converse to the fact that ring homomorphisms induce affine maps (Example 3.3).

Theorem 3.6 *Suppose that $f : Y \to X$ is an affine map between affine spaces. Then there exist coordinate rings R and S for X and Y respectively, a ring homomorphism $\varphi : R \to S$, and a commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \cong\Big\downarrow & & \Big\downarrow\cong \\ \mathsf{Mod}S & \xrightarrow[g]{} & \mathsf{Mod}R, \end{array} \tag{3-1}$$

where g is the map induced by φ.

Proof. Let R and S be arbitrary coordinate rings of X and Y. We fix equivalences $\mathsf{Mod}X \to \mathsf{Mod}R$ and $\mathsf{Mod}Y \to \mathsf{Mod}S$, and identify the categories. By Watt's Theorem, there is an R-S-bimodule ${}_RB_S$ such that $f^* = - \otimes_R B$. Since f_* is exact, B_S is projective. Since f_* has a right adjoint it commutes with direct sums, so B_S is finitely generated. Since f_* is faithful, B_S is a generator. Thus B_S is a progenerator, and we conclude that $\mathsf{Mod}S$ is equivalent to the module category over the endomorphism ring of B_S.

We set $S' = \operatorname{End}_S B \cong B \otimes_S B^*$, where $B^* = \operatorname{Hom}_S(B, S)$. The action of f^* on morphisms induces a ring homomorphism

$$\varphi : R \to \operatorname{Hom}_Y(f^*R, f^*R) = S'.$$

Let g denote the map inducd by φ, and let $\sigma : \mathsf{Mod}S \to \mathsf{Mod}S'$ be the equivalence defined by $\sigma^* = - \otimes_S B^*$. Then

$$g^* = - \otimes_R S' \cong - \otimes_R B \otimes_S B^* = \sigma^* \circ f^*.$$

Thus $g = f \circ \sigma$ and $f = g \circ \sigma^{-1}$. One now obtains the commutative diagram (3-1) with S' in place of S. □

Despite this result the functor from Rings to spaces sending a ring R to the space X with $\mathsf{Mod}X = \mathsf{Mod}R$ and sending a ring homomorphism to the affine map described in Example 3.3 is not an equivalence. The functor is not full: The inclusion $k \to M_n(k)$ induces a map of spaces $\mathsf{Mod}k \to \mathsf{Mod}k$ that is not induced by a ring homomorphism $k \to k$.

We now follow [107, Chapter II, Exercise 5.17] to show that an affine morphism of schemes $f : Y \to X$ is an affine map in our sense. First, f is affine if $f^{-1}(V)$ is affine for every open affine $V \subset X$. A candidate for a right adjoint to f_* is the functor $f^!$ defined by

$$(f^!\mathcal{M})(f^{-1}V) := \mathrm{Hom}_{\mathcal{O}_X(V)}(\mathcal{O}_Y(f^{-1}V), \mathcal{M}(V))$$

for every $\mathcal{M} \in \mathsf{Qcoh}X$ and every open $V \subset X$.

$\boxed{\text{DETAILS}}$

The next result says that affine maps behave like continuous maps with respect to closed subspaces.

Proposition 3.7 *Let $f : Y \to X$ be an affine map. Let Z be a closed subspace of X. Define $\mathsf{Mod}f^{-1}(Z)$ to be the full subcategory of $\mathsf{Mod}Y$ consisting of those M such that f_*M is a Z-module. Then $f^{-1}(Z)$ is a closed subspace of Y.*

Proof. Because f is affine, f_* is exact, and commutes with direct products and direct sums. Write $i : Z \to X$ for the inclusion. Because i_* has both a left and a right adjoint, $\mathsf{Mod}Z$ is closed under direct products and direct sums in $\mathsf{Mod}X$. Write C for $\mathsf{Mod}f^{-1}(Z)$.

First, we show that C is closed under subquotients. If M is in C, and $0 \to L \to M \to N \to 0$ is exact in $\mathsf{Mod}Y$, then $0 \to f_*L \to f_*M \to f_*N \to 0$ is exact in $\mathsf{Mod}X$. Hence f_*N and f_*L are Z-modules, whence N and L are in C.

Since C is closed under subquotients, the inclusion of C in $\mathsf{Mod}Y$ is exact. To show that the inclusion has both a left and a right adjoint it suffices to show that C is closed under direct products and direct sums. If M_λ is a family of Y-modules in C, then $f_*(\prod M_\lambda) = \prod f_*M_\lambda$. This is in $\mathsf{Mod}Z$, so $\prod M_\lambda$ is in C. The argument for direct sums is similar. □

Proposition 3.7 yields a commutative diagram of spaces and maps

$$\begin{array}{ccc} f^{-1}(Z) & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

Therefore, if Y and Z are closed subspaces of X, then $Y \cap Z$ is closed in Y (cf. Proposition 2.6).

Remark. Suppose that X and Y are affine spaces with coordinate rings R and S, and $f : Y \to X$ is induced by a homomorphism $\varphi : R \to S$. Let Z be the closed subspace of X cut out by a two-sided ideal I in R. Then $f^{-1}(Z)$ is cut out by the two-sided ideal of S generated by $\varphi(I)$.

Remark. The definition of $f^{-1}(Z)$ in Proposition 3.7 does not look good for a non-affine map. To see this consider the case of the structure map $f : \mathbb{P}^1 \to \operatorname{Spec} k$ and look at $f^{-1}(\phi)$. We have $f_*(\mathcal{O}(-1)) = 0$, but we would want $f^{-1}(\phi) = \phi$. What is the right definition of $f^{-1}(Z)$?

We now consider the subspaces of an affine space that are related to epimorphisms of rings. A homomorphism $\varphi : R \to S$ is an epimorphism in the category of rings if whenever $\psi_1, \psi_2 : S \to T$ are ring homomorphisms such that $\psi_1\varphi = \psi_2\varphi$, then $\psi_1 = \psi_2$.

It is clear that a surjective ring homomorphism is an epimorphism.

Example 3.8 The standard example of an epimorphism of rings is a localization. If $\varphi : R \to S$ is a homomorphism of rings such that S is generated as a ring by the image of R and inverses of elements in that image, then φ is an epimorphism. The proof is elementary.

A continuous map of topological spaces is an epimorphism if and only if its image is dense. Notice that if x is a regular element in a commutative ring R, then the inclusion $R \to R[x^{-1}]$ corresponds to the morphism of schemes $\operatorname{Spec} R[x^{-1}] \to \operatorname{Spec} R$ and this is an epimorphism in the category of topological spaces. If X is an irreducible variety, then every non-empty open subvariety is dense. $\lozenge$

Example 3.9 The inclusion

$$\begin{pmatrix} k & 0 \\ k & k \end{pmatrix} \to \begin{pmatrix} k & k \\ k & k \end{pmatrix} \tag{3-2}$$

is an epimorphism of rings. To see this, suppose that $\psi_i : M_2(k) \to S$, $i = 1, 2$, are homomorphisms that agree on the lower triangular matrices. Then

$$\begin{aligned}
\psi_1(e_{12}) - \psi_2(e_{12}) &= \psi_1(e_{11})\psi_1(e_{12}) - \psi_2(e_{11})\psi_2(e_{12}) \\
&= \psi_1(e_{11})\big(\psi_1(e_{12}) - \psi_2(e_{12})\big) \\
&= \psi_1(e_{12})\psi_1(e_{21})\big(\psi_1(e_{12}) - \psi_2(e_{12})\big) \\
&= \psi_1(e_{12})\big(\psi_1(e_{21})\psi_1(e_{12}) - \psi_2(e_{21})\psi_2(e_{12})\big) \\
&= \psi_1(e_{12})\big(\psi_1(e_{22}) - \psi_2(e_{22})\big) \\
&= 0
\end{aligned}$$

Hence $\psi_1 = \psi_2$. $\lozenge$

Example 3.10 The inclusion

$$\begin{pmatrix} k & 0 \\ k+kx & k \end{pmatrix} \to M_2(k[x]). \tag{3-3}$$

is an epimorphism of rings. To see this, let f and g be two homomorphisms from $M_2(k[x])$ that agree on the smaller algebra. By Example 3.9, f and g agree on $M_2(k)$. But together $M_2(k)$ and xe_{21} generate $M_2(k[x])$, so $f = g$. $\Diamond$

Our treatment of epimorphisms follows the seminal paper of Silver [211].

Lemma 3.11 *Let $\varphi : R \to S$ be a ring homomorphism. The kernel of the multiplication map $\mu : S \otimes_R S \to S$ is generated as a left or right S-module by $\{s \otimes 1 - 1 \otimes s \mid s \in S\}$. In particular, μ is an isomorphism of S-S-bimodules if and only if $s \otimes 1 = 1 \otimes s$ for all $s \in S$.*

Proof. Clearly each element $s \otimes 1 - 1 \otimes s$ is in the kernel. On the other hand, if $\sum s_i \otimes t_i$ is in the kernel of μ, then

$$\sum s_i \otimes t_i = \sum (s_i \otimes 1 - 1 \otimes s_i)t_i,$$

so the elements $s \otimes 1 - 1 \otimes s$ generate the kernel as a right module. A similar argument shows they also generate it as a left module. $\square$

Theorem 3.12 *The following conditions on a ring homomorphism $\varphi : R \to S$ are equivalent:*

1. *φ is an epimorphism;*
2. *the multiplication map $S \otimes_R S \to S$ is an isomorphism;*
3. *the restriction functor $f_* : \mathsf{Mod}S \to \mathsf{Mod}R$ is fully faithful.*

Proof. (1) $\Rightarrow$ (2) Suppose that M is an S-S-bimodule having an element m such that $mr = rm$ for all $r \in R$, but $ms \neq sm$ for some $s \in S$. Define a ring structure on $S \oplus M$ by

$$(a, u).(b, v) = (ab, av + ub),$$

and ring homomorphisms $\psi_1, \psi_2 : S \to S \oplus M$ by $\psi_1(s) = (s, 0)$ and $\psi_2(s) = (s, sm - ms)$. Then $\psi_1\varphi = \psi_2\varphi$, but $\psi_1 \neq \psi_2$, contradicting the fact that φ is an epimorphism. Hence, no such M exists. Applying this to $M = S \otimes_R S$ and $m = 1 \otimes 1$, since $rm = mr$ for all $r \in R$, we have $sm = ms$ for all $s \in S$. That is, $s \otimes 1 = 1 \otimes s$. Hence the map $S \otimes_R S \to S$ is an isomorphism.

(2) $\Rightarrow$ (1) Let $\psi_1, \psi_2 : S \to T$ be ring homomorphisms such that $\psi_1\varphi = \psi_2\varphi$. Then the map $\psi_1 \otimes \psi_2 : S \otimes_R S \to T$ sending $a \otimes b$ to $\psi_1(a)\psi_2(b)$ is well-defined. By Lemma 3.11, $s \otimes 1 = 1 \otimes s$, so

$$\psi_1(s) = (\psi_1 \otimes \psi_2)(s \otimes 1) = (\psi_1 \otimes \psi_2)(1 \otimes s) = \psi_2(s)$$

so $\psi_1 = \psi_2$.

(1) $\Rightarrow$ (3) The functor f_* is exact and faithful. To show it is full we need to show that every R-module map $\theta : M \to N$ between two S-modules is an S-module map.

Now f_* has a left adjoint $f^* = - \otimes_R S$. If M is an S-module, then

$$f^* f_* M = M \otimes_R S \cong M \otimes_S S \otimes_R S \cong M \otimes_S S \cong M.$$

Therefore

$$\mathrm{Hom}_S(M, N) \cong \mathrm{Hom}_S(f^* f_* M, N) \cong \mathrm{Hom}_R(f_* M, f_* N) = \mathrm{Hom}_R(M, N),$$

as required.

(3) $\Rightarrow$ (2) By hypothesis, f_* embeds $\mathsf{Mod}S$ as a full subcategory of $\mathsf{Mod}R$. Therefore, by Theorem 2.6.15, the adjunction $f^* f_* \to \mathrm{id}_S$ is an equivalence. In particular, $S \otimes_R S = f^* f_* S \to S$ is an isomorphism of right S-modules. It is also a left S-module map. $\square$

Theorem 3.13 *Let R be a ring. Let $f : Y \to X$ be an affine map to an affine space X. Let R and S be coordinate rings of X and Y respectively, and let $\varphi : R \to S$ be the ring homomorphism corresponding to f as in Theorem 3.6. Then*

1. *f_* is full if and only if φ is an epimorphism of rings, and*
2. *if f_* is full and $f_*(\mathsf{Mod}Y)$ is closed under subquotients, the map $\varphi : R \to S$ is surjective.*

Proof. (1) This is a consequence of Theorem 3.12 because $f_* : \mathsf{Mod}S \to \mathsf{Mod}R$ is fully faithful.

(2) Since $R/\ker\varphi$ is an R-submodule of S, the hypothesis implies that $R/\ker\varphi$ is an S-submodule of S. But it contains 1, so must equal S. $\square$

Theorem 3.14 *There is an order reversing bijection between the set of closed subspaces of an affine space $X = \mathsf{Mod}R$ and the set of two sided ideals of R. The ideal corresponding to a closed subspace Z is the largest ideal that annihilates all Z-modules, and the closed subspace corresponding to an ideal I is $\mathsf{Mod}R/I$, the full subcategory of modules annihilated by I.*

Proof. This follows from part (2) of Theorem 3.13. $\square$

Let Y and Z be the closed subspaces of $\mathsf{Mod}R$ cut out by the ideals I and J respectively. Then $X \cap Y$ is cut out by $I + J$. One reason we have not defined the union of closed subspaces is that there are (at least!) two candidates in the affine case—one could take the closed subspace cut out by IJ or that cut out by JI. Possibly one should allow both possibilities and so obtain a non-commutative notion of "union". I have not thought about this.

EXERCISES

3.1 Let $S = k[t]$ and $R = k + I$ where I is the product of two distinct maximal ideals of S. Show that the inclusion $\varphi : R \to S$ is not an epimorphism of rings. [Hint: φ corresponds to a map from the affine line to the nodal cubic that identifies two disinct points.]

3.4 Closed Points

Motivated by the Nullstellensatz, closed points are defined in terms of simple modules.

Notation. If D is a division ring we write $\operatorname{Spec} D$ for the space $\mathsf{Mod}D$.

Definition 4.1 A closed point of a space X is a closed subspace that is isomorphic to $\operatorname{Spec} D$ for some division ring D. We call the point a D-rational point of X. $\Diamond$

A closed point of X will be denoted by a single letter, say p, and we will often write $p \in X$ to indicate this. Let p be a closed point of X. Since $\mathsf{Mod}p$ is closed under subquotients and equivalent to the category of modules over a division ring, it contains a unique (up to isomorphism) simple X-module. We denote this simple by $\mathcal{O}_p$. Thus $\mathsf{Mod}p \cong \mathsf{Mod}D$ where $D = \operatorname{End}_X \mathcal{O}_p$, and every p-module is a direct sum of copies of $\mathcal{O}_p$. We shall sometimes refer to the simple module $\mathcal{O}_p$ as a point.

The category $\mathsf{Sum}S$ associated to a simple module. If S is a simple X-module, we write $\mathsf{Sum}S$ for the full subcategory of $\mathsf{Mod}X$ consisting of all modules that are isomorphic to a direct sum of copies of S. Because $\mathsf{Mod}X$ satisfies Grothendieck's condition Ab5, Propositions 2.5.11 and 2.5.12 ensure that $\mathsf{Sum}S$ is closed under submodules and quotients. It is therefore closed under subquotients.

Proposition 4.2 *Let D be the endomorphism ring of a simple X-module S. Then $\mathsf{Sum}S$ is equivalent to $\mathsf{Mod}D$.*

Proof. It suffices to show that S is a progenerator in $\mathsf{Sum}S$. It is a generator because every module in $\mathsf{Sum}S$ is a direct sum of copies of S. To see that S is projective in $\mathsf{Sum}S$, suppose that $g : M \to N$ is an epimorphism in $\mathsf{Sum}S$, and that $f : S \to N$ is an arbitrary map. We must show there exists a map $h : S \to M$ such that $gh = f$. Every submodule of M, in particular $g^{-1}(f(S))$, is isomorphic to a direct sum of copies of S. Thus g sends at least one of those copies isomorphically onto $f(S)$. The existence of h follows easily. Finally, a simple module is finitely generated, so S is a progenerator in $\mathsf{Sum}S$. $\square$

It is natural to ask whether a simple X-module, S say, gives a closed point of X. In general, the answer is "*no*" because the inclusion functor $\mathsf{Sum}S \to \mathsf{Mod}X$ does not always have a left adjoint.

Lemma 4.3 *Let S be a simple X-module. The inclusion $i_* : \mathsf{Sum}S \to \mathsf{Mod}X$ has a right adjoint $i^!$ given by*

$$i^!N = \textit{the sum of all submodules of } N \textit{ that are isomorphic to } S.$$

Proof. If $f : M \to N$ is a map between X-modules, then $f(i^!M) \subset i^!N$, so $i^!$ can be defined on morphisms by sending a morphism to its restriction. Thus $i^!$ really is a functor, and takes values in $\mathsf{Sum}S$ because a sum of copies of S is isomorphic to a direct sum of copies of S (Proposition 2.5.12). It is clear that $\operatorname{Hom}_X(S, M) = \operatorname{Hom}_X(S, i^!M)$, so $i^!$ is right adjoint to i_*.

Alternatively, since $\mathsf{Sum}S$ is closed under subquotients, i_* is exact, and since $\mathsf{Sum}S$ is closed under direct sums, i_* commutes with direct sums, whence i_* has a right adjoint by Proposition 2.16.7. □

Definition 4.4 An X-module S is tiny if $\operatorname{Hom}_X(M, S)$ is a finitely generated module over $\operatorname{End}_X S$ for all finitely generated X-modules M. ◊

Finitely generated modules in abelian categories are examined in section 2.6.

Lemma 4.5 *A simple X-module S is tiny if and only if every finitely generated submodule of every direct product of copies of S is isomorphic to a finite direct sum of copies of S.*

Proof. Let D be the endomorphism ring of S.

($\Rightarrow$) Let P be a direct product of copies of S, and N a non-zero finitely generated submodule of P. By hypothesis, $\operatorname{Hom}_X(N, S)$ has a finite basis, say $\rho_1, \dots, \rho_n$, as a left D-module. If the intersection of the kernels of all the ρ_i were non-zero, there would be a projection $P \to S$ that did not vanish on that intersection (cf. Exercise 2.4). However, the restriction of that projection to N is in the span of the ρ_i so vanishes on the intersection of the kernels of the ρ_i. This is a contradiction, so we conclude that the common intersection of the kernels of the ρ_i is zero.

By the universal property of $S^{\oplus n}$, there is a map $\rho : N \to S^{\oplus n}$ such that $\rho_i = \gamma_i \rho$ for $1 \leq i \leq n$, where $\gamma_i : S^{\oplus n} \to S$ is the projection onto the i^{th} component. Since $\ker \rho$ is contained in the intersection of all the $\ker \rho_i$, it follows that ρ is monic. Hence N is isomorphic to a finite direct sum of copies of S.

($\Leftarrow$) Suppose that M is a finitely generated X-module. Let K denote the intersection of $\ker f$ taken over all $f \in \operatorname{Hom}_X(M, S)$. Then $\operatorname{Hom}_X(M, S) \cong \operatorname{Hom}_X(M/K, S)$. Now M/K embeds in the product of copies of S indexed by the elements of $\operatorname{Hom}_X(M, S)$, and M/K is finitely generated because M is. Hence M/K is isomorphic to a finite direct sum of copies of S. Thus $\operatorname{Hom}_X(M/K, S)$ is isomorphic to a finite product of copies of $\operatorname{Hom}(S, S)$; this is certainly a finitely generated module over $\operatorname{End}_X S$, so we conclude that S is tiny. □

Theorem 4.6 *Suppose that* $\mathsf{Mod}X$ *is locally finitely generated. Let* S *be a simple* X*-module and let* $i_* : \mathsf{Sum}S \to \mathsf{Mod}X$ *denote the inclusion. The following are equivalent:*

1. *there is a closed point* $p \in X$ *such that* $S \cong \mathcal{O}_p$*;*
2. i_* *has a left adjoint;*
3. *every direct product of copies of* S *is isomorphic to a direct sum of copies of* S*;*
4. S *is tiny.*

Proof. (1) $\Rightarrow$ (2) This is true by the very definition of a closed subspace.

(2) $\Leftarrow$ (1) By Lemma 4.3, i_* has a right adjoint, so the hypothesis ensures that $\mathsf{Sum}S$ satisfies the requirements to be the module category of a closed point.

(2) $\Leftrightarrow$ (3) Since i_* is exact, it has a left adjoint if and only if it commutes with products. That is, if and only if $\mathsf{Sum}S$ is closed under products. But this is equivalent to condition (3).

(3) $\Rightarrow$ (4) Let M be a finitely generated X-module. We have $\mathrm{Hom}_X(M,S) = \mathrm{Hom}_X(M, i_*S) \cong \mathrm{Hom}_X(i^*M, S)$. But i^*M is a quotient of M, so is finitely generated, and belongs to $\mathsf{Sum}S$, so is a *finite* direct sum of copies of S. Thus $\mathrm{Hom}_X(M,S)$ is a finitely generated module over $\mathrm{End}_X S$.

(4) $\Rightarrow$ (3) Let P be a product of copies of S. By hypothesis, P is a direct limit, and hence a sum, of finitely generated submodules. By Lemma 4.5, each of those submodules is semisimple, so P is a sum of semisimple modules. However, in an Ab5 category every sum of semisimple modules is semisimple. Therefore P is semisimple. □

Proposition 4.7 *A simple module* S *over a ring* R *is tiny if and only if* $R/\operatorname{Ann}S$ *has finite length.*

Proof. ($\Rightarrow$) Write $D = \mathrm{End}_X S$ and $I = \operatorname{Ann}S$. It is clear that R/I embeds in a product of copies of S. But R/I is a finitely generated module, so is isomorphic to a finite direct sum of copies of S.

($\Leftarrow$) Since I is a prime ideal, R/I is a matrix algebra over D. In particular, it is isomorphic to a finite direct sum of copies of S. Any R-module map $M \to S$ must vanish on MI, so $\mathrm{Hom}_R(M,S) = \mathrm{Hom}_R(M/MI, S)$. However, if M is a finitely generated R-module, then M/MI is a finitely generated R/I-module. It follows that $\mathrm{Hom}_R(M/MI, S)$ is a finite direct sum of copies of D. □

Over a commutative ring, all simples are tiny.

Let R be an algebra over a field k, and M a simple R-module such that $\dim_k M < \infty$. Then $R/\operatorname{Ann}M$ embeds in $\mathrm{End}_k M$, so is a finite dimensional k-algebra, and hence artinian. Thus M is tiny.

Proposition 4.8 *Let* M *be a tiny simple* R*-module. Then*

1. *$R/\operatorname{Ann} M$ is isomorphic to a matrix algebra over D;*

2. *M is the unique simple $R/\operatorname{Ann} M$-module up to isomorphism.*

Proof. Since $R/\operatorname{Ann} M$ is artinian, there is a finite set of elements $m_1, \dots, m_n$ in M such that $\operatorname{Ann}(m_1) \cap \dots \cap \operatorname{Ann}(m_n) = \operatorname{Ann} M$. Since $R/\operatorname{Ann}(m_i) \cong M$ for all i, it follows that $R/\operatorname{Ann} M$ has a finite composition series with all factors isomorphic to M. Therefore, by uniqueness of composition factors, M is the only simple module annihilated by $\operatorname{Ann} M$. To see that $R/\operatorname{Ann} M$ is simple, let I be a proper ideal containing $\operatorname{Ann} M$. The only possible composition factor of R/I is M, so M is an R/I-module, whence I annihilates M; thus $I \subset \operatorname{Ann} M$.

The Artin-Wedderburn theorem says that every simple artinian ring is a matrix ring over a division ring, and that division ring is the endomorphism ring of its unique simple module. □

Corollary 4.9 *Two tiny simples over a ring R are isomorphic if and only if they have the same annihilator.*

Lemma 4.10 *If A is a countably generated k-algebra, then $\dim_k A$ is at most countable.*

Proof. First recall that a countable union of countable sets is countable. To see this, suppose that the elements of the n^{th} set are listed as $a_{n1}, a_{n2}, a_{n3}, \dots$ for $n \in \mathbb{N}$. Then the elements in the union of those sets may be listed as

$$a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, a_{14}, a_{23}, a_{32}, a_{41}, \dots,$$

thus showing that the union is countable.

Now fix k-algebra generators $x_1, x_2, \dots$ for A. Then A is spanned as a k-vector space by the union $W_1 \cup W_2 \cup \dots$ where

$$W_n = \text{ the words in the } x_i \text{ of length } \leq n.$$

Since the cardinality of W_n is at most that of $\mathbb{N}^n$, W_n is at most countable. By the previous paragraph, the union of the W_ns is at most countable. □

Lemma 4.11 *Let D be a division algebra over a field k. If k is uncountable and D is a countably generated k-algebra, then D is algebraic over k.*

Proof. The hypotheses imply that D has countable dimension. Since k is uncountable, the rational function field $k(x)$ has uncountable dimension over k—the elements $(x-\lambda)^{-1}$, $\lambda \in k$, are linearly independent. Therefore, if $\alpha \in D$, $k(\alpha)$ must be an algebraic extension of k. □

It is an open question whether this is true without the hypothesis that k is uncountable.

Proposition 4.12 *Let R be a countably generated algebra over an uncountable algebraically closed field k. A simple R-module M is tiny if and only if* $\dim_k M < \infty$.

Proof. If M is finite dimensional over k, so is $\mathrm{Hom}_k(M, M)$. But $R/\operatorname{Ann} M$ is a subalgebra of this, so is artinian. Thus M is tiny.

Suppose that M is tiny. Let D be its endomorphism ring. If m is a non-zero element of M, then the map $\alpha \mapsto \alpha(m)$ gives an injection $D \to M$. Since R is countably generated it has countable dimension over k, and since M is a quotient of R it too has countable dimension. Therefore D has countable dimension. By Lemma 4.11, D is algebraic over k. But k is algebraically closed, so $D \cong k$, whence $R/\operatorname{Ann} M \cong M_n(k)$ for some n. Therefore $\dim_k M < \infty$. $\square$

Theorem 4.13 *Let $X = \mathsf{Mod}R$ be an affine space. There is a bijection between the following sets:*

1. *the closed points of X;*
2. *the isomorphism classes of tiny simple R-modules;*
3. *the maximal two-sided ideals $\mathfrak{m}$ such that $R/\mathfrak{m}$ is artinian.*

If R is a countably generated algebra over an algebraically closed field k these sets are in bijection with the set of isomorphism classes of finite dimensional simple R-modules.

Proof. Since a simple artinian ring has a unique simple module up to isomorphism, it follows from Proposition 4.8 and its corollary that the sets in (2) and (3) are in bijection.

By Theorem 3.14, the closed subspaces of $\mathsf{Mod}R$ are the full subcategories $\mathsf{Mod}R/I$ as I ranges over all two-sided ideals. Thus, a closed point in $\mathsf{Mod}R$ is a full subcategory $\mathsf{Mod}R/I$ that is equivalent to $\mathsf{Mod}D$ for some division ring D. But the only rings Morita equivalent to D are the matrix algebras $M_n(D)$, so the closed points are in bijection with the ideals I such that $R/I \cong M_n(D)$ for some D and some n. These are precisely the ideals I such that R/I is simple artinian. This establishes the bijection between the sets in (1) and (3).

Proposition 4.12 shows that under the hypotheses in the last part of the theorem the tiny simples are the finite dimensional simples. $\square$

Determining the closed points is an important, but often difficult, problem. Their existence can hinge on delicate arithmetic properties of the base field and the defining relations of the algebra. The representation theory of semisimple Lie algebras, and the representation theory of the rings $k_q[x, y]$ in Chapter 4 illustrate this.

The next example shows that quite reasonable non-commutative spaces can have no closed points.

Example 4.14 (The Weyl algebra) The ring D of differential operators over a field k of characteristic zero is the subalgebra of the ring $\operatorname{End}_k k[x]$ of vector space endomorphisms generated by multiplication by x and the differentiation operator d/dx. We denote the first operator by x and the second by ∂. Thus $D = k[x, \partial]$. The identity element is the identity operator. It is clear that D consists of all differential operators with polynomial coefficients:

$$p_n(x)\frac{d}{dx} + \cdots + p_1(x)\frac{d}{dx} + p_0(x),$$

where the $p_i(x) \in k[x]$. The following computation is key:

$$(\partial x - x\partial)(f) = \frac{d}{dx}(xf) - xf' = f + xf' - xf' = f.$$

Therefore, computing in the ring D, we have

$$\partial x - x\partial = 1. \tag{4-1}$$

The most natural left D-module is $k[x]$ on which D acts in the natural way as differential operators.

It follows from (4-1) that if V is a finite dimensional simple D-module, then $V = 0$. To see this, consider the trace of the action of ∂ and x on V. Because V has finite dimension, $\operatorname{Tr}(\partial)\operatorname{Tr}(x) - \operatorname{Tr}(x)\operatorname{Tr}(\partial) = 0$; however, if $\dim_k V = n$, then $\operatorname{Tr}(\partial)\operatorname{Tr}(x) - \operatorname{Tr}(x)\operatorname{Tr}(\partial) = \operatorname{Tr}(\partial x - x\partial) = n$. But the characteristic of k is zero, so $\dim_k V = 0$.

This is false in positive characteristic. If $\operatorname{char} k = p > 0$, then the ideal (x^p) is a D-submodule of $k[x]$, and the quotient is (easily checked to be) a simple D-module. $\diamondsuit$

Links between closed points. Perhaps the single most important way in which non-commutative rings differ from commutative ones is that there can be non-split extensions between non-isomorphic simple modules. The non-split extensions between tiny simple modules play an important role in non-commutative geometry (see Lemma 5.20 and section 3.14). If $\operatorname{Ext}^1(\mathcal{O}_p, \mathcal{O}_q)$ is non-zero, we will say that there is a link from p to q.

The existence of links in an affine space is equivalent to an arithmetic property involving the ideals that are the annihilators of the corresponding simple modules.

Lemma 4.15 *Let M and N be tiny simple modules over a ring R. Write $\mathfrak{m} = \operatorname{Ann} M$ and $\mathfrak{n} = \operatorname{Ann} N$. Then $\operatorname{Ext}^1_R(M, N) = 0$ if and only if $\mathfrak{m}\mathfrak{n} = \mathfrak{m} \cap \mathfrak{n}$.*

Proof. ($\Rightarrow$) Because M and N are tiny, $R/\mathfrak{m}$ and $R/\mathfrak{n}$ are simple artinian rings. As a left module over its endomorphism ring, say $D = \operatorname{End}_R N$, N is isomorphic to a finite direct sum of copies of D. Since $R/\mathfrak{m}$ is a finite direct sum of copies of M, $\operatorname{Ext}^1(R/\mathfrak{m}, N) = 0$. Therefore the natural map

$$\operatorname{Hom}_R(R, N) \to \operatorname{Hom}_R(\mathfrak{m}, N)$$

induced by restriction is surjective. The left action of D on N makes this a homomorphism of left D-modules; since $\mathrm{Hom}_R(R,N)$ is isomorphic to N as a left D-module and is therefore a finite direct sum of copies of D, $\mathrm{Hom}_R(\mathfrak{m},N)$ is also a finite direct sum of copies of D. Since $\mathrm{Hom}_R(\mathfrak{m}/\mathfrak{m}\mathfrak{n},N)$ is isomorphic to a D-submodule of $\mathrm{Hom}_R(\mathfrak{m},N)$ it too is a finite direct sum of copies of D. As a right R-module $\mathfrak{m}/\mathfrak{m}\mathfrak{n}$ is isomorphic to a direct sum of copies of N because it is annihilated by $\mathfrak{n}$, so $\mathrm{Hom}_R(\mathfrak{m}/\mathfrak{m}\mathfrak{n},N)$ is a direct product of copies of D as a left D-module. It follows that $\mathrm{Hom}_R(\mathfrak{m}/\mathfrak{m}\mathfrak{n},N)$ must be a *finite* product, whence $\mathfrak{m}/\mathfrak{m}\mathfrak{n}$ is isomorphic to a *finite* direct sum of copies N. Because $\mathrm{Ext}^1_R(M,-)$ commutes with finite direct sums this implies that $\mathrm{Ext}^1_R(M,\mathfrak{m}/\mathfrak{m}\mathfrak{n})$ is zero, and hence that the sequence

$$0 \to \mathfrak{m}/\mathfrak{m}\mathfrak{n} \to R/\mathfrak{m}\mathfrak{n} \to R/\mathfrak{m} \to 0$$

splits. Thus $R/\mathfrak{m}\mathfrak{n}$ is isomorphic to a direct sum of various copies of M and N. It is therefore annihilated by $\mathfrak{m}\cap\mathfrak{n}$, whence $\mathfrak{m}\cap\mathfrak{n}\subset\mathfrak{m}\mathfrak{n}$. The reverse inclusion always holds.

($\Leftarrow$) We will prove the contrapositive. Suppose that $\mathrm{Ext}^1_R(M,N)\neq 0$, and let $0\to N\to Q\to M\to 0$ be a non-split extension. Since $R/\mathfrak{m}\cap\mathfrak{n}$ is semisimple ring, Q is not annihilated by $\mathfrak{m}\cap\mathfrak{n}$. However, $Q\mathfrak{m}\mathfrak{n}=0$ so $\mathfrak{m}\cap\mathfrak{n}\neq\mathfrak{m}\mathfrak{n}$. □

If X and Y are affine varieties over a field k with coordinate rings R and S, then the cartesian product $X\times Y$ can be given the structure of a variety. Its coordinate ring is, by definition, $R\otimes_k S$. The next result is an appropriate non-commutative generalization of this.

Proposition 4.16 *Let k be an uncountable algebraically closed field. Let R and S be countably generated k-algebras. Then the set of closed points in* $\mathsf{Mod}R\otimes_k S$ *is in natural bijection with the cartesian product of the closed points in* $\mathsf{Mod}R$ *and* $\mathsf{Mod}S$.

Proof. The hypotheses ensure that the closed points are in bijection with the finite dimensional simples. We must show that the finite dimensional simple $R\otimes_k S$-modules are precisely the modules $M\otimes_k N$ where M is a finite dimensional simple R-module and N is a finite dimensional simple S-module.

The hypotheses on k, R, and S ensure that the endomorphism ring of every simple module over R, or S, or $R\otimes_k S$ is k.

First we show that $M\otimes_k N$ is a simple $R\otimes_k S$-module whenever M and N are simple modules over R and S (not necessarily of finite dimension). To see this, let $0\neq a\in M\otimes_k N$, and let $m\otimes n\in M\otimes_k N$. Write $a=\Sigma m_i\otimes n_i$ where the n_i are linearly independent over k. By the Jacobson Density theorem there exists $s\in S$ such that $n_0 s=n$ and $n_i s=0$ for $i\neq 0$. Since M is simple there exists $r\in R$ such that $m_0 r=m$. Hence $a\cdot(r\otimes s)=m\otimes n$. It follows that $M\otimes_k N$ is simple.

Conversely, suppose that L is a finite dimensional simple $R\otimes_k S$-module. Let M be a simple R-submodule of L. Then $L=M\cdot(1\otimes S)$, and as the action of S commutes with the action of R, L is isomorphic as an R-module to a direct sum of copies of M. Hence L is annihilated by some maximal two-sided ideal, say I, of R. It is also annihilated by a maximal two-sided ideal, say J, of S. Thus L is a simple module over $R \otimes_k S/(I \otimes S + R \otimes J) = (R/I) \otimes_k (S/J)$. The hypotheses imply that $R/I \cong M_r(k)$ and $S/J \cong M_s(k)$ for some integers r and s. Hence $R/I \otimes_k S/J \cong M_{rs}(k)$. In particular, $R/I \otimes_k S/J$ has a unique simple module, and that module is isomorphic to $M \otimes_k N$, where M and N are simple modules over R/I and S/J respectively. □

The bijection in Proposition 4.16 is such that if p and q are closed points of $\mathsf{Mod}R$ and $\mathsf{Mod}S$, then

$$\mathcal{O}_{(p,q)} \cong \mathcal{O}_p \otimes_k \mathcal{O}_q.$$

The extensions between simple $R \otimes_k S$-modules are controlled by the extensions between simple modules over R and S respectively. By [59, Theorem 3.1, pg. 209], $\operatorname{Ext}^1_{R\otimes S}(U \otimes V, M \otimes N)$ is isomorphic to

$$\operatorname{Hom}_R(U,V) \otimes \operatorname{Ext}^1_S(M,N) + \operatorname{Ext}^1_R(U,V) \otimes \operatorname{Hom}_S(M,N).$$

The degree of a closed point. If R is a finitely generated commutative algebra over an algebraically closed field, then every simple R-module has dimension one by the Nullstellensatz. However, simple modules over a non-commutative ring can have a range of dimensions, and that dimension is an important invariant. But that invariant is not an invariant of the simple module as an object in $\mathsf{Mod}R$; it depends on a choice of progenerator. Thus, we adopt the following strategy to use this invariant in non-commutative geometry.

Definition 4.17 Let k be a field and let $(X, \mathcal{O}_X)$ be an enriched space over $\operatorname{Spec} k$. Thus $\mathcal{O}_X$ is a fixed X-module that we call a structure module for X. The degree of a closed point p in X is defined to be

$$\deg p = \dim_k \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{O}_p).$$

If, as is often the case, X is an affine space with a "preferred" coordinate ring R, it is natural to take the right regular representation R_R as the structure module. In that case, if p is a closed point of X, then

$$\deg p = \dim_k \operatorname{Hom}_R(R, \mathcal{O}_p) = \dim_k \mathcal{O}_p.$$

Thus the degree of a closed point is equal to the dimension of the simple R-module it corresponds to.

EXERCISES

4.1 Determine the points of $\operatorname{Spec} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and observe that this set is not the cartesian product of the set $\operatorname{Spec} \mathbb{C}$ with itself.

4.2 Let $\mathfrak{m}$ and $\mathfrak{n}$ be distinct maximal two-sided ideals in a ring R. Show that

$$\mathfrak{m} \cap \mathfrak{n} = \mathfrak{m}\mathfrak{n} + \mathfrak{n}\mathfrak{m}.$$

3.5 Quivers and path algebras

This section introduces a well-understood family of finite dimensional algebras, namely the path algebras of quivers without cycles.

Definition 5.1 A quiver is a directed graph with a finite number of vertices. It may have multiple arrows, and loops. ◊

The pictures of quivers which appear in this section should be considered as pictures of non-commutative spaces. The vertices are the points of the space. The arrows between the points indicate an interaction between the points. This is a non-commutative phenomenon. In commutative algebra, if p and q are distinct points, then all the extension groups $\mathrm{Ext}^n_X(\mathcal{O}_p, \mathcal{O}_q)$ are zero. This is no longer true for a non-commutative space: an arrow from p to q is drawn if $\mathrm{Ext}^1_X(\mathcal{O}_p, \mathcal{O}_q) \neq 0$.

We denote vertices by Roman letters, and arrows by Greek letters. If α is an arrow from a to b, and β is an arrow from b to c we write $\alpha\beta$ for the path from a to c which is "first traverse α, then traverse β". Composition of paths is defined by concatenation in the obvious way.

If we adjoin an element that we call zero, and denote by 0, the paths form a monoid provided we adopt the rule that $\alpha\beta = 0$ if the endpoint of the path α is different from the starting point of the path β.

At each vertex v we introduce the trivial path ε_v, the effect of which is that if α is a path ending at v, and β is a path beginning at v, then $\alpha\varepsilon_v\beta = \alpha\beta$.

A quiver determines a category in which the objects are the vertices, and the morphisms from u to v are the paths beginning at u and ending at v. Composition of morphisms is given by concatenation of paths. The identity morphism on an object v is the trivial path ε_v. The category determined by a quiver Q will also be denoted by the letter Q.

Definition 5.2 Let k be a field, and Q a quiver. A representation of Q over k is a functor $F : Q \to \mathsf{Mod}k$. A map $\tau : F \to G$ between two representations is a natural transformation of the functors. We write $\mathsf{Mod}Q$ for the category of representations.

Thus a representation of a quiver assigns to each vertex v a vector space M_v, and to each arrow $\alpha : u \to v$ a linear map $M_u \to M_v$. This data determines the representation. A map $\tau : M \to N$ between two representations consists of linear maps $\tau_v : M_v \to N_v$ for each vertex v, such that the obvious diagrams commute; that is, $\tau_v \circ M(\alpha) = N(\alpha) \circ \tau_u$ whenever α is an arrow from u to v.

Definition 5.3 Let k be a field, and Q a quiver. The path algebra kQ is the k-vector space with basis given by all the paths in Q, and multiplication defined by concatenation of paths. ◊

The identity element in kQ is the sum of all the trivial paths ε_v. The elements ε_v are mutually orthogonal idempotents. They are also primitive: this is because each ε_v has degree zero in the natural grading which is defined on page 134.

Example 5.4 Let Q be the quiver with n vertices, and no arrows. Then kQ is isomorphic to a product of n copies of the base field. We can also view kQ as the ring of k-valued functions on the set $\{1, 2, \dots, n\}$. $\lozenge$

Example 5.5 The path algebra of

$$\begin{tikzcd} \bullet \arrow[loop above] \end{tikzcd} \tag{5-1}$$

is isomorphic to the polynomial ring on one variable, $k[x]$, where x^n denotes the path that traverses the loop n times. The path algebra of

$$\begin{tikzcd} \bullet \arrow[loop left] \arrow[loop right] \end{tikzcd} \tag{5-2}$$

is a free algebra on two variables, $k\langle x, y\rangle$. The path algebra of the quiver with one vertex and n arrows is isomorphic to the tensor algebra over k on the n-dimensional vector space. $\lozenge$

Example 5.6 The path algebra of the quiver

$$\overset{1}{\bullet} \longleftarrow \overset{2}{\bullet} \longleftarrow \bullet \ \cdots \ \bullet \longleftrightarrow \overset{n-1}{\bullet} \longleftarrow \overset{n}{\bullet} \tag{5-3}$$

with n vertices is the ring of lower triangular $n \times n$ matrices. The isomorphism is implemented by writing e_{ij} for the path beginning at the i^{th} vertex and ending at the j^{th} vertex. $\lozenge$

The proof of the next result is obvious.

Lemma 5.7 *The path algebra kQ is finite dimensional if and only if the quiver has no cycles (i.e., no path in the quiver begins and ends at the same vertex).*

A grading on the path algebra. The path algebra can be given the structure of a $\mathbb{Z}$-graded algebra by defining the degree of a path to be its length. The trivial paths ε_v are given degree zero. Thus kQ is $\mathbb{N}$-graded, and its degree zero component is the product of fields $\oplus_{v \in Q} k\varepsilon_v$. The degree one component of kQ has basis the arrows of Q. It is a bimodule over $(kQ)_0$. For example, the degree i component of the lower triangular matrix algebra in Example 5.6 is the i^{th} lower diagonal.

We do not need the following result, so we do not prove it.

Proposition 5.8 *Let k be a field, and Q a quiver. Then kQ is isomorphic to the tensor algebra of $(kQ)_1$ viewed as a $(kQ)_0$-bimodule.*

Theorem 5.9 *Let k be a field, and Q a quiver. The category of representations of Q is equivalent to the category of right kQ-modules.*

Proof. If F is a representation of Q, then each $F(v)$ is a k-vector space M_v. The direct sum of all these is denoted by M. If γ is a path in Q starting at the vertex v, then we define the action of $\gamma \in kQ$ on $m \in M$ by

$$m.\gamma = F(\gamma)(m_v),$$

where m_v is the component of m belonging to M_v. It is easily verified that by extending this action linearly to kQ, one makes M a kQ-module.

Conversely, if M is a kQ-module, then $M = \oplus M\varepsilon_v$, where the sum is over all vertices v of Q. We associate to M a functor $F : Q \to \mathsf{Mod}k$ as follows. For each vertex v, $F(v) := M\varepsilon_v$. If γ is a path from u to v, we define

$$F(\gamma) = \varepsilon_u \gamma \varepsilon_v : M_u \to M_v.$$

It is easily verified that F is a functor, and the rule $M \mapsto F$ is inverse to the rule $F \mapsto M$ described in the previous paragraph. $\square$

One consequence of Theorem 5.9 is that the representations of Q form a Grothendieck category, thus making $\mathsf{Mod}Q$ a non-commutative space. Our convention for labelling non-commutative spaces forces us to write $Q = \mathsf{Mod}Q$.

Remark. Classifying the representations of a quiver can be interpreted as a classification problem in linear algebra. For example, the classification of vector spaces endowed with a single linear map is the same thing as the classification of representations of the quiver consisting of one vertex and one arrow (5-1), and Theorem 5.9 says that this is equivalent to the classification of modules over $k[x]$.

The classification of pairs of linear maps on a vector space (up to simultaneous conjugation) is equivalent to the representation theory of the quiver with one vertex and two arrows (5-2). By Theorem 5.9 this is equivalent to the classification of modules over the free algebra on two variables. This is a famously intractable problem and is a benchmark against which other algebraic classification problems are measured. A classification problem that includes the problem of classifying all such pairs of linear maps is said to be *wild.*

The n-subspace problem for $n \geq 5$ is wild: the problem is to classify all n-tuples of subspaces of a given vector space. This is almost the same problem as that of classifying the representations of the quiver

$$\begin{tikzcd}
\overset{1}{\bullet} \arrow[drr] & \overset{2}{\bullet} \arrow[dr] & \cdots & \overset{n-1}{\bullet} \arrow[dl] & \overset{n}{\bullet} \arrow[dll] \\
 & & \bullet & &
\end{tikzcd} \tag{5-4}$$

If M is a representation of this quiver in which one of the arrows is not injective, then $M = M' \oplus K$ where K is the kernel of that non-injective arrow, and M' is a representation of the subquiver for the $(n-1)$-subspace problem. Thus, excluding the $n+1$ simple modules, the indecomposable representations are in bijection with the classification of n-tuples of subspaces of a vector space. If $n \le 3$ there are only finitely many such indecomposables up to isomorphism. If $n = 4$ there are a finite number of one-parameter families of indecomposables [92]; the problem is said to be *tame*. For $n \ge 5$ the problem is wild.

Notation. Let R and S be rings, and let ${}_RM_S$ and ${}_SN_R$ be bimodules. Suppose there are bimodule maps $f : M \otimes_S N \to R$ and $g : N \otimes_R M \to S$. Then we form a ring

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix} := \left\{ \begin{pmatrix} r & m \\ n & s \end{pmatrix} \,\middle|\, r \in R,\, s \in S,\, m \in M,\, n \in N \right\}. \tag{5-5}$$

The multiplication is defined using the maps f and g in the obvious way.

Example 5.10 We call

$$\overset{1}{\bullet} \underset{\longleftarrow}{\overset{\longleftarrow}{}} \overset{2}{\bullet} \tag{5-6}$$

the Kronecker quiver. Its path algebra is isomorphic to

$$R := \begin{pmatrix} k & 0 \\ k^2 & k \end{pmatrix}. \tag{5-7}$$

We call this the Kronecker algebra. Let $A = k[x, y]$ be the commutative polynomial ring with its standard grading. Then R is isomorphic to the subring

$$\begin{pmatrix} A_0 & 0 \\ A_1 & A_0 \end{pmatrix}$$

of $M_2(A)$. The row-module $P_n := (A_n, A_{n-1})$ now becomes a right R-module via right multiplication. Viewed as a representation of the quiver, P_n has the vector space A_{n-1} at the vertex labelled 2, and A_n at the vertex labelled 1, and the two arrows are multiplication by x and y.

The modules P_0 and P_1 are indecomposable projectives. It is an easy exercise to show that every P_n is indecomposable, and that $\operatorname{Hom}_R(P_n, P_{n+m}) \cong A_m$, with $a \in A_m$ acting by left multiplication, $a.(b, c) = (ab, ac)$. A classification of all indecomposable R-modules can be found in [120, Theorem 8.5.7] and in [28, VIII.7]. $\diamondsuit$

If v is a vertex of the quiver Q, we write S_v for the representation that assigns the zero vector space to each vertex other than v, and assigns to v the one-dimensional vector space k, and assigns to every arrow the zero map. It is clear that S_v is an irreducible representation of Q, or, what is the same thing, a simple kQ-module.

Proposition 5.11 *Let Q be a quiver without cycles. Then the closed points of the space with coordinate ring kQ are in bijection with the vertices of Q.*

Proof. The k-linear span of the non-trivial paths is a two-sided ideal, say I. The subalgebra

$$\oplus_{v \in Q} k\varepsilon_v$$

is a complement to I in kQ, so kQ/I is isomorphic to a product of fields. The n^{th} power of I is spanned by the paths of length $\geq n$. Since the quiver has no cycles, there is a bound to the length of the paths, so I is nilpotent. It follows that I is the nilpotent radical of kQ. So the simple kQ-modules are in bijection with the simple modules over $\oplus_{v \in Q} k\varepsilon_v$, and hence with the vertices of Q. All these simple modules are finite dimensional over k. $\square$

This result fails if Q has cycles. For example, consider the polynomial ring $k[x]$, which is the path algebra of the quiver with a single vertex and a single arrow.

Each S_v can be given the structure of a graded kQ-module, and, up to shifting degree, these form a complete set of simple modules in $\mathsf{GrMod} kQ$. This follows from the fact that $(kQ)_0 = \oplus k\varepsilon_v$.

Proposition 5.12 *Let Q be a quiver without cycles. Then the indecomposable projective kQ-modules are in bijection with the vertices of Q.*

The indecomposable projective corresponding to the vertex v is the right ideal P_v of kQ spanned by the paths that begin at v.

Proof. Clearly P_v is the right ideal of kQ generated by ε_v. It is also a graded right ideal. Since every path begins at a unique vertex, kQ is the direct sum of these right ideals. Thus each P_v is projective. And P_v is indecomposable because the ε_v are primitive idempotents. $\square$

Example 5.13 The right regular representation of kQ gives a representation of Q for which the module concentrated at v is the projective *left* ideal $P'_v := (kQ)\varepsilon_v$. Each arrow from u to v gives a linear map $P'_u \to P'_v$ which is a map of left kQ-modules. The component at the vertex u of the representation P_v is $P_v\varepsilon_u = \varepsilon_v(kQ)\varepsilon_u$, which is the linear span of all paths that begin at v and end at u. $\Diamond$

Example 5.14 The path algebra of the quiver

$$\overset{1}{\bullet} \longleftarrow \overset{2}{\bullet} \longleftarrow \bullet \quad \cdots \quad \bullet \quad \cdots \quad \bullet \longleftarrow \overset{n}{\bullet}$$

is the ring of $n \times n$ lower triangular matrices. The idempotent ε_i corresponding to the vertex i is the matrix unit e_{ii}. A representation can be succinctly denoted by drawing a picture that is the same shape as the quiver with each vertex replaced by the corresponding vector space, and each arrow labelled with the sorresponding linear map. For example, the simple modules for this quiver can be denoted

$$S_1 = k0\dots00, \quad S_2 = 0k0\dots00, \quad \dots, S_n = 00.\dots0k,$$

Here S_i consists of the row vectors having zeroes in all except possibly the i^{th} position, the matrix ring acts by right multiplication. The indecomposable projectives are the rows

$$P_1 = \begin{pmatrix} k & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \cdots, P_n = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \\ k & k & \cdots & k \end{pmatrix}$$

of the matrix algebra. More succintly, $P_i = kk\dots k0\dots00$ and all the maps are the identity, or zero. The simple module S_i is a quotient of P_i. Left multiplication by $e_{i,i-1}$ gives the first map in the projective resolution $0 \to P_{i-1} \to P_i \to S_i \to 0$. $\Diamond$

The next result shows this is a general phenomenon.

Theorem 5.15 *Let k be a field, and Q a quiver. Then kQ has global dimension at most one. The global dimension is zero if and only if there are no arrows.*

Proof. If Q has no arrows, then kQ is a product of copies of k, so has global dimension zero. Now suppose that Q has some arrows.

Fix a vertex v. Let Θ denote the set of arrows beginning at v, and for each $\alpha \in \Theta$, write $t(\alpha)$ for the vertex at which the arrow α ends.

Let S_v denote the simple module at v. Then $S_v\varepsilon_u$ is zero if $u \neq v$, and equals S_v if $u = v$. Hence there is a surjective map $\varphi : P_v \to S_v$, the kernel of which is spanned by the non-trivial paths beginning at v. If γ is a non-trivial path beginning at v, then $\gamma = \alpha\gamma'$ for a unique $\alpha \in \Theta$, and a unique path γ' beginning at $t(\alpha)$. Therefore $\ker\varphi$ is the direct sum of the right ideals $\alpha P_{t(\alpha)}$ where α runs through Θ. Left multiplication by $\alpha \in \Theta$ is an injective right kQ-module map $P_{t(\alpha)} \to P_v$. Hence there is an exact sequence

$$0 \to \bigoplus_{\alpha\in\Theta} \alpha P_{t(\alpha)} \to P_v \to S_v \to 0. \tag{5-8}$$

Therefore $\operatorname{pdim} S_v \leq 1$. Since Q has some arrow, $S_v \neq P_v$ for some v, whence $\operatorname{pdim} S_v \geq 1$.

If kQ is finite dimensional we are done since $\{S_v\}$ is a complete set of simple modules, and every finite dimensional module has a composition series.

If kQ is infinite dimensional we note that (5-8) is a projective resolution in the graded category. Thus each graded simple module has projective dimension at most one in $\mathsf{GrMod}(kQ)$. Since kQ is positively graded, it follows from ??? that $\operatorname{gldim} kQ \leq 1$. $\square$

An algebra of global dimension one is said to be **hereditary**.

Lemma 5.16 *Let P_v be the indecomposable projective module, and S_v the simple module corresponding to the vertex v. Then*

$$\operatorname{Hom}_Q(P_v, S_u) = \begin{cases} k & \text{if } u = v, \\ 0 & \text{if } u \neq v. \end{cases}$$

Proof. Let $f : P_v \to S_u$ be a module homomorphism. Then $f(\varepsilon_v) = f(\varepsilon_v^2) = f(\varepsilon_v)\varepsilon_v$. If $u \neq v$, then $S_u\varepsilon_v$=0, so $f(\varepsilon_v) = 0$; but ε_v generates P_v, so $f(P_v) = 0$.

Now suppose that $u = v$. The paths that begin at v give a basis for P_v. If α is a non-trivial such path, then $S_v\alpha = 0$, so $0 = f(\varepsilon_v\alpha) = f(\alpha)$. Therefore, if I is the linear span of all the non-trivial paths beginning at v, then $f(I) = 0$. But $P_v/I \cong S_v$, so the result follows. □

The arrows in a quiver control the homological behavior of the simple modules.

Definition 5.17 An exact sequence of the form

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0 \tag{5-9}$$

is called an extension of N by L. We also say that M is an extension of N by L. The extension is split if there is a map $\gamma : N \to M$ such that $\beta \circ \gamma = \mathrm{id}_N$. Otherwise the extension is said to be non-split. ◊

The trivial extension of N by L is the sequence $0 \to L \to L \oplus N \to N \to 0$ with the obvious maps. It splits. If the extension (5-9) splits, then the image of γ is isomorphic to N, and $M = L \oplus \gamma(N) \cong L \oplus N$.

Lemma 5.18 *Let L and N be simple modules. The extension (5-9) is non-split if and only if $\alpha(L)$ is the only proper submodule of M.*

Proof. If the extension splits via $\gamma : N \to M$, then $\gamma(N)$ is a submodule distinct from $\alpha(L)$. Therefore, if $\alpha(L)$ is the only proper submodule the extension can not split.

Suppose that K is a proper submodule distinct from $\alpha(L)$. Then L and K are the two composition factors of M, so K must be isomorphic to N. In particular, the restriction of β to K is an isomorphism $\psi : K \to N$. Then $\gamma = \psi^{-1}$ splits the sequence. □

A non-split extension between different simples can only exist if the ring is non-commutative.

Lemma 5.19 *Let L and N be non-isomorphic simple R-modules.*

1. *If R is commutative then every extension of N by L splits.*
2. *If z is a central element of R that annihilates L but not N, then every extension of N by L splits.*

Proof. (1) Suppose M is an extension of N by L. Let $\mathfrak{p}$ and $\mathfrak{q}$ be the annihilators of L and N. These are maximal ideals, and are distinct because L and N are not isomorphic. Pick $x \in \mathfrak{p}$ such that $x \notin \mathfrak{q}$. Then multiplication by x is a module map $M \to M$, and its image Mx is isomorphic to $M/\{m \mid mx = 0\} = M/L \cong N$. By Lemma 5.18, the extension splits.

(2) The argument is essentially the same. □

There is an abelian group $\operatorname{Ext}^1_R(N, L)$ that classifies the non-split extensions of N by L. Details can be found in any book on homological algebra. For now it is sufficient to remark that $\operatorname{Ext}^1_R(N, L)$ is zero if and only if every extension of N by L is split.

Lemma 5.20 *Let Q be a quiver. There is an arrow from u to v if and only if there is a non-split extension of S_u by S_v.*

Proof. Suppose there is an arrow $u \to v$. Let M be the representation with k at both u and v, and zeroes elsewhere, and assign to the arrow $u \to v$ the identity map $k \to k$. It is clear that S_v is a sub-representation of M, and that the quotient is isomorphic to S_u. Hence M is an extension of S_u by S_v. Because the map $k \to k$ is non-zero, S_v is the only proper sub-representation of M. Thus, by Lemma 5.18, the extension does not split.

Conversely, suppose that $0 \to S_v \to M \to S_u \to 0$ is a non-split extension. Then $\dim_k M = 2$. We consider S_v as a submodule of M. Since $S_u\varepsilon_v = 0$ and $S_v\varepsilon_v = S_v$, $M\varepsilon_v = S_v$. Since $S_v\varepsilon_u = 0$ and $S_u\varepsilon_u = S_u$, $M\varepsilon_u$ is a one-dimensional subspace of M; it is not necessarily a submodule. Hence the representation of Q associated to M has k at the vertex labelled u and k at the vertex labelled v. If there were no arrow from u to v, then S_u would be a sub-representation of M, so by Lemma 5.18, M would split. This is a contradiction, so we conclude that there must be an arrow from u to v. □

A more precise result is possible once one has the Ext-groups at hand.

Proposition 5.21 *If u and v are two vertices, then $\dim \operatorname{Ext}^1_{kQ}(S_v, S_u)$ is the number of arrows from v to u.*

Proof. We use the notation in the proof of Theorem 5.15. It follows from the projective resolution (5-8) that $\operatorname{Ext}^1_{kQ}(S_v, S_u)$ is the cokernel of the natural map

$$\operatorname{Hom}_Q(P_v, S_u) \to \operatorname{Hom}_Q\Big(\bigoplus_{\alpha\in\Theta} \alpha P_{t(\alpha)}, S_u\Big).$$

But this map is zero because (5-8) is a minimal projective resolution, so

$$\begin{aligned}\dim_k \operatorname{Ext}^1_Q(S_v, S_u) &= \dim_k \operatorname{Hom}_Q(\oplus_{\alpha\in\Theta}\alpha P_{t(\alpha)}, S_u)\\ &= \sum_{\alpha\in\Theta} \dim_k \operatorname{Hom}_Q(S\varepsilon_{t(\alpha)}, S_u).\end{aligned}$$

It is easy to see that this is the cardinality of $\{\alpha \in \Theta \mid t(\alpha) = u\}$, which is what we needed to prove. □

Propositions 5.11 and 5.21 give a recipe for recovering the quiver from a finite dimensional path algebra. One takes a vertex for each simple module, and from the vertex u to the vertex v one draws $\dim_k \operatorname{Ext}^1(S_u, S_v)$ arrows.

The next result, which we do not need in this book, emphasizes again the fundamental nature of path algebras.

Theorem 5.22 *Suppose that k is a perfect field. Then, up to Morita equivalence, the path algebras kQ give all the finite dimensional hereditary algebras.*

We now examine a family of infinite dimensional path algebras consisting of non-maximal hereditary orders.

Proposition 5.23 *Let Q be the following quiver with n vertices and n arrows:*

$$Q \ = \ \begin{tikzcd} \underset{n}{\bullet} \arrow[r] & \underset{n-1}{\bullet} \arrow[r] & \bullet & \cdots & \bullet \arrow[r] & \underset{2}{\bullet} \arrow[r] & \underset{1}{\bullet} \arrow[llllll, bend right] \end{tikzcd} \tag{5-10}$$

Write x_i for the arrow that ends at the vertex labelled i.

1. *The path algebra is isomorphic to the ring*
$$S = \begin{pmatrix} R & xR & \cdots & xR & xR \\ R & R & \cdots & xR & xR \\ \vdots & & & & \vdots \\ R & R & \cdots & R & xR \\ R & R & \cdots & R & R \end{pmatrix}, \tag{5-11}$$
where $R = k[x]$.

2. *Under this isomorphism, $x = \sum_{i=1}^{n} x_i x_{i-1} \ldots x_{i+1}$.*

3. *The element $u = x_1 + \ldots + x_n$ is normal, meaning that $uS = Su$. If we identify kQ with S, then*
$$u = \begin{pmatrix} 0 & 0 & \ldots & 0 & x \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

4. *$u^n = x$.*

5. *If k is algebraically closed, then the simple Q-modules are as follows:*

   (a) *There are n one-dimensional simple modules V_i, the representations defined by placing k at the vertex labelled i and zero at all others. Under the isomorphism $V_i e_{ii} \neq 0$, where $e_i i$ is the primitive diagonal idempotent in the ii-position.*

(b) *There is a family of n-dimensional simples S_λ parametrized by $\lambda \in \mathbb{A}^1 \backslash \{0\}$ obtained by placing k at each vertex, and making each arrow the identity map, except the arrow from 1 to n which is multiplication by λ.*

6. *If $\lambda \neq 0$, then $S/(x-\lambda) \cong M_n(k)$.*

7. *The element u annihilates all the one-dimensional simple S-modules.*

8. *The auto-equivalence $- \otimes_S Su$ of $\mathsf{Mod}S$ sends the one-dimensional simple V_i to V_{i-1} and fixes each n-dimensional simple.*

Proof. To show that x is central it suffices to show that it commutes with the idempotents ε_i (the trivial paths) and with each x_i. This is a routine calculation.

A non-trivial path begins at some vertex i, goes t times around the loop, but not $t+1$ times, and ends at the vertex j. This path is

$$(x_{i-1}x_{i-2}\dots x_i)^t x_{i-1}x_{i-2}\dots x_j,$$

which equals $x^t x_{i-1}x_{i-2}\dots x_j$. Thus, $k[x]$ is a central subalgebra of kQ, and a basis for kQ over $k[x]$ is given by the elements $x_{i-1}x_{i-2}\dots x_j$ together with the idempotents ε_i.

The claimed isomorphism is obtained as follows. If $i=j$, we send ε_i to e_{ii}. If $i>j$, we send $x_{i-1}x_{i-2}\dots x_j$ to e_{ij}. Otherwise, we send $x_{i-1}x_{i-2}\dots x_j$ to xe_{ij}. For example, $x_1 \mapsto e_{21}$, $x_2 \mapsto e_{32}$, and $x_n \mapsto xe_{1n}$.

It is clear that the one-dimensional simples are as claimed. It is also clear that the modules S_λ are simple. So, it remains to show that this is a complete list of the simples.

Let M be a simple module. Then M is annihilated by $x-\lambda$ for some $\lambda \in k$. Let $M = \oplus_{i=1}^n M_i$ be the decomposition with M_i the component at the vertex i. Any non-zero submodule of M must contain some non-zero homogeneous element, say $0 \neq m \in M_i$. We have

$$\lambda m = m.x = mx_{i-1}x_{i-2}\dots x_i.$$

Therefore the elements $m, mx_{i-1}, mx_{i-1}x_{i-2}, \dots$ span a submodule of M, which must, by hypothesis, equal M itself. If $\lambda \neq 0$, this is isomorphic to S_λ (just change bases in the correct way). If $\lambda = 0$, then the last non-zero element in the list $m, mx_{i-1}, mx_{i-1}x_{i-2}, \dots$ spans a one-dimensional submodule, so we conclude that $\dim_k M = 1$.

It follows from the earlier part of the proof that S is spanned by n^2 elements as a $k[x]$-module (in fact these elements form a basis), so $\dim_k S/(x-\lambda) \leq n^2$. However, when $\lambda \neq 0$, this ring has an n-dimensional simple module, so it must be isomorphic to $M_n(k)$.

To see that u is normal, we need only observe that $u\varepsilon_i = x_i = \varepsilon_{i+1}u$, and that $ux_{i-1} = x_i x_{i-1} = x_i u$. A simple calculation shows that $u^n = x$. It is clear that $V_i u = 0$ for all simples (because, for example, $\deg u > 0$).

The action of $- \otimes_S Su$ sends any module annihilated by $x-\lambda$ to another module annihilated by $x-\lambda$, so $S_\lambda \otimes_S Su \cong S_\lambda$. The one-dimensional simple

V_i is annihilated by ε_j for all $j \neq i$. Since $u\varepsilon_j = \varepsilon_{j+1}u$, $V_i \otimes_S Su$ is annihilated by ε_j for all $j \neq i-1$. Therefore $V_i \otimes_S Su \cong V_{i-1}$. □

Proposition 5.24 *Let Q be the quiver in Proposition 5.23. Then* $\mathsf{Mod}kQ$ *is equivalent to the category* $\mathsf{GrMod}k[t]$ *where $k[t]$ is graded by the group $\mathbb{Z}_n$, with* $\deg t = 1$.

Proof. Label the elements of $\mathbb{Z}_n$ by $0, 1, \dots, n-1$ in the obvious way. It is helpful to give the vertices of Q the same labels in such a way that there is an arrow from i to $i+1$ for each i.

This is so obvious that explanation probably obscures the issue. Each representation of Q gives a graded $k[t]$-module with the degree i component of the $k[t]$-module being the component of the representation at the vertex labelled i, and the action of t on the degree i component being given by the action of the map corresponding to the arrow from i to $i+1$. Conversely, a graded $k[t]$-module gives a representation of Q in an obvious way. Morphisms in one category obviously are morphisms in the other category. The functors between the categories are clearly mutually inverse. □

A ring R is prime if $IJ \neq 0$ whenever I and J are non-zero two-sided ideals of R. Equivalently, $xRy \neq 0$ whenever x and y are non-zero elements of R. The path algebra in Proposition 5.23 is a prime ring.

Lemma 5.25 *A path algebra kQ is prime if and only if for every pair of distinct vertices u and v there is a path from u to v.*

Proof. Suppose that there is no path from u to v. Then $\varepsilon_u \alpha \varepsilon_v = 0$ for all paths α. Hence $\varepsilon_u R \varepsilon_v = 0$, so R is not prime.

Conversely, suppose that there is a path between any two vertices. Let x and y be non-zero elements of R, each written as a linear combination of paths. Pick a path α occurring in the expression for x, and a path γ appearing in the expression for y. Let u denote the vertex where α ends, and let v denote the vertex where γ begins. Let β be a path from u to v. Then $\alpha\beta\gamma \neq 0$, and it follows from this that $x\beta y \neq 0$. Thus $xRy \neq 0$, showing that R is prime. □

Our definition of a quiver required there to be only a finite number of vertices and arrows. However, much of the theory extends to quivers having an infinite number of vertices and/or arrows.

If Q has infinitely many vertices, then the path algebra does not have an identity. A representation of Q gives a module over the path algebra in the usual way. If we define $\mathsf{Mod}kQ$ to consist of those right kQ-modules M such that $M = \sum M\varepsilon_v$, where v runs over all the vertices of Q, then each kQ-module gives a representation of Q. Thus one obtains a version of Theorem 5.9 for infinite quivers. The ring kQ itself is a generator and a projective object in $\mathsf{Mod}kQ$ (cf. [88, pp. 346-347]). Furthermore, the category $\mathsf{Mod}kQ$ is hereditary: every module has a projective resolution of length one.

A nice application of this last fact to inverse limits appears in [43, page 96].

As a simple example consider the quiver consisting of vertices indexed by $\mathbb{Z}$ and no arrows. Then the representations are just vector spaces with a specified decomposition $V = \oplus_n V_n$ into subspaces. It is a tautology that the category of representations of the quiver is equivalent to the category of graded modules over the base field.

Example 5.26 The path algebra of the doubly infinite quiver

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots \tag{5-12}$$

is isomorphic to the ring of doubly infinite lower triangular matrices having only finitely many non-zero entries. The category of representations of the quiver is equivalent to the category $\mathsf{GrMod}k[x]$ of graded modules over the polynomial ring in one variable x having degree one. If the vertices are labelled by the integers $\dots, n-1, n, n+1, \dots$ beginning at the left, then a representation $M = \oplus M_n$ of the quiver can be viewed as a graded $k[x]$-module with M_n being the degree n component, and x acting on M_n via the arrow from n to $n+1$. Conversely each graded $k[x]$-module gives a representation by placing the degree n component of the module at the vertex labelled n. ◊

Example 5.27 The path algebra of the quiver

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[loop right] \end{tikzcd}$$

is right, but not left, noetherian. The path algebra of the opposite quiver

$$\begin{tikzcd} \bullet \arrow[loop left] \arrow[r] & \bullet \end{tikzcd}$$

is left, but not right, noetherian. The path algebra of the quiver

$$\begin{tikzcd} \bullet \arrow[loop left] \arrow[r] & \bullet \arrow[loop right] \end{tikzcd}$$

is neither right nor left noetherian. ◊

Maps between quivers. One source of maps between non-commutative spaces are the "obvious" maps between quivers. These are the maps sending vertices to vertices and arrows to arrows in a compatible way.

Definition 5.28 A $\mathsf{morphism}$, or a map, $f : Q \to Q'$ between two quivers is a set map from the vertices of Q to the vertices of Q' together with a set map from the arrows of Q to the arrows of Q' such that if α is an arrow in Q from u to v, then $f(\alpha)$ is an arrow from $f(u)$ to $f(v)$.

The objects in the $\mathsf{category\ of\ quivers}$ are all quivers, and the morphisms are as above. We denote this category Quiver. ◊

A quiver determines a category whose objects are the vertices, and whose morphisms are the paths. A map of quivers induces a functor between the associated categories. However, a functor between the categories that sends some non-trivial path to a trivial path is not induced by a map of quivers.

Proposition 5.29 *The rule $Q \mapsto kQ$ extends to a contravariant functor from the category of quivers to k-algebras.*

Proof. Let $f : Q \to Q'$ be a map of quivers. We define $\varphi : kQ' \to kQ$ to be the linear map such that for each path α' in Q',

$$\varphi(\alpha') := \sum_{f(\alpha)=\alpha'} \alpha.$$

In particular, if $\alpha = \varepsilon_u$ is the trivial path at u, then $\varphi(\varepsilon_u) = \sum_{f(v)=u} \varepsilon_v$, from which it follows that $\varphi(1) = 1$.

To show that φ is an algebra homomorphism we must show that if α' and β' are paths in Q', then $\varphi(\alpha'\beta') = \varphi(\alpha')\varphi(\beta')$.

First suppose that $\alpha'\beta' = 0$. Then $s(\beta') \neq t(\alpha')$. Therefore $s(\beta) \neq t(\alpha)$ for any paths α and β for which $f(\alpha) = \alpha'$ and $f(\beta) = \beta'$; hence $\alpha\beta = 0$. Thus $\varphi(\alpha')\varphi(\beta') = 0$.

Now suppose that $\alpha'\beta' \neq 0$. Set $v' = t(\alpha') = s(\beta')$, and let $v_1, \dots, v_n$ be the distinct vertices in Q such that $f(v_i) = v'$. Let A_i be the set of arrows in Q that end at v_i and are mapped to α'. Let B_i be the set of arrows in Q that start at v_i and are mapped to β'. Then

$$\varphi(\alpha')\varphi(\beta') = \sum_{i=1}^{n} \Big(\sum_{\alpha \in A_i} \alpha \Big) \Big(\sum_{b \in B_i} \beta \Big). \tag{5-13}$$

On the other hand,

$$\varphi(\alpha'\beta') = \sum_{f(\gamma)=\alpha'\beta'} \gamma. \tag{5-14}$$

Each such γ passes through a unique v_i. Because f sends each arrow to an arrow, there is a unique way to write $\gamma = \alpha\beta$ with $\alpha \in A_i$ and $\beta \in B_i$. It follows that the sums (5-13) and (5-14) are equal. □

Corollary 5.30 *A map of quivers $f : Q \to Q'$ in the sense of Definition 5.28 is also an affine map of spaces in the sense of Definition 3.1.*

Proof. The induced algebra homomorphism $kQ' \to kQ$ gives rise to a map as in Example 3.3. □

The adjoint pair of functors $f^* : \mathsf{Mod}Q' \to \mathsf{Mod}Q$ and $f_* : \mathsf{Mod}Q \to \mathsf{Mod}Q'$ associated to a map $f : Q \to Q'$ of quivers can be described directly.

A representation of Q' is the same thing as a functor from Q' to $\mathsf{Mod}k$, where Q' is viewed as a category in which the morphisms are the paths. Hence each Q'-module M, viewed as a functor, determines a functor $M \circ f : Q \to \mathsf{Mod}k$. We have $f^*M = M \circ f$. Explicitly, if $v \in Q$, then $(f^*M)_v = M_{f(v)}$, and if α is an arrow from u to v in Q, then the action of α on f^*M sends $M_{f(u)}$ to $M_{f(v)}$ via the action of $f(\alpha)$.

On the other hand, if N is a Q-module, then f_*N is Q'-module with $(f_*N)_{v'} = \oplus_{f(v)=v'} N_v$, and if α' is an arrow in Q' then α' acts on f_*N as the direct sum of the actions of $f(\alpha)$ as α runs through all arrows such that $f(\alpha) = \alpha'$.

Example 5.31 We call Q' a subquiver of Q if there is a map $f : Q' \to Q$ that is injective on vertices and arrows. In this case the induced map $kQ \to kQ'$ gives an isomorphism $kQ' \cong kQ/I$, where I is the ideal generated by those ε_v for which v is not in Q' and by those arrows α which are not in Q'. Thus Q' is a closed subspace of the non-commutative space Q in the sense of Definition 2.1.

A special case of this arises when Q is a quiver with n vertices, and Q' is the subquiver consisting of the same vertices, but no arrows. The induced map $kQ \to kQ' \cong k^{\times n}$ is gotten by modding out the ideal spanned by the arrows; that is, one quotients out the homogenous ideal of kQ consisting of positive degree elements; if kQ is finite dimensional, this is the same as modding out the nilpotent radical.

As a second special case, let Q be the quiver

$$\overset{n}{\bullet} \longrightarrow \overset{n-1}{\bullet} \longrightarrow \bullet \ \cdots \ \bullet \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{1}{\bullet} \tag{5-15}$$

Let Q' be the subquiver of Q obtained by deleting the first vertex and arrow (or the last vertex and arrow). Then the induced map $kQ \to kQ'$ is that obtained by modding out the last row (or the first column) of the lower triangular $n \times n$ matrix ring. ♢

We end this section with an example involving our favorite infinite dimensional path algebra.

Example 5.32 (cf. Proposition 5.23 and Example 14.4) Consider the unique map of quivers $h : Q \to Q'$ from

$$Q \ = \ \begin{tikzcd} \underset{n}{\bullet} \arrow[r] & \underset{n-1}{\bullet} \arrow[r] & \bullet \ \cdots \ \bullet \arrow[r] & \underset{2}{\bullet} \arrow[r] & \underset{1}{\bullet} \arrow[llll, bend right] \end{tikzcd} \tag{5-16}$$

to

$$Q' \ = \ \begin{tikzcd} \bullet \arrow[loop above] \end{tikzcd} \tag{5-17}$$

Thus h sends the vertices of Q to the vertex of Q', and sends the arrows of Q to the single arrow of Q'.

The path algebra kQ' is isomorphic to a polynomial ring $k[u]$, and kQ is isomorphic to the subalgebra of $M_n(k[x])$ in (5-11). The ring homomorphism

$kQ' \to kQ$ can be viewed as the inclusion $k[u] \to kQ$ that sends u to the matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 & x \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

This inclusion induces a map of spaces $\mathsf{Mod}kQ \to \mathbb{A}^1$.

For $0 \neq \lambda \in k$ let S_λ denote the simple kQ-module annihilated by $x - \lambda$. Then $\dim S_\lambda = n$. The action of u on $v = (v_1, \dots, v_n) \in S_\lambda$ is given by

$$(v_1, \dots, v_n).u = (v_2, \dots, v_n, \lambda v_1).$$

It follows that $h_* S_\lambda \cong k[u]/(u^n - \lambda)$. If k has n distinct n^{th} roots of unity, then $h_* S_\lambda$ is a direct sum of n distinct simple $k[u]$-modules; thus h breaks the closed point corresponding to S_λ into n distinct closed points on the affine line parametrized by u. If M is a one-dimensional simple S-module, then $h_* M \cong k[u]/(u)$. On the other hand h^* sends $k[u]/(u - \lambda)$ to S_λ, and sends $k[u]/(u)$ to S/Su which is isomorphic to the direct sum of the n one-dimensional simples; to see this last isomorphism observe that the row space k^n becomes a cyclic module, with generator $(1, 1, \dots, 1)$, under the natural action of $S \subset M_n(k[x])$ with x acting as zero.

Let $g : \mathbb{A}^1 \to \mathbb{A}^1$ be the map $\alpha \mapsto \alpha^n$, and let $f : Q \to \mathbb{A}^1$ be the composition $f = g \circ h$. The image to have in mind is that $f : Q \to \mathbb{A}^1$ is a covering of the affine line by a non-commutative curve. A picture of this curve appears at (14-5) in Example 14.4.

Since g is induced by the inclusions $k[u^n] \to k[u]$, and since $u^n = x$ (see Proposition 5.23), the map f is induced by the inclusion of $k[x]$ as the center of kQ. The map f sends closed points of kQ to closed points of $\mathbb{A}^1$. The fiber over 0 consists of the n points of Q corresponding to the one-dimensional simples, and the fiber over any other closed point of $\mathbb{A}^1$ consists of a single point, one corresponding to an n-dimensional simple kQ-module. $\Diamond$

EXERCISES

5.1 Suppose that we adopt the opposite convention for the concatenation of paths—that is, we write $\alpha\beta$ to mean first traverse β, then traverse α. Write $k * Q$ for the path algebra obtained in this way. Show that $k * Q \cong (kQ)^{\text{op}}$.

5.2 Let Q be the quiver $1 \to 2 \to \cdots \to n$. Show that $k * Q$ is isomorphic to the ring of lower triangular matrices in such a way that e_{ij} corresponds to the path from j to i. With this convention, the representations of the path algebra correspond to the left modules over $k * Q$.

5.3 If Q is the disjoint union of quivers Q_1 and Q_2, show that $kQ \cong kQ_1 \oplus kQ_2$.

5.4 Show that the indecomposable projectives for the path algebra $k * Q$ in the previous exercise are the columns of the triangular matrix ring. More generally, show that the left ideal of $k * Q$ consisting of all paths that begin at a vertex v is an indecomposable projective.

5.5 If Q is a quiver, let Q^{op} denote the "same" quiver with the arrows reversed. Show that $kQ^{\mathrm{op}} \cong (kQ)^{\mathrm{op}}$.

5.6 A vertex is called a **source** if there are no arrows ending at it. A vertex is called a **sink**, if there are no arrows beginning at it.

(a) Show that v is sink if and only if S_v is projective.

(b) Show that v is source if and only if S_v is injective.

5.7 Show that $\mathrm{Hom}_Q(P_u, S_v)$ is zero if $u \neq v$, and is isomorphic to k if $u = v$.

5.8 Suppose that Q has no cycles. Let B denote the linear span of all arrows in Q. Determine the $(kQ)_0$-bimodule structure of B for some of the examples in this section. Since $(kQ)_0$ is a product of fields $\prod_v k\varepsilon_v$, so is $(kQ)_0 \otimes_k (kQ)_0$. The simple $(kQ)_0$ bimodules are indexed by pairs of vertices, and we denote them by ${}_uS_v$, this being the bimodule that is *not* annihilated by ε_u on the left, and *not* annihilated by ε_v on the right.

(a) For the quiver

$$\begin{tikzcd} \overset{2}{\bullet} \arrow[r, shift left] & \overset{1}{\bullet} \arrow[l, shift left] \end{tikzcd} \tag{5-18}$$

show that $B \cong {}_1S_2 \oplus {}_2S_1$, that $B^{\otimes(2n+1)} \cong B$ for all n, and that $B^{\otimes 2n} \cong {}_1S_1 \oplus {}_2S_2$ for all n.

(b) Compare this with what happens for the Kronecker quiver (4-5).

5.9 Let Q be the quiver

$$\begin{tikzcd} \overset{0}{\bullet} \arrow[rr, "x"] \arrow[dr, "z"'] & & \overset{1}{\bullet} \arrow[dl, "y"] \\ & \underset{2}{\bullet} & \end{tikzcd}$$

Let $M_{\alpha\beta\gamma}$ be the representation

$$\begin{tikzcd} k \arrow[rr, "\alpha"] \arrow[dr, "\gamma"'] & & k \arrow[dl, "\beta"] \\ & k & \end{tikzcd}$$

where the maps are multiplication by the non-zero scalars $\alpha, \beta, \gamma \in k$. Show that $M_{\alpha\beta\gamma} \cong M_{\mu\nu\omega}$ if and only if $\alpha\beta\gamma^{-1} = \mu\nu\omega^{-1}$.

5.10 Let Q be a quiver. Let M be the matrix with rows and columns indexed by the vertices of Q, and entry m_{uv} the number of arrows from u to v. Show that the uv-entry in M^n is the number of paths of length n from u to v.

The sum of the entries in M^n is the dimension of the degree n component of kQ.

5.11 Show that the Kronecker algebra R is isomorphic to the ring of global sections of the sheaf of non-commutative algebras

$$\mathcal{R} := \begin{pmatrix} \mathcal{O} & \mathcal{O}(-1) \\ \mathcal{O}(1) & \mathcal{O} \end{pmatrix} \tag{5-19}$$

on the projective line $\mathbb{P}^1$. Fix $n \in \mathbb{Z}$. Show that the action of $\mathcal{R}$ on $\mathcal{O}(n) \oplus \mathcal{O}(n-1)$ by left matrix multiplication induces an isomorphism $\mathcal{R} \cong \mathcal{E}nd(\mathcal{O}(n) \oplus \mathcal{O}(n-1))$.

5.12 Retain the notation in Exercise 11. Write $\mathcal{P} = \mathcal{O}(0) \oplus \mathcal{O}(-1)$. Show that that there is an exact functor
$$\mathcal{H}om(\mathcal{P}, -) : \mathsf{Mod}\mathbb{P}^1 \to \mathsf{Mod}\mathcal{R}$$
from quasi-coherent $\mathcal{O}_{\mathbb{P}^1}$-modules to sheaves of right $\mathcal{R}$-modules.

5.13 Retain the notation in Exercise 11. Show that the global section functor $H^0(\mathbb{P}^1, -)$ gives a left exact functor $\mathsf{Mod}\mathcal{R} \to \mathsf{Mod}R$. Let F denote the left exact functor $\mathsf{Mod}\mathbb{P}^1 \to \mathsf{Mod}R$ that is the composition of the global sections functor with the functor in the previous exercise. Thus $F = \operatorname{Hom}_{\mathbb{P}^1}(\mathcal{P}, -)$.

(a) For each invertible $\mathcal{O}_{\mathbb{P}^1}$-module $\mathcal{O}(n)$ describe $F(\mathcal{O}(n))$ as a representation of the quiver
$$\bullet \rightrightarrows \bullet \,. \tag{5-20}$$

(b) Do the same thing for each simple $\mathbb{P}^1$-module $\mathcal{O}_p$, $p \in \mathbb{P}^1$.

5.14 Retain the notation in Exercise 11. Let x be a coordinate function on $\mathbb{P}^1$, and let U be the copy of the affine line where x has no pole. Show that $\mathcal{R}(U)$, the ring of sections of $\mathcal{R}$ over U, is isomorphic to
$$\begin{pmatrix} k[x] & (x) \\ (x^{-1}) & k[x] \end{pmatrix},$$
where (x) denotes the ideal of $k[x]$ generated by x, and (x^{-1}) denotes the $k[x]$-submodule of $k(x)$ generated by x^{-1}. Show that this ring is isomorphic to $M_2(k[x])$. [Hint: conjugation by a suitable unit will give an explicit isomorphism; one can also argue by using $\mathcal{P}(U)$.] Over the open set U' where x has no poles, $\mathcal{R}(U')$ is isomorphic to
$$\begin{pmatrix} k[x^{-1}] & (x^{-1}) \\ (x) & k[x^{-1}] \end{pmatrix},$$
which is naturally isomorphic to 2×2 matrices over $k[x^{-1}]$.

5.15 Retain the notation in Exercise 11. For each $\mathcal{R}$-module $\mathcal{F}$, $\mathcal{F}(U)$ is a module over $\mathcal{R}(U)$, so by Morita equivalence gives a module over $k[x]$. For the $\mathbb{P}^1$-modules $\mathcal{M} = \mathcal{O}(n)$ and $\mathcal{M} = \mathcal{O}_p$ describe the modules over $k[x]$ that are obtained from the $\mathcal{R}$-modules $\mathcal{H}om(\mathcal{P}, \mathcal{M})$.

5.16 Define projective cover, and give a related exercise.

5.17 Let R be a finite dimensional k-algebra. Then $\operatorname{Hom}_k(-, k)$ is a duality between left and right R-modules. Show that it interchanges projective modules and injective modules. Show that it interchanges injective envelopes and projective covers; in particular, that it sends the injective envelope of a simple module S to the projective cover of the corresponding simple module $S^* := \operatorname{Hom}_k(S, k)$. Also show that $\operatorname{Ann} S = \operatorname{Ann} S^*$.

5.18 Define representations U_i, $1 \leq i \leq n$, for the quiver (5-10) as follows. To each vertex assign a copy of k, and assign to each arrow the identity map, except for the arrow x_i ending at i. Assign the zero map to that arrow. Show that U_i is uniserial (i.e., it has a unique composition series), and that its composition factors are the one-dimensional simples $V_i, V_{i-1}, \ldots, V_{i+1}$ starting from the top.

5.19 Verify the claims in Example 5.27.

3.6 Maps between non-commutative spaces

Ring homomorphisms are not sufficiently plentiful or flexible enough to suffice for the deeper study of the relations between rings and their modules. Similarly, the notion of an affine map is too restrictive to allow a fine analysis of the relations between various spaces. In this section we examine the weaker, but more appropriate, notion of a map.

Another way to see that the notion of affine map is too restrictive is to recall that the direct image functor associated to a map of schemes is not usually right exact, so need not have a right adjoint. The standard example is the structure map $\mathbb{P}^1 \to \operatorname{Spec} k$.

Example 6.1 There exist non-affine maps between affine spaces. Let B be an R-S-bimodule. Then B determines a map $f : \mathsf{Mod}S \to \mathsf{Mod}R$ defined by

$$f^* = - \otimes_R B, \quad \text{and} \quad f_* = \operatorname{Hom}_S(B, -).$$

However, if B is not a projective right S-module, then f_* is not right exact, so cannot have a right adjoint. Watt's Theorem shows that every map between affine spaces is induced by a bimodule in this fashion. $\Diamond$

A map $f : Y \to X$ is regular if f_*M is non-zero for all simple Y-modules M. The map f in Example 6.1 is regular provided that $\operatorname{Hom}_S(B, M) \neq 0$ for all simple S-modules M.

Other kinds of bimodules also provide maps.

Definition 6.2 Let R be a ring, and X a space. An R-X-bimodule, or an X-valued R-module, is a pair (M, α) consisting of an X-module M and a ring homomorphism $\alpha : R \to \operatorname{Hom}_X(M, M)$. $\Diamond$

Example 6.3 If (M, α) is an R-X-bimodule, then $\operatorname{Hom}_X(M, -)$ is a left exact functor $\mathsf{Mod}X \to \mathsf{Mod}R$. The natural right R-module structure on $\operatorname{Hom}_X(M, N)$ is given by

$$\lambda.r = \lambda \circ \alpha(r)$$

for $\lambda \in \operatorname{Hom}_X(M, N)$ and $r \in R$ (cf. Proposition 2.3.8). Since $\operatorname{Hom}_X(M, -)$ is left exact and commutes with products, it has a left adjoint. We use the notation $- \otimes_R M$ to denote such a left adjoint. If we label this adjoint pair (f^*, f_*), then these define a map $f : X \to \operatorname{Spec} R$ of spaces. $\Diamond$

The next result shows that the notation $- \otimes_R M$ behaves in a way that is compatible with its usual meaning for modules over a ring.

Proposition 6.4 *Let R be a ring, X a space, and let (M, α) be an R-X-bimodule. Let $f : X \to \operatorname{Spec} R$ be the associated map. Then*

1. *$R \otimes_R M \cong M$;*
2. *under the isomorphism in (1), if $\delta : R \to R$ is left multiplication by r, then $f^*(\delta) = \alpha(r)$.*

Proof. For each X-module N and each right R-module K, we denote by ν_{KN} the adjoint isomorphism

$$\nu_{KN} : \operatorname{Hom}_R(K, \operatorname{Hom}_X(M, N)) \xrightarrow{\sim} \operatorname{Hom}_X(K \otimes_R M, N).$$

We will show that the map $\nu_{RM}(\alpha) : R \otimes_R M \to M$ is an isomorphism.

By the Yoneda lemma, to show that $\nu_{RM}(\alpha)$ is an isomorphism it suffices to show for every X-module N that the induced map $\operatorname{Hom}_X(M, N) \to \operatorname{Hom}_X(R \otimes_R M, N)$ defined by

$$\lambda \mapsto \lambda \circ \nu_{RM}(\alpha) \tag{6-1}$$

is an isomorphism. By equation (1.6-4),

$$\lambda \circ \nu_{RM}(\alpha) = \nu_{RN}(f_*\lambda \circ \alpha). \tag{6-2}$$

Since $f_*\lambda : \operatorname{Hom}_X(M, M) \to \operatorname{Hom}_X(M, N)$ is defined by $(f_*\lambda)(\theta) = \lambda \circ \theta$,

$$(f_*\lambda \circ \alpha)(r) = \lambda \circ \alpha(r) = \lambda.r, \tag{6-3}$$

for $r \in R$ and $\lambda \in \operatorname{Hom}_X(M, N)$.

There is an isomorphism $\Phi : \operatorname{Hom}_X(M, N) \to \operatorname{Hom}_R(R, \operatorname{Hom}_X(M, N))$ of right R-modules defined by $\Phi(\lambda)(r) = \lambda.r$. Therefore (6-2) and (6-3) give

$$\lambda \circ \nu_{RM}(\alpha) = \nu_{RN}(\Phi(\lambda))$$

which implies that the map defined by (6-1) is $\nu_{RN} \circ \Phi$. This is certainly an isomorphism so (1) is true.

(2) Fix $r \in R$ and let $\delta : R \to R$ be left multiplication by r. We must show that the diagram

$$\begin{tikzcd} R \otimes_R M \arrow[r, "f^*(\delta)"] \arrow[d, "\nu(\alpha)"'] & R \otimes_R M \arrow[d, "\nu(\alpha)"] \\ M \arrow[r, "\alpha(r)"'] & M \end{tikzcd}$$

commutes. For $x \in R$, we have

$$(\alpha \circ \delta)(x) = \alpha(rx) = \alpha(r)\alpha(x) = (f_*(\alpha(r)) \circ \alpha)(x).$$

Equations (1.6-3) and (1.6-4) give

$$\begin{aligned} \nu(\alpha) \circ f^*(\delta) &= \nu(\alpha \circ \delta) \\ &= \nu(f_*(\alpha(r)) \circ \alpha) \\ &= \alpha(r) \circ \nu(\alpha), \end{aligned}$$

as required. $\square$

We continue to assume that M is an R-X-bimodule via a ring homomorphism $\alpha : R \to \operatorname{Hom}_X(M, M)$. Composition $\operatorname{Hom}_X(M, M) \times \operatorname{Hom}_X(N, M) \to \operatorname{Hom}_X(N, M)$ gives $\operatorname{Hom}_X(N, M)$ the structure of a *left* R-module via

$$r.\lambda = \alpha(r) \circ \lambda.$$

Because $\operatorname{Hom}_X(M, M)$ is an R-R-bimodule, if K is a right R-module, then $\operatorname{Hom}_R(K, \operatorname{Hom}_X(M, M))$ is a *left* R-module via

$$(r.\theta)(x) = \alpha(r).\theta(x) = \alpha(r) \circ \theta(x)$$

for $x \in K$ and $\theta \in \operatorname{Hom}_R(K, \operatorname{Hom}_X(M, M))$.

Lemma 6.5 *Let M be an R-X-bimodule via a ring homomorphism $\alpha : R \to \operatorname{Hom}_X(M, M)$. The adjunction isomorphism*

$$\nu : \operatorname{Hom}_R(K, \operatorname{Hom}_X(M, M)) \xrightarrow{\sim} \operatorname{Hom}_X(K \otimes_R M, M)$$

is an isomorphism of left R-modules.

Proof. Let $\theta \in \operatorname{Hom}_R(K, \operatorname{Hom}_X(M, M))$. We must show that $\nu(r.\theta)$ is equal to $r.\nu(\theta)$ for all $r \in R$. The actions of r are described just prior to the lemma.

As in the previous proof, we write $f_* = \operatorname{Hom}_X(M, -)$. Since $\alpha(r) : M \to M$, $f_*(\alpha(r)) : f_*M \to f_*M$. Explicitly,

$$f_*(\alpha(r))(\psi) = \alpha(r) \circ \psi.$$

Hence, if $x \in K$, then $(r.\theta)(x) = \alpha(r)\circ\theta(x) = f_*(\alpha(r))(\theta(x))$, so $r.\theta = f_*(\alpha(r))\circ\theta$. Therefore $\nu(r.\theta) = \nu(f_*(\alpha(r)) \circ \theta)$, and this is equal to $\alpha(r) \circ \nu(\theta)$ by (6-4) in Chapter 2. But $\alpha(r) \circ \nu(\theta) = r.\nu(\theta)$, thus proving the result. □

Example 6.6 The projective line is a subspace of the Kronecker space. Let R denote the path algebra of the Kronecker quiver

$$\overset{1}{\bullet} \;\underset{\longleftarrow}{\longleftarrow}\; \overset{2}{\bullet} \tag{6-4}$$

The $\mathbb{P}^1$-module $\mathcal{O} \oplus \mathcal{O}(1)$ can be given an R-$\mathbb{P}^1$-bimodule structure by letting the arrows act as multiplication by the two homogeneous coordinate functions on $\mathbb{P}^1$. More explicitly,

$$\operatorname{End}_{\mathbb{P}^1} \mathcal{O} \oplus \mathcal{O}(1) \cong \begin{pmatrix} k & 0 \\ k^2 & k \end{pmatrix}.$$

This provides a map of spaces $f : \mathbb{P}^1 \to \mathsf{Mod}Q$. The direct image functor is $f_* = \operatorname{Hom}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1), -)$. Each $p \in \mathbb{P}^1$ gives a 2-dimensional representation $f_*\mathcal{O}_p$ of Q; all these are simple Q-modules except for $f_*\mathcal{O}_0$ and $f_*\mathcal{O}_\infty$, which are non-split extensions between the two one-dimensional simple Q-modules. Notice that f_* is exact on finite length modules. ♢

The maps in our sense are more general than the morphisms in algebraic geometry. For example, some correspondences give rise to maps in our sense.

Example 6.7 Let X and Y be schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{Y\times X}$-module and suppose that its support, say Z, is finite over Y. Let $\mathrm{pr}_1 : Z \to Y$ and $\mathrm{pr}_2 : Z \to X$ be the projections. Since pr_1 is a finite morphism it is affine, whence pr_{1*} has a right adjoint $\mathrm{pr}_1^!$. It follows that the functor $f^* : \mathsf{Qcoh}X \to \mathsf{Qcoh}Y$ defined by

$$f^*M := \mathrm{pr}_{1*}(\mathrm{pr}_2^* M \otimes_{\mathcal{O}_Z} \mathcal{F})$$

has a right adjoint, namely f_* defined by

$$f_*N = \mathrm{pr}_{2*}\mathcal{H}om_Z(\mathcal{F}, \mathrm{pr}_1^! N),$$

so (f^*, f_*) defines a map $f : Y \to X$ in our sense. Recall that a correspondence from Y and X is simply a closed subscheme $Z \subset Y \times X$. Hence, if $\mathrm{pr}_2 : Z \to Y$ is finite, then one obtains a map in our sense by taking $\mathcal{F} = \mathcal{O}_Z$. $\Diamond$

Definition 6.8 Let $f : Y \to X$ be a map of spaces, and let p be a closed point of Y. If there exists a closed point $q \in X$ such that $f_*\mathcal{O}_p \in \mathsf{Mod}q$, we call q the image of p under f, and denote it by $f(p)$. We also say the p lies in fiber over q. $\Diamond$

On page 111 we observed that if $f : Y \to X$ is a morphism of affine algebraic varieties, then $f_*\mathcal{O}_p$ is a finite direct sum of copies of $\mathcal{O}_{f(p)}$ for all closed points $p \in Y$. The next example shows that this does not hold for maps between non-commutative spaces. An even simpler example is provided by Example 11.4.

Example 6.9 Let Q be the Kronecker quiver (4-5). Write Q for the space $\mathsf{Mod}kQ$. Let $f : \mathbb{A}^1 \to Q$ be the map induced by the inclusion of algebras

$$\varphi : kQ = \begin{pmatrix} k & 0 \\ k + kx & k \end{pmatrix} \to M_2(k[x]). \tag{6-5}$$

There are two closed points, say p and q, in Q. We label them so that $\mathcal{O}_q$ is the projective simple. The points $\lambda \in \mathbb{A}^1$ are in bijection with the simple modules over $M_2(k[x])$. The restriction of the simple module $\mathcal{O}_l$ to kQ gives a non-split extension

$$0 \to \mathcal{O}_q \to f_*\mathcal{O}_\lambda \to \mathcal{O}_p \to 0.$$

In particular, $f_*\mathcal{O}_\lambda$ is not a direct sum of simples, nor are all its composition factors isomorphic to a single simple module. By Example 3.10, φ is an epimorphism of rings, so f_* is full and faithful by Theorem 3.12. However, the "image" of $\mathsf{Mod}\mathbb{A}^1$ in $\mathsf{Mod}Q$ does not give a closed subspace, nor, once we have defined the terms, does it give an open subspace or a weakly closed subspace. $\Diamond$

The next two examples show that the behavior of maps on closed points can be quite counter-intuitive.

Example 6.10 Let k be a field with a primitive n^{th} root of unity ξ. Let $v = \operatorname{diag}(1, \xi, \xi^2, \ldots, \xi^{n-1})$ be the diagonal $n \times n$ matrix, and consider the inclusion map $\varphi : k[v] \to M_n(k)$. Notice that $k[v] \cong k^{\times n}$, and $\mathsf{Mod} M_n(k) \cong \mathsf{Mod} k$, so φ induces a map of spaces that can be thought of as a map $f : \operatorname{Spec} k \to \operatorname{Spec} k^{\times n}$. Because the simple $M_n(k)$-module is isomorphic to the direct sum of the n distinct simple $k[v]$-modules, this map can be thought of as sending the single point of $\operatorname{Spec} k$ to the n distinct points of $\operatorname{Spec} k^{\times n}$. We will sometimes refer to this as *breaking a single point into n distinct points.* I think of this as splitting the atom. $\diamondsuit$

Example 6.11 Let $f : \mathbb{A}^1 \to \mathbb{A}^1$ be the map $\lambda \mapsto \lambda^2$. This map is induced by the inclusion map $\varphi : k[x^2] \to k[x]$. We have associated functors f^*, f_*, and $f^!$. Now $f^! = \operatorname{Hom}_{k[x^2]}(k[x], -)$ is naturally equivalent to the functor f^*. This follows from the corollary to Watt's Theorem that states that $\operatorname{Hom}_R(P, -)$ is naturally equivalent to $- \otimes_R P^\vee$ whenever P is a finitely generated projective right R-module. Hence f^* is both a left and a right adjoint to f_*. Therefore, we may define a new map (in the non-commutative sense) $g : \mathsf{Mod} k[x] \to \mathsf{Mod} k[x^2]$ by setting $g^* = f_*$ and $g_* = f^*$.

One way to exclude g is to deal only with enriched spaces and to define a map $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ to be an adjoint pair (f^*, f_*) as before, but with the additional requirement that $f^*\mathcal{O}_X$ is isomorphic to $\mathcal{O}_Y$. Such a definition excludes g because $g^*(k[x]) \cong k[x^2] \oplus k[x^2]$. $\diamondsuit$

Example 6.12 Suppose that R and S are k-algebras satsifying the hypotheses in Proposition 4.16. The inclusions $R \to R \otimes_k S$ and $S \to R \otimes_k S$ induce maps from $\mathsf{Mod} R \otimes_k S$ to $\mathsf{Mod} R$ and $\mathsf{Mod} S$ respectively. We refer to these as the projections onto the first and second components respectively. If π_1 is the projection to $\mathsf{Mod} R$, then $\pi_{1*}\mathcal{O}_{(p,q)}$ is a direct sum of $\dim_k \mathcal{O}_q$ copies of $\mathcal{O}_p$, so p is the image of (p, q) under π_1, and (p, q) is in the fiber over p. $\diamondsuit$

Induced modules and fibers. Let R be a subring of a ring S. If M is an R-module we call $M \otimes_R S$ an induced module. The inclusion $R \to S$ induces an affine map of affine spaces $f : \mathsf{Mod} S \to \mathsf{Mod} R$, and $M \otimes_R S = f^*M$. When p is a closed point in $\mathsf{Mod} R$ we call $f^*\mathcal{O}_p$ a fiber module. One should think of it as lying in the "fiber over p". Fiber modules are used extensively in the representation theory of Lie algebras. For example, if $\mathfrak{b}$ is a Borel subalgebra (i.e., a maximal solvable subalgebra) of a finite dimensional semisimple Lie algebra $\mathfrak{g}$, and f is induced by the inclusion $U(\mathfrak{b}) \to U(\mathfrak{g})$, then Verma modules are defined as those modules of the form $f^*\mathcal{O}_p$ where p is a closed point in $\mathsf{Mod} U(\mathfrak{b})$. When $\mathfrak{g}$ is solvable, and $\lambda \in \mathfrak{g}^*$, one chooses a Lie subalgebra $\mathfrak{b}$ that is maximal subject to the condition that λ vanish on $[\mathfrak{b}, \mathfrak{b}]$; then λ defines a closed point p in $\mathsf{Mod} U(\mathfrak{b})$, and the module $f^*\mathcal{O}_p$ is a vehicle for carrying information from $U(\mathfrak{b})$ to $U(\mathfrak{g})$.

Example 6.13 Let k be an uncountable algebraically closed field. Let x be an element in a countably generated k-algebra R. Suppose that $x - \lambda$ is a regular element of R for all $\lambda \in k$. The inclusion $k[x] \to R$ induces a map

$f : \mathsf{Mod}R \to \mathbb{A}^1 = \operatorname{Spec} k[x]$. For each closed point $\lambda \in \mathbb{A}^1$ we call $f^*\mathcal{O}_\lambda$ the fiber module over λ. By analogy with the commutative case, one would expect that $\mathsf{Mod}R$ is the "union" of the fibers of f. We have not defined non-commutative versions of these terms. However, if x is a function on an affine scheme X, it is a tautology that X is the disjoint union of the level hypersurfaces $x = \lambda$, $\lambda \in k$. If X_λ is a family of closed subschemes of an affine scheme X, then a closed point $p \in X$ is in their union if and only if $\operatorname{Ext}^i_X(\mathcal{O}_{Y_\lambda}, \mathcal{O}_p)$ is non-zero for some i and some λ. This suggests that one might be able to use Ext-groups as a way to measure the analogous non-commutative ideas. As evidence for this, under the above hypotheses, one can show that if M is a non-zero, finitely generated R-module, then $\operatorname{Ext}^i_R(f^*\mathcal{O}_\lambda, M)$ is non-zero for some i and some λ. To see this, observe that

$$0 \longrightarrow R \xrightarrow{x-\lambda} R \longrightarrow f^*\mathcal{O}_\lambda \longrightarrow 0$$

is a projective resolution, so $\operatorname{Ext}^i_R(f^*\mathcal{O}_\lambda, M)$ is computed as the homology of the sequence

$$0 \longleftarrow M \xleftarrow{x-\lambda} M \longleftarrow 0.$$

If all the Ext-groups were zero then these maps from M to M would all be isomorphisms, so viewing M as a $k[x]$-module it would in fact be a $k(x)$-module. However, $k(x)$ has uncountable dimension over k, whereas M has countable dimension. Hence the action of some $x - \lambda$ on M fails to be bijective. It follows that one of Ext-groups is non-zero. $\Diamond$

EXERCISES

6.1 Some questions ??

3.7 Open subspaces

Although open subspaces of a non-commutative space X correspond to localizations of $\mathsf{Mod}X$ (Corollary 7.3) we prefer a definition that resembles the definition of a closed subspace.

Definition 7.1 Let X be a non-commutative space. A subspace U of X is open if the inclusion functor $j_* : \mathsf{Mod}U \to \mathsf{Mod}X$ has an exact left adjoint j^*. We call the map $j : U \to X$ determined by the adjoint pair (j^*, j_*) the inclusion of U.

If U is an open subspace of X we call j^* the restriction to U. If M is an X-module we write $M|_U$ for j^*M, and if f is a morphism in $\mathsf{Mod}X$ we write $f|_U$ for $j^*(f)$.

A map $f : Y \to X$ is an open immersion if it is an isomorphism onto an open subspace of X. $\Diamond$

If x is a regular element in a commutative ring R, the natural ring homomorphism $R \to R_x = R[x^{-1}]$ induces a map $j : \operatorname{Spec} R_x \to \operatorname{Spec} R$ for which j^* is the functor $M \mapsto M \otimes_R R_x$.

Theorem 7.2 *Let $j : U \to X$ be a map of spaces. Suppose that j^* is exact and that the adjunction $\eta : j^* j_* \to \mathrm{id}_U$ is a natural equivalence. Let T be the full subcategory of $\mathsf{Mod}X$ consisting of the X-modules M such that $j^*M = 0$. Then T is a localizing subcategory of $\mathsf{Mod}X$ and*

$$\mathsf{Mod}U \cong \mathsf{Mod}X/\mathsf{T}.$$

Proof. Since j^* is exact, T is a Serre subcategory of $\mathsf{Mod}X$. Let $\pi : \mathsf{Mod}X \to \mathsf{Mod}X/\mathsf{T}$ be the quotient functor. By Theorem 2.13.9, there is a unique functor $g^* : \mathsf{Mod}X/\mathsf{T} \to \mathsf{Mod}U$ such that $j^* = g^*\pi$. By *loc. cit.*, g^* is exact.

Since $g^*\pi j_* = j^* j_* \cong \mathrm{id}_U$, g^* is full and every U-module is isomorphic to g^*N for some X-module N. Hence, to show g^* is an equivalence of categories it suffices to show it is faithful (Theorem 1.6.21).

Let θ be an X-morphism. Because j^* is exact, $j^*(\theta)$ is an isomorphism if and only if $j^*(\ker\theta) = j^*(\operatorname{coker}\theta) = 0$. By definition of T this is the same as the condition that $\pi(\ker\theta) = \pi(\operatorname{coker}\theta) = 0$. This is equivalent to $\pi(\theta)$ being an isomorphism.

Let $\tau : \mathrm{id}_X \to j_* j^*$ be the natural transformation. Let M be an X-module, and apply the previous paragraph to $\theta = \tau_M : M \to j_* j^* M$. By hypothesis, $j^*(\tau_M)$ is an isomorphism, so

$$\pi(\tau_M) : \pi M \to \pi j_* j^* M = \pi j_* g^* \pi M$$

is an isomorphism. Since every object in $\mathsf{Mod}X/\mathsf{T}$ is of the form πM for some M, this produces a natural equivalence $\mathrm{id}_{\mathsf{Mod}X/\mathsf{T}} \to \pi j_* g^*$. It follows that g^* is faithful, and hence an equivalence. In fact, we have $g^*\pi j_* = j^* j_* \cong \mathrm{id}_U$, so $(g^*, \pi j_*)$ is the adjoint pair implementing the equivalence.

Because $\mathsf{Mod}X$ is a Grothendieck category, it has enough injectives. Therefore, by Theorem 2.14.12, to prove that T is localizing it suffices to show that each X-module M has a largest submodule which is in T. Since j^* is a left adjoint it commutes with colimits. Therefore j^* vanishes on the sum of all submodules of M on which j^* vanishes; this is the largest such submodule of M.

If ω is a right adjoint to π, then $(j^*\omega, \pi j_*) \cong (g^*, g_*)$. □

Corollary 7.3 *Let $j : U \to X$ be the inclusion of an open subspace. Then*

$$\mathsf{Mod}U \cong \mathsf{Mod}X/\mathsf{T}$$

where T is the localizing subcategory consisting of those M for which $j^*M = 0$.

Proof. By Theorem 2.6.15, the adjunction $j^* j_* \to \mathrm{id}_U$ is a natural equivalence. Hence Theorem 7.2 applies. □

If $F : \mathsf{Mod}X \to \mathsf{C}$ is an exact functor, then the full subcategory, say T, consisting of the modules M such that $FM = 0$ is a Serre subcategory. Let $j : U \to X$ be the open subspace defined by $\mathsf{Mod}U = \mathsf{Mod}X/\mathsf{T}$. Then there is a unique functor $G : \mathsf{Mod}U \to \mathsf{A}$ such that $F = Gj^*$. There are two natural examples.

If I is an injective X-module, then

$$\mathsf{T} = \{M \mid \mathrm{Hom}_X(M, I) = 0\}.$$

Every open subspace arises in this way: given a localizing subcategory, take I to be the direct product of all the torsion-free indecomposable injectives.

If P is a projective X-module, then

$$\mathsf{T} = \{M \mid \mathrm{Hom}_X(P, M) = 0\}.$$

Proposition 7.4 *Let P be a projective module over a ring R. Let I denote its trace ideal. Then*

1. $\mathrm{Hom}_R(P, M) = 0$ *if and only if M is annihilated by I;*
2. $I = I^2$.

Proof. (1) By definition of the trace ideal, the natural map $\mathrm{Hom}_R(P, I) \to \mathrm{Hom}_R(P, R)$ is surjective. Hence $\mathrm{Hom}_R(P, R/I) = 0$. Thus R/I is a torsion module. Since T is closed uder direct sums and quotients, $\mathrm{Hom}_R(P, M) = 0$ for all R/I-modules M. Hence each R/I-module is torsion.

Conversely, suppose that M is an R-module such that $MI \neq 0$. Then $ma \neq 0$ for some $m \in M$ and $a \in I$. There some $\varphi \in \mathrm{Hom}_R(P, I)$ such that $a \in \mathrm{im}\,\varphi$. Define $\psi : P \to M$ by $\psi(p) = m\varphi(p)$. This is a right R-module map, and $ma \in \mathrm{im}\,\psi$. Hence $\mathrm{Hom}_R(P, M) \neq 0$.

(2) Since I/I^2 is annihilated by I, $\mathrm{Hom}_R(P, I/I^2) = 0$. Hence the image of any $\varphi \in \mathrm{Hom}_R(P, I)$ is contained in I^2. Hence, the trace ideal of P is contained in I^2. That is, $I = I^2$. $\square$

We have not defined "unions" or "intersections" of closed or open subspaces. However, the next result will allow us to define unions of open subspaces, and it shows that an arbitrary union of open subspaces is again an open subspace.

Lemma 7.5 *Let X be a space. Let T_λ, $\lambda \in \Lambda$, be a family of localizing subcategories of $\mathsf{Mod}X$. Then their intersection $\mathsf{T} = \cap \mathsf{T}_\lambda$ is also a localizing subcategory of $\mathsf{Mod}X$.*

Proof. Let τ_λ be the torsion functor associated to T_λ. By definition T is the full subcategory of $\mathsf{Mod}X$ consisting of those modules M that are in every T_λ. It is clear that T is a Serre subcategory of $\mathsf{Mod}X$.

If M is an arbitrary module, define $\tau M := \cap_\lambda \tau_\lambda M$. It is contained in each T_λ, so is in T. Moreover, if N is a submodule of M that belongs to T, then it must be contained in every $\tau_\lambda M$. Therefore τM is the largest submodule of

M that is in T. Because $\mathsf{Mod}X$ has enough injectives, Theorem 2.14.12 implies that T is localizing. □

By Proposition 3.7, if $j : U \to X$ is the inclusion of an open subspace, and j is an affine map, then $U \cap Z$ is closed in U.

Definition 7.6 Let U_λ, $\lambda \in \Lambda$, be a family of open subspaces of X. Write $\mathsf{Mod}U_\lambda = \mathsf{Mod}X/\mathsf{T}_\lambda$. Their union, $\cup_\lambda U_\lambda$, is defined by declaring that

$$\mathsf{Mod} \cup_\lambda U_\lambda := \mathsf{Mod}X/\mathsf{T},$$

where $\mathsf{T} = \cap_\lambda \mathsf{T}_\lambda$. Lemma 7.5 ensures that T is a localizing subcategory, so $\cup U_\lambda$s is an open subspace of X. If $\mathsf{T} = 0$, then we say that the U_λ's provide an open cover of X. If each U_λ is affine we say that the U_λs provide an open affine cover of X. ♢

The next result says that the U_λs provide an open cover if and only if every X-module is determined by its restrictions to the U_λs.

Lemma 7.7 *Let U_λ, $\lambda \in \Lambda$, be a family of open subspaces of X, and write U for their union. Let $j : \mathsf{Mod}U \to \mathsf{Mod}X$ be the inclusion map. The U_λs provide an open cover of X if and only if the functor $j^* : \mathsf{Mod}X \to \mathsf{Mod}U$ is faithful.*

Proof. Adopting the notation in Definition 7.6, j^* is faithful if and only if $\mathsf{T} = 0$. But this is the condition that the U_λs provide an open cover. □

Thus an X-module M is zero if and only if $M|_{U_\lambda} = 0$ for all λ. Similarly, a morphism of X-modules is zero if and only if its restriction to each U_λ is zero. And a sequence of X-modules is exact if and only if its restriction to each U_λ is exact.

Proposition 7.8 *Let $j : U \to X$ be an open immersion. Let M be an X-module, and N a U-module. Then there is a spectral sequence*

$$\mathrm{Ext}^p_X(M, R^q j_* N) \Rightarrow \mathrm{Ext}^{p+q}_U(j^*M, N) \tag{7-1}$$

Proposition 7.9 *Let $j : U \to X$ be an open immersion. If X is noetherian and* $\operatorname{gldim} X = 1$*, then j is an affine map.*

Proof. Let τ be the associated torsion functor that sends an X-module to its largest submodule that j^* kills. Then

$$(R^1 j_*) j^* \cong R^2 \tau = 0.$$

Because j is an open immersion, $j^* j_* \cong \mathrm{id}_U$, whence $0 = (R^1 j_*) j^* j_* = R^1 j_*$. Thus j_* is exact. By 2.16.4, j_* commutes with direct sums so has a right adjoint. Finally, because $j^* j_* \cong \mathrm{id}_U$, j_* is full and faithful. Therefore j is an affine map. □

Corollary 7.10 (Goodearl) *Every open subspace of a noetherian affine space of global dimension one is affine.*

Proof. This follows from Propositions 7.9 and 3.4. □

The corollary can be stated in a way that is closer to Goodearl's original foormulation. Suppose R is a prime noetherian ring of global dimension one and S_i, $i \in I$, is some collection of simple right R-modules. Set $X = \mathrm{Sp}(R)$ and let Y denote the weakly closed subspace of X defined by declaring $\mathsf{Mod}Y$ to consist of all direct sums of the S_is. Then $X \backslash Y$ is an open subspace of X. Goodearl showed that there is a ring R_Y containing R such that $- \otimes_R R_Y$ kills the simple modules S_i and no others. The equivalent geometric statement is that $X \backslash Y$ is an affine space.

$\boxed{\text{Paul}}$ Compare the simples and closed points in X and $X \backslash Y$. Show that if Y is all except one simple, then $\mathsf{Mod}X \backslash Y$ is local.

EXERCISES

7.1 Some questions???

3.8 Weakly closed subspaces

Theorem 3.14 showed that the closed subspaces of an affine space $\mathsf{Mod}R$ are in bijection with the two-sided ideals of R. In general a non-commutative ring can have few two-sided ideals, and for such a ring knowledge of its two-sided ideals is not helpful in the study of $\mathsf{Mod}R$. Similarly, many non-commutative spaces have few closed subspaces, and for that reason we introduce a weaker notion that has proven to be effective.

Definition 8.1 [88, p. 395], [258] A subspace Y of X is weakly closed if $\mathsf{Mod}Y$ is closed under subquotients, and the inclusion functor $i_* : \mathsf{Mod}Y \to \mathsf{Mod}X$ has a right adjoint $i^!$. (The inclusion functor i_* is exact.) ◊

Definition 8.2 A weak map $f : W \to X$ between non-commutative spaces is a natural equivalence class of a left exact functor $f_* : \mathsf{Mod}W \to \mathsf{Mod}X$.

The inclusion of a weakly closed subspace in its ambient space is a weak map.

If Y is weakly closed, then $\mathsf{Mod}Y$ is closed under direct limits in $\mathsf{Mod}X$.

A closed subspace is weakly closed.

Example 8.3 If X is a commutative scheme, then the modules supported on any (possibly infinite) union of closed subschemes is a weakly closed subspace. If i_* is the inclusion functor, then its right adjoint $i^!$ sends a module to its largest submodule having its support on that union. If X is affine, say $X = \mathrm{Spec}\, R$, we can appeal to Theorem 3.14 to see that i_* does not usually have a left adjoint;

if it did, then the category of modules supported on that union would be of the form $\mathsf{Mod}R/I$ for some ideal I. This is usually not the case. $\diamondsuit$

Lemma 8.4 *Every localizing subcategory of* $\mathsf{Mod}X$ *is of the form* $\mathsf{Mod}Y$ *for some weakly closed subspace* Y *of* X*.*

Proof. Let T be a localizing subcategory of $\mathsf{Mod}X$. Since T is closed under quotients and subobjects, the inclusion $i_* : \mathsf{T} \to \mathsf{Mod}X$ is exact. By Theorem 2.16.3, T is a Grothendieck category. The torsion functor $\tau : \mathsf{Mod}X \to \mathsf{T}$ is a right adjoint to i_*, so T is weakly closed. $\square$

Definition 8.5 Let X be a noetherian space. If Y is a weakly closed subspace of X, we define $\mathsf{Mod}_Y X$, the category of X-modules supported on Y, to be the full subcategory consisting of modules M having a filtration $0 = M_0 \subset M_1 \subset M_2 \subset \cdots$ such that $M = \cup_{i=1}^{\infty} M_i$ and each M_i/M_{i-1} is a Y-module. $\diamondsuit$

Lemma 8.6 *Let* X *be a noetherian space. If* Y *is a weakly closed subspace of* X*, then* $\mathsf{Mod}_Y X$ *is a localizing subcategory of* $\mathsf{Mod}X$*.*

Proof. First, $\mathsf{Mod}_Y X$ is closed under submodules and quotient modules because $\mathsf{Mod}Y$ is. Let $0 \to L \to M \to N \to 0$ be exact, and suppose that L and N are in $\mathsf{Mod}_Y X$. Write L and N as unions of ascending chains of submodules L_i and N_i such that the quotients are Y-modules. If the number of L_i's is finite we are done.

If M' is a noetherian X-module, then $M' \in \mathsf{Mod}_Y X$ if and only if M' has a *finite* filtration with the slices being Y-modules. Since $\mathsf{Mod}X$ is noetherian, a module is in $\mathsf{Mod}_Y X$ if and only if every noetherian submodule of it is.

So we need only show that every noetherian submodule M' of M is in $\mathsf{Mod}_Y X$. Set $L' = M' \cap L$, and let N' be the image of M' in N. Hence there is an exact sequence $0 \to L' \to M' \to N' \to 0$ in which L' and N' have *finite* filtrations with slices in $\mathsf{Mod}Y$. It follows that M' also has this property, so is in $\mathsf{Mod}_Y X$. $\square$

The intersection of a family of weakly closed subspaces is weakly closed.

$\boxed{\text{Paul}}$ Need some interesting examples e.g. fibers in Ore extensions.

EXERCISES

8.1 questions??

3.9 Open complements

It is a tautology that every Zariski-open subscheme of a commutative scheme is the complement of a closed subscheme. The non-commutative analogue is that every open subspace of a noetherian space is the open complement of a weakly closed subspace (Theorem 9.3).

Definition 9.1 Let X be a noetherian space. If Y is a weakly closed subspace of X, we define its open complement $X \backslash Y$ by

$$\mathsf{Mod}X \backslash Y := \mathsf{Mod}X / \mathsf{Mod}_Y X,$$

the quotient category. (Lemma 8.6 ensures that $\mathsf{Mod}_Y X$ is a localizing category, so the quotient category may be formed.) ♢

Proposition 9.2 *If Y is a weakly closed subspace of X, then $X \backslash Y$ is an open subspace of X.*

Proof. Since $\mathsf{Mod}_Y X$ is a localizing subcategory, the quotient functor $j^* : \mathsf{Mod}X \to \mathsf{Mod}X \backslash Y$ has a right adjoint j_*. By Theorem 2.13.8, j^* is exact. Because $j^* j_* \cong \mathrm{id}_{X \backslash Y}$, Theorem 2.6.15 ensures that j_* is fully faithful. Hence we may view $\mathsf{Mod}X \backslash Y$ as a full subcategory of $\mathsf{Mod}X$. Finally, $\mathsf{Mod}X \backslash Y$ is a Grothendieck category by Theorem 2.16.3. □

Theorem 9.3 *Let X be a noetherian space. Every open subspace of X is the complement of a weakly closed subspace.*

Proof. Let $j : U \to X$ be an open subspace. By Theorem 7.2, the full subcategory T consisting of the modules M for which $j^* M = 0$ is a localizing subcategory of $\mathsf{Mod}X$, and $\mathsf{Mod}U \cong \mathsf{Mod}X/\mathsf{T}$. By Lemma 8.4, there is a weakly closed subspace Y such that $\mathsf{T} = \mathsf{Mod}Y$. So it remains to see that $\mathsf{T} = \mathsf{Mod}_Y X$. It is clear that $\mathsf{T} \subset \mathsf{Mod}_Y X$, so let $M \in \mathsf{Mod}_Y X$. If M' is a noetherian submodule of M, then $M' \in \mathsf{T}$ because T is closed under extensions. Because X is noetherian, M is the direct limit of its noetherian submodules. Since T is localizing, the inclusion of T in $\mathsf{Mod}X$ has a right adjoint, and by Proposition 2.16.7 this implies that the inclusion functor commutes with direct limits. Therefore T is closed under direct limits, and we conclude that M is in T. □

Different weakly closed subspaces can have the same open complement. This happens even in the Zariski topology because the open complement to the zero locus of an ideal I is the same as the open complement to the zero locus of I^2.

Proposition 9.4 *Let $Z \subset Y \subset X$ be closed subspaces. Then*

1. $\mathsf{Mod}_Z Y = \mathsf{Mod}Y \cap \mathsf{Mod}_Z X$*;*
2. *$Y \backslash Z$ is a closed subspace of $X \backslash Z$;*
3. *the open complement to $Y \backslash Z$ in $X \backslash Z$ is isomorphic to $X \backslash Y$.*

Proof. We use Proposition 2.13.11. □

The intersection of weakly closed subspaces was defined in ??? and the union of open subspaces was given in Definition 7.6. Now that we have also defined open complements we wish to check that these notions are compatible with the usual topological terminology.

Proposition 9.5 *Let $W_1, \dots, W_n$ be weakly closed subspaces of a noetherian space X. Then*

$$\bigcup_{i=1}^{n} X\backslash W_i = X\backslash(\bigcap_{i=1}^{n} W_i). \tag{9-1}$$

Proof. The result for $n = 1$ is a tautology, so we consider the case $n = 2$. Both sides of (9-1) are obtained by localizing $\mathsf{Mod}X$, so it suffices to prove that the Serre subcategories of $\mathsf{Mod}X$ involved in defining each localization are the same. Thus, we must show that if M is a non-zero noetherian X-module, then M is supported on $W_1 \cap \dots \cap W_n$ if and only if it is supported on each W_i. It is clear that if M is supported on $W_1 \cap \dots \cap W_n$ then it is supported on each W_i. Thus, we suppose that M is supported on W_1 and on W_2. Hence M has two finite filtrations such that all the slices of one are W_1-modules and all slices of the second are W_2-modules. We must show there is a single finite filtration such that all the slices are simultaneously W_1- and W_2-modules. By noetherian induction we may suppose that this can be done for every proper quotient module of M. Thus it suffices to show that M has a non-zero submodule which is both a W_1-module and a W_2-module. Let $i_j : W_j \to X$, $j = 1, 2$, denote the inclusions. By hypothesis, $i_1^! M \neq 0$. Being a submodule of a module supported on W_2 it must have a non-zero submodule which is a W_2-module; that submodule is now both a W_1- and a W_2-module. This completes the proof for the case $n = 2$. The general case reduces to the $n = 2$ case by induction. □

I haven't been able to prove the previous result for an infinite number of W_i. Part of the problem is that even if $\mathsf{Mod}X$ is locally noetherian then one need not have the descending chain condition on weakly closed subspaces (or even on closed subspaces). For example, in $\operatorname{Spec}\mathbb{Z}$ there is a chain of wekaly closed subspaces

$$\{2, 3, 5, 7, \dots\} \supset \{3, 5, 7, \dots\} \supset \{5, 7, \dots\} \supset \dots.$$

And, if $\mathsf{Mod}X$ is the category of representations of the quiver consisting of infinitely many vertices and no arrows then $\mathsf{Mod}X$ is locally noetherian (the simple modules at each vertex provide a set of generators), but there is an infinite descending chain of closed subspaces. (Maybe this suggests that the locally noetherian condition is not quite strong enough if we want a good replacement for the usual noetherian condition.)

3.10 Divisors and hypersurfaces

Consider an affine scheme $\operatorname{Spec} R$ where R is a commutative ring. The open complement to the closed subscheme $\operatorname{Spec} R/(x)$ is $\operatorname{Spec} R[x^{-1}]$. The inclusion map

$$j : \operatorname{Spec} R[x^{-1}] \to \operatorname{Spec} R$$

is induced by the natural ring homomorphism $R \to R[x^{-1}]$. Since $R[x^{-1}]$ is a flat R-module, the inverse image functor j^* is exact.

Definition 10.1 An element x in a ring R is normal if $xR = Rx$. $\Diamond$

Proposition 10.2 *Let x be a normal regular element in a ring R. Set $X =$ $\mathsf{Mod}R$ and $Y = \mathsf{Mod}R/(x)$. Then Y is a closed subspace of X, and*

$$X \backslash Y \cong \mathsf{Mod}R[x^{-1}].$$

Proof. The canonical map $R \to R[x^{-1}]$ induces a map of spaces $f : \mathsf{Mod}R[x^{-1}] \to X$. If N is an $R[x^{-1}]$-module, then $N \otimes_R R[x^{-1}] \cong N$, so $f^* f_*$ is naturally equivalent to the identity functor.

The X-modules supported on Y are precisely those M such that every element of M is annihilated by a power of x. These are precisely the modules such that $M \otimes_R R[x^{-1}]$ is zero. Therefore f^* vanishes precisely on $\mathsf{Mod}_Y X$. It follows from Theorem 7.2 that $\mathsf{Mod}R[x^{-1}] \cong \mathsf{Mod}X/\mathsf{Mod}_Y X$. But this is $\mathsf{Mod}(X \backslash Y)$ by definition, so the proof is complete. $\square$

Questions. What is the right definition of a hypersurface? Let $i : Y \to X$ be a closed immersion. We could say that Y is a regularly embedded hypersurface if $R^2 i^! = 0$, as James and I do. But this is equivalent to asking for an affine space $\mathsf{Mod}R$ that we have a two sided ideal I that is projective. This is obviously too strong. But maybe that is adequate to begin with. Perhaps it would be better to focus on divisors, and adopt Michel's definition. Possibly, one could try to phrase the definition in terms of $j : X \backslash Y \to X$. One might also want $\mathsf{Mod}_Y X$ to be closed under injective envelopes.

Let Y be a weakly closed subspace, and set $U = X \backslash Y$. Let $i : Y \to X$ and $j : U \to X$ be the immersions. Let τ be the torsion functor "supported on Y". Suppose that $R^2 \tau = 0$. (Is that a consequence of $R^2 i^! = 0$?) If $\mathsf{Mod}_Y X$ is closed under injective envelopes, then j_* is exact, so the spectral sequence (7-1) collapses to give

$$\mathrm{Ext}^1_X(M, j_* N) \cong \mathrm{Ext}^1_U(j^* M, N). \tag{10-2}$$

In particular, for closed points $p, q \in X$,

$$\mathrm{Ext}^1_X(\mathcal{O}_p, j_* j^* \mathcal{O}_q) \cong \mathrm{Ext}^1_U(j^* \mathcal{O}_p, j^* \mathcal{O}_q).$$

If they are non-zero, then $j^* \mathcal{O}_p$ and $j^* \mathcal{O}_q$ are simple U-modules. (Presumably they give closed points.) This should enable us to relate the links between points in X to the links between points in U.

3.11 Points and subspaces

Suppose that X is a commutative scheme, Z a closed subscheme, and U its open complement. If p is a closed point on X, then p is either in Z or U. For non-commutative spaces, the situation is more complicated.

Definition 11.1 If Z is a subspace of X, we say that a closed point $p \in X$ lies on Z, and we write $p \in Z$, if $\mathcal{O}_p \in \mathsf{Mod}Z$. If p does not lie on Z we write $p \notin Z$. $\Diamond$

Lemma 11.2 *Let p be a closed point in X, and let Y be a subspace of X. If p lies on Y, then it is a closed point of Y.*

Proof. Let $i : Y \to X$ and $\beta : p \to X$ be the inclusions. The hypothesis says that $\mathsf{Mod}p \in \mathsf{Mod}Y$. Therefore if $\alpha_* : \mathsf{Mod}p \to \mathsf{Mod}Y$ denotes the inclusion, then $\beta_* = i_*\alpha_*$.

To show that p is closed in Y it suffices to exhibit left and right adjoints to α_*. We define $\alpha^* = \beta^* i_*$ and $\alpha^! = \beta^! i_*$. Let M be a Y-module, and V a p-module. Then

$$\begin{aligned}\mathrm{Hom}_p(\alpha^* M, V) &= \mathrm{Hom}_p(\beta^* i_* M, V)\\ &\cong \mathrm{Hom}_X(i_* M, \beta_* V)\\ &= \mathrm{Hom}_X(i_* M, i_*\alpha_* V)\\ &= \mathrm{Hom}_Y(M, \alpha_* V).\end{aligned}$$

Thus α^* is left adjoint to α_*. And

$$\begin{aligned}\mathrm{Hom}_p(V, \alpha^! M) &= \mathrm{Hom}_p(V, \beta^! i_* M)\\ &\cong \mathrm{Hom}_X(\beta_* V, i_* M)\\ &= \mathrm{Hom}_X(i_*\alpha_* V, i_* M)\\ &= \mathrm{Hom}_Y(\alpha_* V, M).\end{aligned}$$

Thus $\alpha^!$ is right adjoint to α_*. And □

Let p be a closed point on X. If Y is weakly closed in X, and $j : X\backslash Y \to X$ is the inclusion, then $j^*\mathcal{O}_p = 0$ if and only if $p \in Y$.

Lemma 11.3 *Let $i : Z \to X$ be a closed immersion. Then i sends closed points of Z to closed points of X.*

Proof. Let p be a closed point of Z. Let $\alpha : p \to Z$ be the inclusion map. Then the adjoint triples $(i^*, i_*, i^!)$ and $(\alpha^*, \alpha_*, \alpha^!)$, yield an adjoint triple $(\alpha^* i^*, i_*\alpha_*, \alpha^! i^!)$ which gives a map $i\alpha : p \to Z$, making p a closed point of X. Since $\mathsf{Mod}p$ and $\mathsf{Mod}Z$ are closed under subquotients in $\mathsf{Mod}Z$ and $\mathsf{Mod}X$ respectively, $\mathsf{Mod}p$ is closed under subquotients in $\mathsf{Mod}X$. Hence p is a closed point of X. □

Open immersions do not send closed points to closed points. The next example exhibits an open subspace $j : U \to X$, and closed points $p' \in U$ and $p \in X$ such that $j^*\mathcal{O}_p \cong \mathcal{O}_{p'}$, but $j_*\mathcal{O}_{p'} \not\cong \mathcal{O}_p$; thus p' is a k-valued point of U, but $j(p')$ is not a closed point in X. (Example 6.9 provides another such an example.)

Example 11.4 The space $X = \mathsf{Mod}Q$ of representations of the quiver

$$\overset{p}{\bullet} \longleftarrow \overset{q}{\bullet} \tag{11-3}$$

is an affine space with coordinate ring isomorphic to the ring T of lower triangular 2×2 matrices. There are two closed points, p and q, with corresponding simple modules $\mathcal{O}_p$ and $\mathcal{O}_q$. The corresponding maximal ideals are

$$\mathfrak{p} = \operatorname{Ann}\mathcal{O}_p = \begin{pmatrix} 0 & 0 \\ k & k \end{pmatrix}, \quad \text{and} \quad \mathfrak{q} = \operatorname{Ann}\mathcal{O}_q = \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix}.$$

Both are idempotent. By Example 5.14, $\mathcal{O}_p$ is projective. Hence $\mathfrak{p}$ is a projective right T-module. On the other hand, $\mathfrak{q}$ is a projective left T-module. Both $\mathcal{O}_q$ and $\mathfrak{p}$ are injective right T-modules. As a right T-module $\mathfrak{q} \cong \mathcal{O}_p \oplus \mathcal{O}_p$.

We write N for the nilpotent radical of T. It equals ke_{21}. We have $\mathfrak{pq} = N$, and $\mathfrak{qp} = 0$. As a left T-module, $N \cong T/\mathfrak{q}$. As a right T-module, $N \cong T/\mathfrak{p}$. It is easy to check that $\mathcal{O}_p \otimes_T N = 0$, and $\mathcal{O}_q \otimes_T N \cong \mathcal{O}_p$.

The inclusion of N in $\mathfrak{p}$ gives a non-split exact sequence

$$0 \to \mathcal{O}_p \to \mathfrak{p} \to \mathcal{O}_q \to 0. \tag{11-4}$$

The obvious geometric feature of X is that it has two closed points p and q. The point q is both open and closed because the inverse image functor associated to the inclusion $f : q \to X$ is exact: it is given by $f^* = - \otimes_T T/\mathfrak{q}$, and $T/\mathfrak{q}$ is a projective left T-module. Nevertheless, p is *not* the open complement to q. To see this, let $j : X \backslash q \to X$ denote the inclusion. Since (11-4) is the minimal injective resolution of $\mathcal{O}_p$, $j_*j^*\mathcal{O}_p \cong \mathfrak{p}$. Of course $j_*j^*\mathcal{O}_q = 0$. Therefore $\mathsf{Mod} X \backslash q$ consists of all direct sums of $\mathfrak{p}$. Since $\mathcal{O}_p$ is not an $(X\backslash q)$-module, $\mathsf{Mod} X\backslash q$ is not closed under subquotients. Thus, although q is open in X, $X\backslash q$ is not closed in X. In particular, $X\backslash q \neq p$.

In contrast, $X \backslash p = q$. To see this, let $i : X\backslash p \to X$ be the inclusion. Then $i_*i^*\mathcal{O}_q \cong \mathcal{O}_q$ because $\mathcal{O}_q$ is injective. Thus $\mathsf{Mod} X\backslash p$ consists of all direct sums of $\mathcal{O}_q$, whence $\mathsf{Mod} X\backslash p = \mathsf{Mod} q$.

The endomorphism ring of $\mathfrak{p}$ is isomorphic to k, so $X\backslash q$ is isomorphic to $\operatorname{Spec} k$. In particular, $X\backslash q$ has a unique point. We will label it p'. Thus $\mathcal{O}_{p'} = \mathfrak{p}$. Notice that $j_*\mathcal{O}_{p'}$ is *not* a simple X-module; although $j^*\mathcal{O}_p \cong \mathcal{O}_{p'}$, $j_*\mathcal{O}_{p'} \ncong \mathcal{O}_p$.

Let $S = M_2(k)$. The inclusion $\varphi : T \to S$ induces a map of spaces. Morita equivalence gives $\mathsf{Mod} S \cong \operatorname{Spec} k$. By Example 3.9, ϕ is an epimorphism, so the induced map $g : \operatorname{Spec} k \to X$ is such that g_* is a full embedding (Theorem 3.12). The simple right S-module is isomorphic to $\mathfrak{p}$ as a right T-module. We therefore identify $\mathsf{Mod} S$ with the full subcategory of $\mathsf{Mod} X$ consisting of all direct sums of $\mathfrak{p}$. In particular, g coincides with the inclusion of $X\backslash q$ in X. Thus j is an affine map.

The exactness of g^* is equivalent to the fact that S is projective as a *left* T-module. It is also projective as a right T-module. $\lozenge$

The next result is a corollary to Theorem 4.6.

Proposition 11.5 *Let X be a noetherian space over* $\operatorname{Spec} k$. *Let Y be a weakly closed subspace of X, and let $j : X\backslash Y \to X$ be the inclusion of its open complement. Let p be a closed point in X such that $p \notin Y$. If $j_*j^*\mathcal{O}_p$ is tiny, then there is a closed point p' in $X\backslash Y$ such that $j^*\mathcal{O}_p \cong \mathcal{O}_{p'}$.*

Proof. Since $\mathcal{O}_p$ is simple, so is $j^*\mathcal{O}_p$. So, it suffices to show that $j^*\mathcal{O}_p$ is a tiny U-module. Let M be a noetherian U-module. By Lemma 2.14.19, there is a noetherian X-module N such that $M \cong j^*N$. Therefore $\operatorname{Hom}_U(M, j^*\mathcal{O}_p) \cong \operatorname{Hom}_X(N, j_*j^*\mathcal{O}_p)$. This is finite dimensional by hypothesis, so $j * \mathcal{O}_p$ is tiny. $\square$

If q is another closed point of X with the same properties as p, and q' is the corresponding closed point of U, then $q' \neq p'$. This is a conseqence of the fact that if $\mathcal{O}_p$ is not isomorphic to $\mathcal{O}_q$, then $j^*\mathcal{O}_p$ is not isomorphic to $j^*\mathcal{O}_q$ (see Lemma 2.14.18).

Example 11.6 The path algebra of the quiver

$$Q \; = \quad \begin{tikzcd} \underset{n}{\bullet} \arrow[r] & \underset{n-1}{\bullet} \arrow[r] & \bullet & \cdots & \bullet \arrow[r] & \underset{2}{\bullet} \arrow[r] & \underset{1}{\bullet} \arrow[llllll, bend right] \end{tikzcd}$$

is naturally a subalgebra of $M_n(k[x])$, so there is an associated map of spaces $f : \mathbb{A}^1 \to Q$. The central element x of $M_n(k[x])$ is in kQ. The inclusion $kQ \to M_n(k[x])$ induces a map $kQ[x^{-1}] \to M_n(k[x, x^{-1}])$. This map is an isomorphism. It follows that the open subspace of Q that is the complement to the locus where x is zero is isomorphic to the open subspace of $\mathbb{A}^1$ that is the complement to the locus where x is zero. Thus $Q \backslash \mathcal{Z}(x) \cong \mathbb{A}^1 \backslash \{0\}$. $\lozenge$

Proposition 11.7 *Let M be a simple R-module, and let x be a normal element in R. Then either $Mx = 0$, or x acts bijectively on M. In the latter case M becomes an $R[x^{-1}]$-module.*

Proof. Since x is normal, Mx is a submodule of M, so is either zero or M. The normality of x also ensures that the kernel of the multiplication map $x : M \to M$ is a submodule of M. Hence if $Mx \neq 0$, x acts injectively, and $Mx = 0$.

When x acts bijectively, we define the action of x^{-1} on M by defining $m.x^{-1}$ to be the unique element $m' \in M$ such that $m'x = m$. This makes M and $R[x^{-1}]$-module. $\square$

Corollary 11.8 *Let p be a closed point of an affine space $X = \mathsf{Mod}R$. If x is a normal element of R, then either p is in the zero locus of x, or in its open complement $\mathsf{Mod}R[x^{-1}]$.*

Questions.

If $i : Y \to X$ is a weakly closed subspace, and $\alpha : p \to Y$ is a closed point, then $i_*\alpha_* : \mathsf{Mod}p \to \mathsf{Mod}X$ has a right adjoint $\alpha^! i^!$, but does not appear to always have a left adjoint. I would like a simple example of this. It suggests that we might need some notion of a "weakly closed point". One such example would be to take a big simple R-module M, and define $\mathsf{Mod}Y = \mathsf{Mod}p$ to be the full subcategory consisting of all R-modules that are isomorphic to a direct sum of copies of M.

Maybe we should observe that big simples are those such that an infinite direct product of them is not isomorphic to a direct sum of them.

I have not tried to define a weakly closed complement to an open subspace. Nor have I tried to say what we might mean by "dense open".

EXERCISES

11.1 Questions??

3.12 The graded line

This section concerns a commonly occuring non-commutative space, the graded line. It, and natural generalizations of it, occur frequently as weakly closed subspaces of non-commutative spaces that are far from being commutative (see [228]). For example, section 4.1 exhibits a non-commutative analogue of the affine plane containing many copies of the graded line.

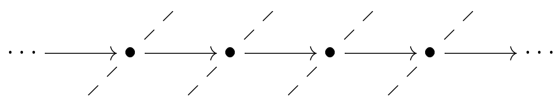
Definition 12.1 The graded line is the non-commutative space $\mathbb{L}^1 = \mathsf{GrMod}k[x]$ where $k[x]$ is the polynomial ring with grading given by $\deg x = 1$. (cf. Example 5.26.) $\lozenge$

The closed points in $\mathbb{L}^1$. The obvious simple $\mathbb{L}^1$-module is $k[x]/(x)$. We denote this by k and call it the trivial module. If $n \in \mathbb{Z}$, we write $k(n)$ for the shift of the trivial module, which is $k[x]/(x)$ concentrated in degree $-n$. The descending filtration $k[x] \supset (x) \supset (x^2) \supset \dots$ has slices k, $k(-1)$, $\dots$, descending from the top.

We label the closed points of $\mathbb{L}^1$ by $\dots, -1, 0, 1, \dots$ where

$$\mathcal{O}_n = k(-n).$$

Our picture of $\mathbb{L}^1$ is the following:



(12-1)

The vertex labelled n denotes the closed point n. There is an arrow from n to $n+1$ because $\mathrm{Ext}^1_{\mathbb{L}^1}(\mathcal{O}_n, \mathcal{O}_m) = k$ if $m = n+1$ and is zero otherwise. The dashed line through the point n represents the shifted module $A(-n)$.

The picture of the graded line is, in some sense, a limit of the pictures (5-3) of the quivers whose path algebras are the lower triangular matrix rings. It suggests that the spaces with coordinate rings the lower triangular matrix rings should be closed subspaces of the graded line, and that $\mathsf{GrMod}k[x]$ should be like the category of modules over larger and larger triangular matrix algebras. The next two results make this precise.

Lemma 12.2 *Let T be the ring of doubly infinite lower triangular matrices over k having only a finite number of non-zero entries. It does not have an identity. Let $\mathsf{Mod}T$ denote the category of right T-modules M satisfying $M = \sum_i Me_{ii}$. Then the category $\mathsf{GrMod}k[x]$ is equivalent to $\mathsf{Mod}T$.*

Proof. This is essentially proved in Example 5.26. □

Lemma 12.3 *Fix an integer $n \geq 1$. Let $a \in \mathbb{Z}$. The full subcategory of $\mathsf{GrMod}k[x]$ consisting of the modules having support in $\{a, a+1, \dots, a+n-1\}$ is a closed affine subspace of $\mathsf{GrMod}k[x]$ with coordinate ring the ring of $n \times n$ lower triangular matrices over k.*

Proof. The argument in Example 5.31 applies. □

Proposition 12.4 Spec k *is an open subspace of the graded line. It is the open complement to the closed points.*

Proof. Let Y be the weakly closed subspace of the graded line defined by declaring $\mathsf{Mod}Y$ to be the full subcategory of $\mathsf{GrMod}k[x]$ consisting of all semisimple modules. Then $\mathsf{mod}_Y\mathbb{L}^1$ consists of all the finite dimensional $k[x]$-modules. Therefore $\mathsf{Mod}_Y\mathbb{L}^1$ consists of all modules M such that every element of M is annihilated by some power of x. Therefore

$$\mathsf{Mod}\mathbb{L}^1\backslash Y \cong \mathsf{GrMod}k[x, x^{-1}],$$

and the inclusion $\mathbb{L}^1\backslash Y \to \mathbb{L}^1$ is the affine map induced by the graded ring homomorphism $k[x] \to k[x, x^{-1}]$. Since $k[x, x^{-1}]$ has a unit in every degree it is a progenerator in $\mathsf{GrMod}k[x, x^{-1}]$. Hence $\mathsf{GrMod}k[x, x^{-1}]$ is equivalent to the category of modules over the endomorphism ring of $k[x, x^{-1}]$. This endomorphism ring is isomorphic to k, so $\mathsf{GrMod}k[x, x^{-1}]$ is equivalent to $\mathsf{Mod}k$. □

The dashed lines in (12-1) are meant to represent the generic point Spec k of the graded line.

A localization of a ring is rarely isomorphic to the original ring. Similarly, if one removes a point from a variety the resulting variety is never isomorphic to the original variety. It is therefore a surprise that the open complement to a closed point on the graded line is itself isomorphic to the graded line.

Let A denote $\mathsf{GrMod}k[x]$, and let T denote the full subcategory of A consisting of all direct sums of the trivial module $k = k(0)$. Let $\pi : \mathsf{A} \to \mathsf{A}/\mathsf{T}$ be the quotient functor, and ω its right adjoint. We denote the counit by η. Thus, for each $M \in \mathsf{A}$, there is a canonical map

$$\eta_M : M \to \omega\pi M.$$

If this map is an isomorphism we say that M is saturated and torsion-free. We denote the full subcategory of such modules by B. Since $\pi\omega \cong \mathrm{id}_{\mathsf{A}/\mathsf{T}}$, π and ω implement an equivalence between B and A/T.

Lemma 12.5 *Let M be a graded $k[x]$-module. Then M is saturated and torsion-free if and only if the action of $x : M_0 \to M_1$ is a vector space isomorphism.*

Proof. There is an exact sequence

$$0 \to \tau M \to M \to \omega\pi M \to R^1\tau M \to 0,$$

where τM is the sum of all submodules of M isomorphic to $k(0)$. Therefore η_M is injective if and only if the map $x : M_0 \to M_1$ is injective. Since $\omega\tau M$ is the largest essential extension of $M/\tau M$ with the quotient belonging to T, $R^1\tau M$ is zero if and only if $\operatorname{Ext}^1_{\mathsf{A}}(k, M) = 0$. But $\operatorname{Ext}^1_{\mathsf{A}}(k, M)$ is isomorphic to M_1/xM_0, so η_M is surjective if and only if $M_1 = xM_0$. □

Theorem 12.6 *If p is a closed point of the graded line $\mathbb{L}^1$, then $\mathbb{L}^1\backslash\{p\} \cong \mathbb{L}^1$.*

Proof. We must show that the categories A/T and A are equivalent.

It is well-known that A is equivalent to the representations of the quiver having vertices labelled by the integers and arrows $n \to n+1$. For the purposes of this proof it is helpful to adopt a different labelling of this quiver.

Let Q be the quiver with vertices $\mathbb{Z}\backslash\{0\}$, and arrows $n \to n+1$ if $n \notin \{-1, 0\}$ and $-1 \to 1$. To simplify notation, if $n \in \mathbb{Z}\backslash\{0\}$ we define

$$n \oplus 1 = \begin{cases} n+1 & \text{if } n \neq -1, \\ 1 & \text{if } n = -1. \end{cases}$$

Thus for every vertex n in Q, there is an arrow $n \to n \oplus 1$. Therefore A is equivalent to the category RepQ of representations of Q. A representation of Q will be denoted by (V, ρ) where V is a sequence of vector spaces V_n, $n \in \mathbb{Z}\backslash\{0\}$, and ρ is a sequence of linear maps $\rho_n : V_n \to V_{n\oplus 1}$, $n \in \mathbb{Z}\backslash\{0\}$.

We will prove the theorem by constructing mutual quasi-inverse functors $F : \mathsf{B} \to \mathsf{RepQ}$ and $G : \mathsf{RepQ} \to \mathsf{B}$.

If M is a graded $k[x]$-module, we define a representation (FM, ρ) of Q as follows. If $n \in \mathbb{Z}\backslash\{0\}$, we set

$$(FM)_n := M_n,$$

and

$$\rho_n = \begin{cases} \text{multiplication by } x & \text{if } n \neq -1, \\ \text{multiplication by } x^2 & \text{if } n = -1. \end{cases}$$

If $f : M \to N$ is a morphism in A we define $F(f) : FM \to FN$ to be the restriction of f to FM.

If (V, ρ) is a representation of Q, we define the $\mathbb{Z}$-graded k-vector space GV by

$$(GV)_n = \begin{cases} V_n & \text{if } n \neq 0, \\ V_1 & \text{if } n = 0, \end{cases}$$

if $n \in \mathbb{Z}$. We give GV the structure of a graded $k[x]$-module by defining the action of x on $(GV)_n$ as follows:

- if $n \notin \{-1,0\}$, then $x : (GV)_n = V_n \to (GV)_{n+1} = V_{n\oplus 1}$ is ρ_n;
- $x : (GV)_{-1} = V_{-1} \to (GV)_0 = V_1$ is ρ_{-1};
- $x : (GV)_0 = V_1 \to (GV)_1 = V_1$ is the identity.

The last of these requirements ensures that GV is torsion-free and saturated, so $GV \in \mathsf{B}$. If $\theta : (U,\mu) \to (V,\rho)$ is a morphism in RepQ, we define $G(\theta) : GU \to GV$ on each component to be the corresponding component of θ. Thus G is a functor $\mathsf{RepQ} \to \mathsf{B}$.

We define a natural transformation $\tau : \mathrm{id}_{\mathsf{B}} \to G \circ F$ by defining $\tau_M : M \to GFM$ as follows:

$$(\tau_M)_n : M_n \to (GFM)_n = \begin{cases} \mathrm{id} & \text{if } n \neq 0 \\ x : M_0 \to (GFM)_0 = M_1 & \text{if } n = 0. \end{cases}$$

To see that τ_M is a morphism in A requires checking that τ_M commutes with multiplication by x. If $n \notin \{0,-1\}$ it is clear that $x \circ (\tau_M)_n = (\tau_M)_{n+1} \circ x : M_n \to (GFM)_{n+1}$. The only two interesting cases, $n = -1$ and $n = 0$, are checked by verifying the commutativity of the diagrams

$$\begin{array}{ccc} M_{-1} & \xrightarrow{(\tau_M)_{-1}=\mathrm{id}} & (GFM)_{-1} = M_{-1} \\ {\scriptstyle x}\big\downarrow & & \big\downarrow{\scriptstyle x^2} \\ M_0 & \xrightarrow{(\tau_M)_0 = x} & (GFM)_0 = M_1 \end{array}$$

and

$$\begin{array}{ccc} M_0 & \xrightarrow{(\tau_M)_0=x} & (GFM)_0 = M_1 \\ {\scriptstyle x}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ M_1 & \xrightarrow{(\tau_M)_1 = \mathrm{id}} & (GFM)_1 = M_1. \end{array}$$

Thus τ_M is a morphism in A. If $f : M \to N$ is a graded $k[x]$-module homomorphism, then the diagrams

$$\begin{array}{ccc} M_n & \xrightarrow{(\tau_M)_n} & (GFM)_n \\ {\scriptstyle f_n}\big\downarrow & & \big\downarrow{\scriptstyle (GFf)_n} \\ N_n & \xrightarrow{(\tau_N)_n} & (GFN)_n \end{array}$$

commute for all n, so τ is a natural transformation. It is clear that τ_M is bijective (because M is saturated $(\tau_M)_0$ is bijective), so τ is a natural isomorphism. $\square$

Corollary 12.7 *If $p_1, \dots, p_n$ are closed points of the graded line $\mathbb{L}^1$, then $\mathbb{L}^1 \backslash \{p_1, \dots, p_n\} \cong \mathbb{L}^1$.*

Proof. This is proved by induction using one simple at a time and appealing to the theorem. □

The local ring $\mathcal{O}_{X,p}$ at a point p on a reduced scheme X, or rather its module category, may be obtained as the quotient category of $\mathsf{Mod}X$ modulo the localizing subcategory consisting of all direct limits of those finite length X-modules that do not have $\mathcal{O}_p$ as a composition factor.

We can carry out the analogous construction for the graded line. First we fix a closed point $q \in \mathbb{L}^1$. Without loss of generality we can and will assume that $\mathcal{O}_q = k[x]/(x)$.

Proposition 12.8 *Let W be the weakly closed subspace of $\mathbb{L}^1$ such that $\mathsf{mod}W$ consists of the finite length $\mathbb{L}^1$-modules that do not have $\mathcal{O}_q$ as a composition factor. Let T be the ring of lower triangular 2×2 matrices over k. Then there is an open immersion $j : \mathrm{Sp}(T) \to \mathbb{L}^1$ sending $\mathrm{Sp}(T)$ isomorphically to $\mathbb{L}^1 \backslash W$. Furthermore, j is an affine map, and $j^*\mathcal{O}_q \cong ??$ and*

Proof. Write $A = k[x]$. Let

$$\pi : \mathsf{Mod}\mathbb{L}^1 \to \mathsf{Mod}\mathbb{L}^1/\mathsf{Mod}_W\mathbb{L}^1 = \mathsf{Mod}\mathbb{L}^1\backslash W$$

be the quotient functor. We will show that $\mathcal{P} := \pi A \oplus \pi A(-1)$ is a progenerator in $\mathsf{Mod}\mathbb{L}^1\backslash W$ and that its endomorphism ring is isomorphic to T. Then, viewing $\mathcal{P}$ as a T-($\mathbb{L}^1\backslash W$)-bimodule, it induces a map $\mathbb{L}^1\backslash W \to \mathrm{Sp}(T)$ which is an isomorphism.

Since $\{A(n) \mid n \in \mathbb{Z}\}$ is a set of generators for $\mathsf{Mod}\mathbb{L}^1$, their images under π form a set of generators for $\mathsf{Mod}\mathbb{L}^1\backslash W$ (see the proof of Theorem 2.16.3).

The $\mathbb{L}^1$-module $E := k[x, x^{-1}]$ is uniserial, its only proper submodules being $\{x^{-n}k[x] \cong A(n) \mid n \in \mathbb{Z}\}$. It follows immediately that

$$\omega\pi A(n) \cong \begin{cases} k[x, x^{-1}] & \text{if } n \geq 0, \\ x^{-n}k[x] & \text{if } n < 0. \end{cases}$$

Since $\pi\omega \cong \mathrm{id}_{\mathbb{L}^1\backslash W}$,

$$\{\pi A(n) \mid n \in \mathbb{Z}\} = \{\pi A, \pi A(-1)\},$$

so $\mathcal{P}$ is a generator for $\mathsf{Mod}\mathbb{L}^1\backslash W$.

Since $\mathrm{gldim}\,\mathbb{L}^1 = 1$, the open immersion $\mathbb{L}^1\backslash W \to \mathbb{L}^1$ is an affine map and ω is an exact functor (Proposition 7.9). Hence π preserves projectives. In particular, $\mathcal{P}$ is a projective $\mathbb{L}^1\backslash W$-module. Since it has a right adjoint, ω commutes with direct sums. It follows from this that $\mathrm{Hom}_{\mathbb{L}^1\backslash W}(\mathcal{P}, -)$ commutes with direct sums, so we conclude that $\mathcal{P}$ is a progenerator in $\mathsf{Mod}\mathbb{L}^1\backslash W$, and

$$\mathsf{Mod}\mathbb{L}^1\backslash W \cong \mathsf{Mod}(\mathrm{End}_{\mathbb{L}^1\backslash W}\,\mathcal{P}).$$

It is clear that

$$\mathrm{End}_{\mathbb{L}^1\backslash W}\,\mathcal{P} \cong \begin{pmatrix} \mathrm{Hom}(\pi A(-1), \pi A(-1)) & \mathrm{Hom}(\pi A, \pi A(-1)) \\ \mathrm{Hom}(\pi A(-1), \pi A)) & \mathrm{Hom}(\pi A, \pi A)) \end{pmatrix}$$

It is now an elementary task to check that this ring is isomorphic to T.

By applying $\omega\pi$ to the short exact sequence

$$0 \to A(-1) \to A \to \mathcal{O}_q \to 0,$$

one obtains an exact sequence

$$0 \to \mathcal{O}_p \to \pi A \to \pi\mathcal{O}_q \to 0.$$

It is clear that $\pi A(-1)$ is a simple $\mathbb{L}^1\backslash W$-module, and that $\pi A(n)$ has length two for $n \geq 0$. □

3.13 Projective modules and open subspaces

Proposition 13.1 *Consider an affine space $X = \mathsf{Mod}R$. Let P_R be a finitely generated projective R-module, and let T be its trace ideal in R. Let $Z = \mathcal{Z}(T)$ be the closed subspace of X where T vanishes. Then the open complement to Z in X is isomorphic to $\mathsf{Mod}S$, where S is the endomorphism ring of the left R-module $P^* = \operatorname{Hom}_R(P, R)$.*

Proof. Since P^* is an R-S-bimodule it induces a map $j : \mathsf{Mod}S \to \mathsf{Mod}R$ with $j_* = \operatorname{Hom}_S(P^*, -)$ and $j_* = - \otimes_R P^* \cong \operatorname{Hom}_R(P, -)$. The adjunction $\eta : j^*j_* \operatorname{id}_S$ is a natural equivalence because if N is an S-module, then

$$\begin{aligned} j^*j_* &= \operatorname{Hom}_R(P, \operatorname{Hom}_S(P^*, -)) \\ &\cong \operatorname{Hom}_S(P \otimes_R P^*, -) \\ &\cong \operatorname{Hom}(S, -) \quad \text{by Proposition 2.10.7.} \end{aligned}$$

By Proposition 7.4, j^* vanishes exactly on $\mathsf{Mod}Z$. The result now follows from Theorem 7.2. □

Proposition 13.2 *Let $f : Y \to X$ be a map between the affine spaces $Y = \mathsf{Mod}S$ and $X = \mathsf{Mod}R$. Suppose that the corresponding bimodule $B = f^*(R)$ is finitely generated and projective as an R-module. Let T be the trace ideal in R of $B^\vee = \operatorname{Hom}_R(B, R)$, and let $Z = \mathcal{Z}(T)$ be the closed subspace of X where T vanishes. If p is a closed point of X, then $f^*\mathcal{O}_p = 0$ if and only if $p \in Z$.*

Proof. We will prove more. We will show that an X-module M satisfies $f^*M = 0$ if and only if M is a Z-module.

By definition, T is the image of the natural map $B \otimes_S B^\vee \to R$ of R-R-bimodules. Hence there is a commutative diagram

$$\begin{tikzcd} B \otimes_S B^\vee \arrow[d, "\alpha"'] & \\ T \arrow[r] & R \end{tikzcd}$$

in which α is surjective. This gives a commutative diagram

$$\begin{array}{ccc} M \otimes_R B \otimes_S B^\vee & & \\ {\scriptstyle M\otimes\alpha}\big\downarrow & & \\ M \otimes_R T & \longrightarrow & M \otimes_R R \cong M \end{array}$$

in which $M \otimes \alpha$ is surjective. The image of the bottom map is MT. Since M is a Z-module if and only if $MT = 0$, M is a Z-module if and only if the map $M \otimes B \otimes B^\vee \to M$ is zero.

But $f^*M \cong M \otimes_R B$, so M is a Z-module if $f^*M = 0$.

Conversely, suppose that M is a Z-module. Then the map $M \otimes B \otimes B^\vee \to M$ is zero, and hence the induced map

$$M \otimes B \otimes B^\vee \otimes_R B \to M \otimes_R B \tag{13-1}$$

is also zero. By hypothesis, ${}_RB$ is finitely generated projective, so we may choose dual bases $b_\lambda \in B$ and $\beta_\lambda \in B^\vee$. The effect of the map (13-1) is

$$\sum_\lambda m \otimes b \otimes \beta_\lambda \otimes b_\lambda \mapsto \sum_\lambda m \otimes \beta_\lambda(b) b_\lambda = m \otimes b.$$

It follows that $M \otimes_R B = 0$. Thus $f^*M = 0$. □

Lemma 13.3 *Let B be a finitely generated projective left R-module. Set $S = \operatorname{End}_R B$. Let $Y = \mathsf{Mod}S$ and $X = \mathsf{Mod}R$ be the associated affine spaces, and let $f : Y \to X$ be the map corresponding to the R-S-bimodule B. Then the natural transformation $f^*f_* \to \mathrm{id}_Y$ is a natural equivalence.*

Proof. Let $B^\vee = \operatorname{Hom}_R(B, R)$ have its natural S-R-bimodule structure. By Proposition 2.10.7, $B^\vee \otimes_R B$ is isomorphic to S as an S-S-bimodule. Therefore

$$\begin{aligned} f^*f_* &= \operatorname{Hom}_S(B, -) \otimes_R B \\ &\cong \operatorname{Hom}_R(B^\vee, \operatorname{Hom}_S(B, -)) \\ &\cong \operatorname{Hom}_S(B^\vee \otimes_R B, -) \\ &\cong \operatorname{Hom}_S(S, -). \end{aligned}$$

□

Lemma 13.4 *Let $X = \mathsf{Mod}R$ be an affine space. Let e be an idempotent in R, and define $Y = \mathsf{Mod}(eRe)$. Then Y is isomorphic to the open complement in X of the closed subspace $\mathcal{Z}(e)$.*

Proof. Write $S = eRe$. Let $f : Y \to X$ be the map corresponding to the R-S-bimodule Re. This bimodule is a finitely generated projective R-module, so f^* is exact and there is a natural equivalence $f^*f_* \to \mathrm{id}_Y$. Hence, the result will follow from Theorem 7.2 once we show that f_* embeds $\mathsf{Mod}S$ as a full subcategory of $\mathsf{Mod}R$. But

$$f_* = \mathrm{Hom}_S(Re, -) = \mathrm{Hom}_S((1-e)Re, -) \oplus \mathrm{Hom}_S(eRe, -),$$

and this is full and faithful because its summand $\mathrm{Hom}_S(eRe, -)$ is the identity functor. $\square$

We wish to apply this to a space having a skew group ring as its coordinate ring.

Definition 13.5 Let A be a ring on which a group G acts as automorphisms. We write the action multiplicatively: the effect of $g \in G$ acting on $a \in A$ is denoted by a^g.

The skew group ring $A * G$ is the free left A-module with basis the elements of G, and multiplication rule $ag.bh = ab^g gh$ for $g, h \in G$ and $a, b \in A$. A typical element of $A * G$ may be written uniquely as $\sum_{g \in G} r_g g$, where $r_g \in A$. $\lozenge$

The elements of G also provide a basis for $A * G$ as a right A-module.

Example 13.6 Let k be a field with a primitive n^{th} root of unity, ξ. Let G be the cyclic group of order n, with generator σ. Define an action of G as automorphisms of the polynomial ring $A = k[u]$ by declaring $u^\sigma = \xi u$. Let $R = A * G$. Then the category $\mathsf{Mod}A * G$ is equivalent to the category of representations of the quiver

$$Q = \begin{tikzcd} \underset{n}{\bullet} \arrow[r] & \underset{n-1}{\bullet} \arrow[r] & \bullet & \cdots & \bullet \arrow[r] & \underset{2}{\bullet} \arrow[r] & \underset{1}{\bullet} \arrow[llllll, bend right] \end{tikzcd}$$

To see this, let M be an $A * G$-module. Since k has n distinct n^{th} roots of unity, M decomposes as a direct sum of its σ-eigenspaces, say $M = M_0 \oplus M_1 \oplus \ldots \oplus M_{n-1}$ where $M_i = \{m \in M \mid m.\sigma = \xi^i m\}$. It is easy to see that $M_i.u \subset M_{i+1}$, so we assign to M the representation of the quiver that assigns the vector space M_i to the i^{th} vertex, and assigns to the arrows the action of u on each component. Conversely, a representation of the quiver can be made into a $k[u]$-module in the obvious way. There are some details to check, but the idea is clear. $\lozenge$

When a group G acts as automorphisms of a ring A, the subring of invariants is

$$A^G := \{a \in A \mid a^g = a\}.$$

Lemma 13.7 *Let A be a ring, and G a finite group acting as automorphisms of A. Write $R = A * G$. Suppose that $|G|$ is a unit in A. Define*

$$e = \frac{1}{|G|} \sum_{g \in G} g. \tag{13-2}$$

This is an idempotent in R, and there is a ring isomorphism

$$eRe \cong A^G.$$

Proof. Write $n = |G|$. Since $eg = e$ for all $g \in G$, $Re = Ae$. Similarly, $eR = eA$. Hence $eRe = eAe$. For $a \in A$, we define

$$\bar{a} = \frac{1}{|G|} \sum_{g \in G} a^g.$$

It is clear that $\bar{a} \in A^G$, and that every element in A^G is of this form. We will use the fact that $eae = e\bar{a} = \bar{a}e$ for all $a \in A$.

Define $\varphi : eAe \to A^G$ by $\varphi(eae) = \bar{a}$. If $\bar{a} = 0$, then eae is zero because it equals $e\bar{a}$. Hence φ is injective. It is surjective by an earlier remark, so we only need to check that $\varphi(eae.ebe)$ equals $\bar{a}\bar{b}$. We have

$$\varphi(eae.ebe) = \varphi(e\bar{a}be) = \frac{1}{|G|} \sum_{g \in G} (\bar{a}b)^g = \frac{1}{|G|} \sum_{g \in G} \bar{a}b^g = \bar{a}\bar{b}$$

as required. $\square$

We are now in a position to apply Lemma 13.4.

Example 13.8 Let A be a ring, and G a finite group acting as automorphisms of A. Write $R = A * G$. Suppose that $|G|$ is a unit in A, and let e be the idempotent in (13-2). Lemma 13.4 gives a map of spaces

$$j : \mathsf{Mod}A^G \to \mathsf{Mod}A * G \tag{13-3}$$

which is an isomorphism outside $\mathcal{Z}(e)$. If R is a simple ring, then $ReR = R$, so $\mathcal{Z}(e) = \phi$, and the two spaces are isomorphic.

Now reconsider Example 13.6. The n one-dimensional representations of Q give n one-dimensional modules over $A * G$. If we denote by V_i the one-dimensional representation on which σ acts via multiplication by ξ^i, then $V_i.e = 0$ for $i \neq 0$, and e acts on V_0 as the identity. Therefore $\mathcal{Z}(e)$ contains the points corresponding to the simples $V_1, \dots, V_{n-1}$. If S is one of the n-dimensional simples, then $Se \neq 0$. Now $A^G = k[u^n]$, so the map j in (13-3) embeds the affine line $\mathbb{A}^1$ as the open complement to the closed subspace consisting of the $n-1$ points corresponding to $V_1, \dots, V_{n-1}$. That is, we have a map

$$f : \mathbb{A}^1 \to \mathsf{Mod}kQ.$$

By symmetry, if Z is the closed subspace consisting of any $n-1$ of the V_i's, then the open complement to V_i is isomorphic to $\mathbb{A}^1$. More precisely, if σ_i is the rotation of Q sending the vertex labelled j to that labelled $j+i$, then there is an induced algebra automorphism $\sigma_i : kQ \to kQ$, and hence an automorphism of the space $\sigma_i : \mathsf{Mod}kQ \to \mathsf{Mod}kQ$. Thus each $\sigma_i \circ f$ gives a map from $\mathbb{A}^1$ to $\mathsf{Mod}kQ$. $\lozenge$

EXERCISES

13.1 Jazz up Example 11.4, and describe the geometry of the affine space with coordinate ring equal to the ring of $n \times n$ lower triangular matrices.

13.2 If $A = k[x]$ and $B = k[x^2]$ show that the inclusion

$$\begin{pmatrix} B & xB \\ xB & B \end{pmatrix} \to M_2(A)$$

is an epimorphism in the category of rings.

13.3 Discuss the epimorphism in the previous exercise in terms of the geometry of the spaces.

13.4 Adopting the notation in Example 13.6, show that there is an isomorphism $A * G \cong kQ$.

13.5 $\boxed{\text{Paul}}$ Give an example to show that there are problems with torsion in the non-noetherian case even for a commutative ring. Take R commutative, and I a non-noetherian ideal. Say that $m \in M$ is I-torsion, if $ma = 0$ for some $a \in I$

3.14 Pictures of non-commutative spaces

We want to draw pictures of non-commutative spaces. The picture is intended as a heuristic device which encodes some of the structure of the module category. It is valuable to the extent that it suggests, or reflects, the existence of maps between different spaces. The examples in section 3.6 and in this section show that the pictures carry information about such maps.

The pictures need not be drawn according to a strict set of rules, but there should be some conventions we all agree on. The pictures that are drawn when talking about an algebraic variety usually contain only the data that is relevant to the discussion at hand. When discussing a surface, we draw a picture of a surface defined over $\mathbb{R}$, and we might draw some singular points, perhaps one or two curves illustrating their intersection points, and so on.

Convention 1. The conventions for drawing non-commutative spaces should be compatible with the unwritten rules that are followed in the commutative case.

For example, many non-commutative spaces contain subspaces that are commutative, and the pictures should make this apparent.

Convention 2. The points we draw in the picture correspond to simple modules.

This convention is compatible with what is done in the commutative case (cf. the Nullstellensatz, and section 3.4). We do not always draw all the points. If we did, then for a commutative surface, the picture would be completely black, and we could not draw any special curves. We draw some points and leave the reader to fill in the blanks. The white space in the picture is as important as the parts that are filled in.

For finite spaces it is reasonable to draw all the points.

Convention 3. If kQ is a finite dimensional path algebra, then Q itself is a good picture of the non-commutative space $\mathsf{Mod}kQ$.

The pictures of the quivers that were drawn in section 3.5 contain enough information to reconstruct the category $\mathsf{Mod}kQ$ (cf. Theorem 5.9). The vertices of Q are in bijection with the simple kQ-modules, and, by Proposition 5.21, the arrows allow one to recover the extension groups between the simples.

The data contained in the arrows is *non-commutative.* If X if a commutative scheme and p and q are distinct closed points, then $\mathrm{Ext}^1_X(\mathcal{O}_p, \mathcal{O}_q) = 0$. This is false in the non-commutative case.

Convention 4. If p and q are distinct points such that $\mathrm{Ext}^1_X(\mathcal{O}_p, \mathcal{O}_q) \neq 0$, then we draw an arrow $p \to q$.

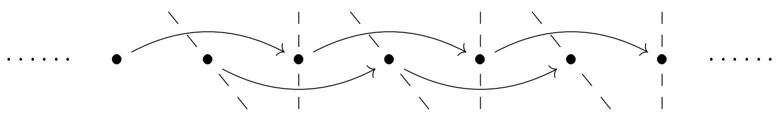
For example, the picture of the space having coordinate ring the $n \times n$ lower triangular matrices is the quiver

$$\overset{1}{\bullet} \longleftarrow \overset{2}{\bullet} \longleftarrow \bullet \cdots \bullet \longleftarrow \overset{n-1}{\bullet} \longleftarrow \overset{n}{\bullet} \tag{14-1}$$

For infinite dimensional path algebras the quiver is not the appropriate picture of the space. For example, the path algebra of the quiver with one vertex and one arrow is the polynomial ring $k[x]$, and a better picture is that which is usually drawn for the affine line, namely

The points on the line represent the simple modules or closed points but, the picture carries no information about the base field. We draw the same picture whether the field is $\mathbb{R}$, or $\mathbb{Q}$, or $\mathbb{C}$, or $\mathbb{C}(t)$, or $\mathbb{F}_{p^n}$.

Example 14.1 A variation on the graded line is given by the space $\mathsf{GrMod}k[y]$ with $\deg y = n > 1$. There is a non-split extension $0 \to k(-n) \to k[y]/(y^2) \to k \to 0$, and it follows easily that the picture of $\mathsf{GrMod}k[y]$ is similar to (12-1) except that each arrow now goes n steps to the right. For example, if $n = 2$, the picture is



For simplicity we only discuss the case $n = 2$, but everything generalizes to arbitrary n in a straightforward way. The fact that there are no arrows from the even vertices to the odd vertices, and vice versa, suggests that the space should decompose as two copies of the graded line. This is indeed the case. Each module decomposes into two submodules, namely its even degree and odd degree components, and there are no non-zero maps between modules concentrated in even degrees and modules concentrated in odd degrees. More formally, $\mathsf{GrMod}k[y]$ is equivalent to the category of graded modules over

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \otimes k[x]$$

where $\deg x = 1$, and $\deg e_{11} = \deg e_{22} = 0$. $\Diamond$

If Q' is a subquiver of Q, then there is an obvious sense in which the picture of Q' is a "sub-picture" of the picture of Q. More generally, one might expect that the picture of a weakly closed subspace should be a "sub-picture" of the picture of its ambient space. Corollary 14.3 below says the arrows in the picture of a weakly closed subspace remain in the picture of its ambient space.

Proposition 14.2 *Let $i : Y \to X$ be the inclusion of a weakly closed subspace. Let N be a Y-module, and M an X-module. There is a Grothendieck spectral sequence*

$$E_2^{pq} = \operatorname{Ext}^p_Y(N, R^q i^! M) \Rightarrow \operatorname{Ext}^n_X(i_*N, M). \tag{14-2}$$

Proof. Let $F = \operatorname{Hom}_Y(N, -)$. A right adjoint to an exact functor preserves injectives so, if E is injective in $\mathsf{Mod}X$, $i^!E$ is injective in $\mathsf{Mod}Y$. Thus $i^!$ is right acyclic for F. Hence there is a third quadrant Grothendieck spectral sequence

$$(R^pF)(R^q i^!)(M) \Rightarrow R^n(F \circ i^!)(M).$$

But $F \circ i^! = \operatorname{Hom}_Y(N, -) \circ i^! \cong \operatorname{Hom}_X(i_*N, -)$, thus giving the result. $\square$

The five term exact sequence arising from this spectral sequence is

$$\begin{aligned} 0 \to \operatorname{Ext}^1_Y(N, i^!M) &\to \operatorname{Ext}^1_X(i_*N, M) \to \operatorname{Hom}_Y(N, R^1 i^! M) \to \\ &\to \operatorname{Ext}^2_Y(N, i^!M) \to \operatorname{Ext}^2_X(i_*N, M) \end{aligned} \tag{14-3}$$

Corollary 14.3 *Let Y be a weakly closed subspace of X. Suppose that p and q are closed points of X that lie on Y. If $\operatorname{Ext}^1_Y(\mathcal{O}_p, \mathcal{O}_q) \neq 0$, then $\operatorname{Ext}^1_X(\mathcal{O}_p, \mathcal{O}_q) \neq 0$.*

Proof. By hypothesis, $i^!\mathcal{O}_p = \mathcal{O}_p$ and $i^!\mathcal{O}_q = \mathcal{O}_q$, so this follows immediately from the five term exact sequence. $\square$

Hence, if Y is weakly closed in X, and p and q are closed points of X that lie on Y, if we draw an arrow $p \to q$ to indicate that $\operatorname{Ext}^1_Y(\mathcal{O}_p, \mathcal{O}_q) \neq 0$, then that arrow remains in the picture for X.

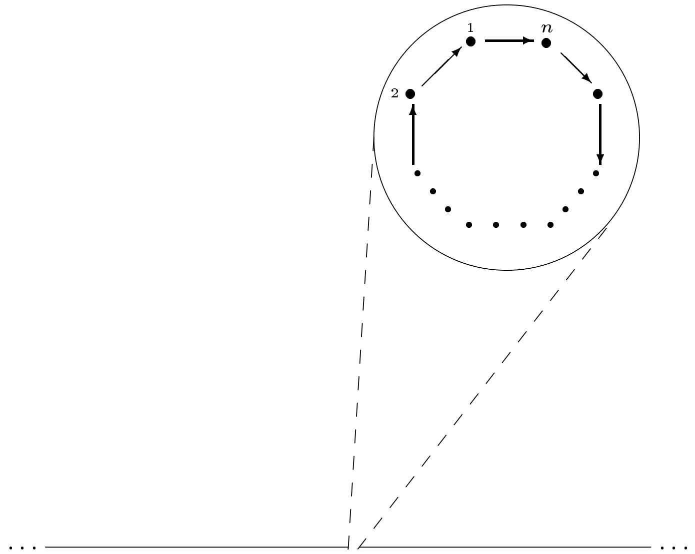
The next example, an affine space of the form $\mathsf{Mod}S$, is a typical non-commutative affine curve. The curve has a dense open subset that is a commutative curve. This is because the localization $S[x^{-1}]$ is a matrix algebra over a commutative ring, so its module category is equivalent to that of its center.

Because S is a finite module over its center, say Z, the inclusion of the center induces a finite map from the non-commutative curve to the commutative curve $\operatorname{Spec} Z$, and that map is an isomorphism on a dense open set. In Chapter **??** we will see that this behavior is typical for non-commutative affine curves.

Example 14.4 (The broken line) Let $n \geq 2$ be an integer. Let $X = \mathsf{Mod}S$ where S is the path algebra of the quiver

$$\begin{tikzcd} \underset{n}{\bullet} \arrow[r] & \underset{n-1}{\bullet} \arrow[r] & \bullet & \cdots & \bullet \arrow[r] & \underset{2}{\bullet} \arrow[r] & \underset{1}{\bullet} \arrow[llllll, bend right] \end{tikzcd} \tag{14-4}$$

The representation theory of S is described in Proposition 5.23. There are n one-dimensional simple modules, and there is a family of n-dimensional simples parametrized by $\mathbb{A}^1 \backslash \{0\}$. Our picture of X is as follows.



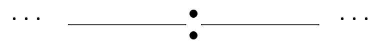
(14-5)

The picture is an affine line with the origin replaced by a cycle of n points arranged as in the magnified circle. For this reason we call X a broken line. The points on $\mathbb{A}^1 \backslash \{0\}$ are in bijection with the n-dimensional simples, and the n points that replace the origin are the n one-dimensional simples that correspond to the irreducible representations of the quiver gotten by placing k at one vertex and zeroes at the other vertices.

The arrows in the quiver give one-dimensional Ext-groups between the one-dimensional simples, and hence give the arrows in the picture (14-5). The fact that there are no other arrows follows from a consideration of the central elements that annihilate the simples. The center of S is isomorphic to $k[x]$, and a one-dimensional simple has central annihilator (x) whereas an n-dimensional simple has central annihilator $(x - \lambda)$ for some non-zero $\lambda \in k$. It therefore follows from the remarks after Lemma 5.19 that any extension between a one-dimensional and an n-dimensional simple splits, and also that extensions between non-isomorphic n-dimensional simples split.

The inclusion $k[x] \to S$ of the center of the algebra gives a map $f : X \to \mathbb{A}^1$. As explained in Example 5.32, the map f collapses the n special points of X to the origin, and sends the other points bijectively to $\mathbb{A}^1 \backslash \{0\}$. $\Diamond$

The picture of the space X in the previous example in the case $n = 2$ is similar to the picture of the affine line with a double origin. The affine line with a double origin, which we denote by Z, is the standard example of a non-separated scheme. It is usually drawn as



The difference is that in X the two origins are linked by arrows. The relation between Z and X is made precise in Example **??**.**??** where we construct a map $f : X \to Z$. There is a sheaf of rings $\mathcal{R}$ on Z such that $H^0(Z, \mathcal{R})$ is isomorphic to S, the coordinate ring of X.

Example 14.5 (Example 14.4 continued). The ring S is not a maximal order. One maximal order properly containing it is $M_n(k[x])$. Conjugating this by the elements u^i, $1 \le i \le n$ gives n distinct maximal orders $T_1, \dots, T_n$ containing S. Each T_i is isomorphic to $M_n(k[x])$. Now let Z denote the affine line with n origins. Let $U_1, \dots, U_n$ be the n different affine lines covering Z and construct a sheaf of $\mathcal{O}_Z$-algebras $\mathcal{R}$ by defining $\mathcal{R}(U_i) = T_i$, and glue these in the obvious way using the inclusions of all T_i in $M_n(k(x))$. Then $H^0(Z, \mathcal{R}) = T_1 \cap \dots \cap T_n = S$. The inclusion $S \to T_i$ induces a map of non-commutative spaces $f_i : \mathbb{A}^1 \to \mathsf{Mod}S$. Since S is hereditary, each T_i is a projective S-module, so each f_i^* is an exact functor. It follows that f_i embeds $\mathbb{A}^1$ as an open subspace of $\mathsf{Mod}S$.

Let P_i be the i^{th} row of S. This is an indecomposable projective right S-module, and it maps onto the one-dimensional simple labelled V_i in Proposition 5.23. It is also a $k[x]$-S-bimodule, so induces a map $\mathbb{A}^1 \to \mathsf{Mod}S$. This is probably the map f_i. The direct image functor for this map is $N \mapsto N \otimes_R P_i$. This sends $k[x]$ to P_i, and sends the exact sequence

$$0 \longrightarrow k[x] \xrightarrow[x]{} k[x] \longrightarrow k[x]/(x) \longrightarrow 0$$

to

$$0 \longrightarrow P_i \xrightarrow[x]{} P_i \longrightarrow P_i/xP_i \longrightarrow 0$$

$\boxed{\text{Paul}}$ Let Q_i be the subquiver of Q gotten by removing the vertex labelled i. Then $\mathsf{Mod}Q_i$, which is the affine space with coordinate ring the $n \times n$ lower triangular matrices, is a closed subspace of X. The points of $\mathsf{Mod}Q_i$ are the special points except the one at the vertex labelled i. Show that the open complement of this is the "image" of f_i. $\Diamond$

Next is an example of an affine space that is presented in a way such that its affinity is not immediately obvious. One way to show it is affine is to exhibit a progenerator.

Example 14.6 Consider the ring $A = k[t]$ endowed with the $\mathbb{Z}_n$-grading in which $\deg t = 1$. Proposition 5.24 shows that the space $\mathsf{GrMod}k[t]$ is isomorphic to the space $\mathsf{Mod}kQ$ where Q is the quiver (14-4) in Example 14.4. It is instructive to see how drawing a picture of $\mathsf{GrMod}k[t]$ suggests this isomorphism.

There is a family of n-dimensional graded simple $k[t]$-modules parametrized by $\mathbb{A}^1 \backslash \{0\}$, namely

$$S_\lambda = A/(t^n - \lambda), \qquad 0 \neq \lambda \in k.$$

Multiplication by t gives an isomorphism $S_\lambda \to S_\lambda(1)$. There are n one-dimensional simple modules, namely $k = A/(t)$, and its shifts $k(i)$ for $i = 1, \ldots, n-1 \in \mathbb{Z}_n$. The non-split extension $0 \to k(-1) \to A/(t^2) \to k \to 0$, and its shifts, show that $\operatorname{Ext}^1_{\mathsf{Gr}}(k(i), k(i-1)) \neq 0$. It is easy to see that there are no non-split extensions between two non-isomorphic n-dimensional simples. Therefore the picture of $\mathsf{GrMod}k[t]$ looks just like that for kQ (14-5).

An alternative way of showing that $\mathsf{GrMod}k[t]$ is isomorphic to $\mathsf{Mod}Q$ is to observe that $P = A \oplus A(1) \oplus \ldots \oplus A(n-1)$ is a progenerator in $\mathsf{GrMod}A$. Then, by Theorem 2.9.6, $\mathsf{GrMod}k[t]$ is equivalent to the category of modules over the endomorphism ring of P.

The endomorphism ring of P can be computed by thinking of P as a row module with its endomorphism ring acting by right matrix multiplication. Since $\operatorname{Hom}_{\mathsf{Gr}}(A(i), A(j)) = A_{j-i}$, the endomorphism ring is the $n \times n$ matrix ring with A_{j-i} in the ij-position. If one writes this out explicitly, one sees that one gets a ring that doesn't look like the matrix ring in Proposition 5.23; it has a more symmetric appearance. Nevertheless, it is not difficult to check that there is an isomorphism (cf., Example 9.1).

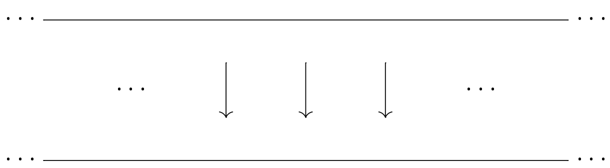
It is also instructive to examine this broken line from the graded module point of view. The map from it to $\mathbb{A}^1$ comes from the inclusion $k[t^n] \to A$. The functor from $\mathsf{GrMod}A$ to $\mathsf{Mod}k[t^n]$ given by taking the degree zero part of a graded module has a left adjoint $- \otimes_{k[t^n]} A$, and a right adjoint $\operatorname{Hom}_{k[t^n]}(A, -)$, where the grading on A is used to induce a grading on the module. For example, the degree i component of $N \otimes_{k[t]} A$ is $N \otimes A_i$, and the degree i component of $\operatorname{Hom}_{k[t^n]}(A, N)$ is $\operatorname{Hom}_{k[t^n]}(A_{-i}, N)$.

The fiber of this map over zero is the closed subspace $\mathsf{GrMod}k[t]/(t^2)$ of $\mathsf{GrMod}k[t]$, and the open complement of this is $\mathsf{GrMod}k[t, t^{-1}]$, which is easily seen to be isomorphic to $\mathbb{A}^1 \backslash \{0\}$ because $k[t, t^{-1}]$ is a progenerator with endomorphism ring its degree zero component $k[t^n, t^{-n}]$. $\Diamond$

Example 14.7 Let $Y = \mathsf{Mod}R$, where

$$R = \begin{pmatrix} k[x] & 0 \\ k[x] & k[x] \end{pmatrix}.$$

is the tensor product of the lower triangular 2×2 matrices over k with $k[x]$. In commutative algebraic geometry the tensor product of the coordinate rings of two affine schemes is the coordinate ring of the product of the schemes. Since our picture of the lower triangular matrices over k is $\bullet \to \bullet$, this suggests that our picture of R should be



This is a picture of two copies of the affine line, with an arrow from a point p in the top line to the point p in the bottom line. This is reasonable since R has two families of simple modules, each parametrized by $\mathbb{A}^1$, and there are non-split extensions in the obvious way. Indeed, each vertical slice of the picture can be thought of as $\mathsf{Mod}R/(x-\lambda)$, $\lambda \in k$, which is isomorphic to the 2×2 lower triangular matrices. Alternatively, the inclusion of the center $k[x] \to R$ induces a map of affine schemes $f : Y \to \mathbb{A}^1$ with the fiber above λ being $\mathsf{Mod}R/(x-\lambda)$. $\lozenge$

Pictures of commutative schemes do not usually include any data related to non-closed points. The pictures in Mumford's red book [165, pages 102-103], and those in Eisenbud and Harris' book [78, pages 40-41], show the mess that results from drawing too many open points.

However, for non-commutative spaces it can be helpful to include a representation of non-closed points in a picture of the space. For example, the dashed lines in (12-1) represent the generic point of the graded line.

In the commutative world it is common practice to omit data relating to nilpotents in the structure sheaf. The pictures on [78, pages 52-53] are standard depictions of schemes with nilpotents. For example, the standard picture of $\operatorname{Spec} k[x]/(x^2)$ is

$$\bullet \to \tag{14-6}$$

But it soon becomes difficult to distinguish such pictures of $\operatorname{Spec} k[x]/(x^n)$ for various n.

Nilpotent ideals can play a different role in non-commutative rings, and sometimes information about them can be encoded in the picture. For example, the nilpotent radical of the ring of 2×2 lower triangular matrices links together the two points in the picture

$$\overset{2}{\bullet} \longrightarrow \overset{1}{\bullet} \tag{14-7}$$

Nevertheless, as the next example illustrates, our pictures do not contain much information about the nilpotent ideals.

Example 14.8 Let T be the ring of $n \times n$ lower triangular matrices over k. Let I be an ideal contained in the square of the radical. Then T/I still has n simple modules $S_1, \dots, S_n$, and $\operatorname{Ext}^1_T(S_i, S_{i+1}) \neq 0$ for $i = 1, \dots, n-1$, so the picture we draw for T/I is the same as that for T. $\Diamond$

The limitations of the pictures. Although these pictures are useful, we should be aware of their limitations. For example, if R is a finite dimensional local k-algebra, the picture of $\mathsf{Mod}R$ consists of a single point, and an arrow from it to itself if $R \neq k$. Even when R is commutative this data gives no insight into the potentially complicated structure of $\mathsf{Mod}R$. A finer study of $\mathsf{Mod}R$ must rely on identifying "important" families of modules. Such families might sometimes be viewed as non-commutative spaces.

Ubiquity. Several of the examples which have appeared so far turn up in a wide range of situations. For example, any non-commutative space having an arrow between two points has a closed subspace isomorphic to $\mathsf{Mod}T_2$. If Q' is any quiver having an infinite dimensional path algebra, then Q' has a subquiver of the form

$$\begin{tikzcd} \underset{n}{\bullet} \arrow[r] & \underset{n-1}{\bullet} \arrow[r] & \bullet & \cdots & \bullet \arrow[r] & \underset{2}{\bullet} \arrow[r] & \underset{1}{\bullet} \arrow[llllll, bend right] \end{tikzcd}$$

and hence a closed subspace that looks like (14-5).

There is a theory of blowing up for non-commutative surfaces [258]. The fiber over the point blown up need not be isomorphic to $\mathbb{P}^1$, but the only other possibilities are the graded line (12-1) and the projective completion of the broken line (14-5). The latter is discussed in detail in Example 9.1.

EXERCISES

14.1 Let A be a ring graded by a finite group G. Show that $P = \oplus_{g \in G} A(g)$ is a progenerator in $\mathsf{GrMod}A$, and hence that $\mathsf{GrMod}A$ is an affine space.

14.2 Consider Example 14.6. Show directly that the endomorphism ring of the module P in that example is isomorphic to kQ.

14.3 Let $A = k[x,y]/(x^2 - y^3)$ be the commutative ring graded by the group $\mathbb{Z}_2 \times \mathbb{Z}_3$ with $\deg x = (1,0)$ and $\deg y = (0,1)$. Find all the simple modules in $\mathsf{GrMod}A$, and draw a picture of $\mathsf{GrMod}A$. Use the previous exercise to find a coordinate ring of $\mathsf{GrMod}A$. Directly describe the representation theory of the ring.

14.4 Draw a picture of the affine space with coordinate ring

$$R = \begin{pmatrix} k[x] & (x-1) \\ k[x] & k[x] \end{pmatrix} \otimes_{k[x]} \begin{pmatrix} k[x] & (x) & (x) \\ k[x] & k[x] & (x) \\ k[x] & k[x] & k[x] \end{pmatrix}$$

[Hint: use Proposition 4.16, but observe that this tensor is over $k[x]$, not over k.]

14.5 Draw a picture of the affine space with coordinate ring

$$T = \begin{pmatrix} k[x] & (x-1) \\ k[x] & k[x] \end{pmatrix} \otimes_{k} \begin{pmatrix} k[x] & (x) & (x) \\ k[x] & k[x] & (x) \\ k[x] & k[x] & k[x] \end{pmatrix}$$

Since there is a surjective ring homomorphism $T \to R$, $\mathsf{Mod}R$ is a closed subspace of $\mathsf{Mod}T$ (it is a curve on the surface). Show it in your picture. [Hint: think about the kernel of the map $T \to R$.]

14.6 Repeat the previous two exercises for the ring

$$R = \begin{pmatrix} k[x] & (x) \\ k[x] & k[x] \end{pmatrix} \otimes_{k[x]} \begin{pmatrix} k[x] & (x) & (x) \\ k[x] & k[x] & (x) \\ k[x] & k[x] & k[x] \end{pmatrix}$$

3.15 Questions

Is there a definition of map that would exclude the map g in Example 6.11 without requiring us to restrict our attention to enriched spaces?

Can we define the locus where a map $f : Y \to X$ is regular? Is there a maximal open subspace of Y on which f is regular.

Can we define fibers? If $f : Y \to X$ is a map, and Z is a closed subspace of X, how should $f^{-1}(Z)$ be defined? Keep in mind the problem of $f^{-1}(\phi)$ when $f : \mathbb{P}^1 \to \operatorname{Spec} k$. There is a commutative diagram

$$\begin{tikzcd} \mathsf{Mod}f^{-1}(Z) \arrow[r, "j_*"'] \arrow[d, "g_*"'] & \mathsf{Mod}Y \arrow[d, "f_*"] \\ \mathsf{Mod}Z \arrow[r, "i_*"'] & \mathsf{Mod}X \end{tikzcd}$$

where $i : Z \to X$ is the inclusion, j_* is the inclusion functor, and g_* is the restriction of f_* to $\mathsf{Mod}f^{-1}(Z)$. If f_* is exact, $\mathsf{Mod}f^{-1}(Z)$ is closed under submodules, quotient modules, and direct sums. Therefore, if we assume that Y is noetherian, j_* has a right adjoint $j^!$. With this definition, the fiber in $\mathsf{Mod}(R \otimes_k S)$ over $p \in \mathsf{Mod}R$ as in Example 6.12 is equal to $\mathsf{Mod}(R/\mathfrak{m}_p \otimes_k S)$, and this is isomorphic to $\mathsf{Mod}S$.

Should the "image" of a closed point $p \in Y$ be the smallest (weakly?) closed subspace Z of X such that $f_*\mathcal{O}_p$ is a Z-module, and should $f_*\mathcal{O}_p$ be the natural structure module of the image? In Example 6.10 I would like to say that the image of the closed point $\operatorname{Spec} k$ is all of $\operatorname{Spec} k^{\times n}$.

Given a space X, there is a unique map $f : \phi \to X$. It includes ϕ as a closed subspace, and when we define open subspaces in the next section, ϕ will be the open complement of X in X.

When $f : Y \to X$ is a map, possibly the fiber over ϕ should be the subspace of Y where f is not defined. For example, if p is a closed point of Y such that $f_*\mathcal{O}_p = 0$ we probably want to say that f is not defined at p.

Let $X = \mathsf{Mod}R$ and $Y = \mathsf{Mod}S$ be affine spaces. Let B be an R-S-bimodule. Let $f : Y \to X$ be the map associated to B, and $g : X \to Y$ the map associated to $B^\vee = \operatorname{Hom}_R(B, R)$. Discuss $f \circ g$ and $g \circ f$. What is the role of the zero locus of the trace ideals, and their open complements. We also have the S-R-bimodule $B^* = \operatorname{Hom}_S(B, S)$, and this gives another map $h : X \to Y$ that should also be examined. What is the role of the trace ideal?

Suppose that X is a space having an open affine cover U_λ. That is, each U_λ is an affine space. Can we show that a map $f : Y \to X$ is affine if and only if $f^{-1}(V)$ is affine for every open affine subspace V of X? For the classical case see [107, Exercise II.5.17]. Rosenberg might have proved this.

Would it make sense to say that f is finite if f^* preserves finitely generated modules?

Chapter 4

Some non-commutative surfaces

This chapter examines a family of non-commutative analogues of the commutative affine plane. One reason for an extensive discussion of such examples is to raise our geometric intuition about non-commutative planes to a level that is comparable with our intuition for the commutative affine plane. The commutative plane $\mathbb{A}^2$ has been a familiar feature of our mathematical lives since elementary school. Basic notions such as points and lines are so ingrained that they hardly need to be defined. We have an intimate understanding of how lines intersect, of conic sections, and so on.

The position we are in with regard to non-commutative geometry is akin to that of a mathematician before the advent of Cartesian geometry. We have a non-commutative ring R that is like the commutative polynomial ring in two variables, so we expect that the space $\mathsf{Mod}R$ should be like the affine plane. We might have an intimate understanding of the category $\mathsf{Mod}R$, but we must build from scratch some geometric notions that reflect those algebraic features. We do not have any primitive notions of things like non-commutative points, lines, or conics to fall back on. We must invent them. On the positive side, this gives us a certain license and liberty. We must also try to invent notions that are reminiscent of important commutative notions like intersection.

The first non-commutative analogue of $\mathbb{A}^2$, in section 4.1, has as a coordinate ring the enveloping algebra of the two-dimensional non-abelian Lie algebra. We discuss the lines in it and their intersections, obtaining results that resemble the commutative case reasonably well. Nevertheless, that example is rather far from the commutative case, so in section 4.3 we examine a family of non-commutative analogues of $\mathbb{A}^2$, the quantum affine planes $\mathbb{A}^2_q$, $q \in k$, that are much more like $\mathbb{A}^2$ when q is a root of unity. This is because they have coordinate rings that are finite modules over their centers, and the centers are polynomial rings in two variables. To prepare for those examples section 4.2 examines rings that are finite over their centers.

In section four we consider products of pairs of non-commutative curves.

In section five we consider some non-commutative quadrics. They lie in the non-commutative analogue of $\mathbb{A}^3$ having coordinate ring the enveloping algebra of the Lie algebra $\mathfrak{sl}_2$ over an algebraically closed field of characteristic zero. The closed points in this space are in bijection with the finite dimensional irreducible representations of $\mathfrak{sl}_2$. The quadrics in question are the zero loci of the central elements $\Omega - \lambda$, $\lambda \in k$, where Ω is the Casimir element. One sees in these examples a futher echo of the commutative theory—there are rulings of the non-commutative quadrics by lines.

The degree of a closed point. The affine spaces in this chapter are defined by first specifying a coordinate ring. That ring, viewed as a right module over itself, is a natural candidate for a structure module, so we consider the examples in this chapter as enriched spaces

$$(X, \mathcal{O}_X) = (\mathsf{Mod}R, R).$$

The degree of a closed point $p \in X$ is therefore

$$\deg p = \dim_k \operatorname{Hom}_R(R, \mathcal{O}_p) = \dim_k \mathcal{O}_p.$$

4.1 A non-commutative affine plane

In this section k denotes an algebraically closed field of characteristic zero.

We will examine the following non-commutative analogue of the affine plane. Let U denote the enveloping algebra of the two-dimensional non-abelian Lie algebra. Thus $U = k[x, y]$ with defining relation

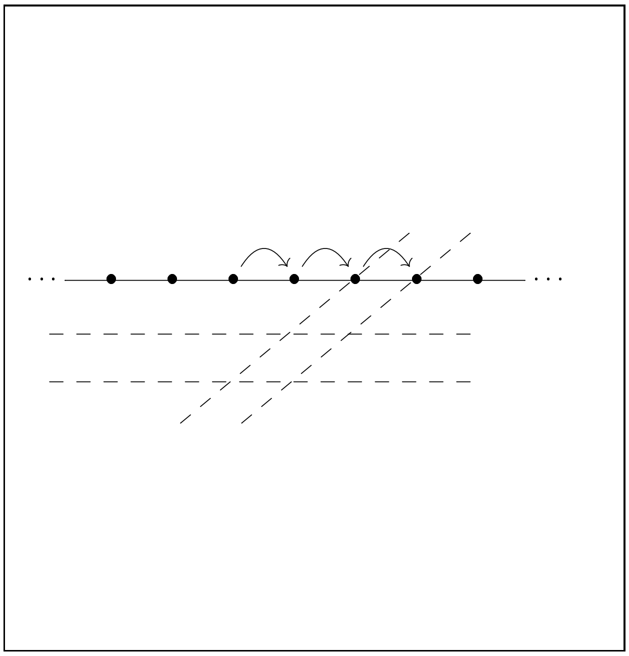
$$xy - yx = x.$$

A k-vector-space basis for U is provided by

$$\{x^i y^j \mid i, j \geq 0\}.$$

The polynomial ring on two indeterminates x and y is defined by the relation $xy - yx = 0$, and the same monomials provide a basis. Thus U looks like a reasonable non-commutative analogue of the polynomial ring. This analogy is reinforced by the fact that U, like the polynomial ring, is a noetherian domain of global dimension two. These properties follow from the fact that U is an Ore extension of the polynomial ring in one variable. A general treatment of this construction appears in chapter ???.

We will argue that the following is a reasonable picture of $\mathsf{Mod}U$.



(1-1)

The solid horizontal line in the picture is the zero locus of x. It is a copy of the affine line because $U/(x)$ is a polynomial ring in one variable. The arrows, which correspond to non-split extensions, go from each point λ on the line to the point $\lambda+1$, $\lambda \in k$. Lemma 1.3 shows that these are the only closed points in $\mathsf{Mod}U$.

The dashed lines in the picture will represent lines in $\mathsf{Mod}U$.

Lemma 1.1 *Let R be an algebra over a field k of characteristic zero. If x is a normal element in a ring R, and $xr - rx = x$ for some $r \in R$, then x annihilates every finite dimensional simple R-module.*

Proof. Let M be a finite dimensional simple R-module. Then $Ma = 0$ for some non-zero $a \in k[x]$. We may choose such an a of minimal degree. Now $0 = M(ar - ra) = Mxa'$, where a' denotes the derivative of a. Because $\operatorname{char} k = 0$, $a' \neq 0$. Since M is and x is normal, Mx is a submodule of M. If $Mx = M$ then $Ma' = 0$, contradicting the choice of a, so we conclude that $Mx = 0$. $\square$

Lemma 1.2 *If M is a simple U-module, then either $Mx = 0$ or* $\operatorname{Ann} M = 0$.

Proof. By writing an element of U as a linear combination of the basis elements $x^i y^j$ it makes sense to define the y-degree of an element as the highest power of y appearing in the expression for the element.

Suppose that $Mu = 0$ for some $u \neq 0$. Choose such a u of minimal y-degree. Notice that $M(xu - ux) = 0$. Write $u = \sum_{i,j} \alpha_{ij} x^i y^j$. A simple calculation shows that

$$xu - ux = \sum_{i,j} \alpha_{ij} x^{i+1}(y^j - (y-1)^j).$$

If $u \notin k[x]$ this element is non-zero and has lower y-degree than u, contradicting the choice of u. Therefore $u \in k[x]$.

Now we may choose a non-zero element $u \in k[x]$ of minimal degree such that $Mu = 0$. A simple calculation shows that $uy - yu = xu'$, where u' denotes the derivative of u. However, $Mxu' = M(uy - yu) = 0$, and either $Mx = 0$ or $Mx = M$ because M is simple and x is normal. Because u was chosen to have minimal degree, we conclude that $Mx = 0$. □

Lemma 1.3 *The finite dimensional simple U-modules are*

$$\mathcal{O}_\lambda := U/(x, y - \lambda), \qquad \lambda \in k.$$

Proof. Let M be a finite dimensional simple module. By Lemma 1.1, $Mx = 0$, so M is a module over $U/(x)$ which is isomorphic to the polynomial ring $k[y]$. The result follows. □

For each $\lambda \in k$, we draw an arrow from λ to $\lambda + 1$ because there is a non-split extension $0 \to \mathcal{O}_{\lambda+1} \to k^2 \to \mathcal{O}_\lambda \to 0$. This extension can be realized explicitly through the map from U to lower triangular 2×2 matrices given by

$$x \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad y \mapsto \begin{pmatrix} \lambda+1 & 0 \\ 0 & \lambda \end{pmatrix}.$$

These are the only non-split extensions (see Exercise 1) among the finite dimensional simples except for those of the form $U/(x, (y-\lambda)^2)$. But these are extensions of $U/(x, y - \lambda)$ by itself, so we do not put in any arrows to indicate them. Comparing the arrows in the picture (1-1) with the arrows in the space having coordinate ring the ring of lower triangular matrices, the next result is no surprise.

Lemma 1.4 *Fix $\lambda \in k$. Set $\mathfrak{m}_i = \operatorname{Ann}\mathcal{O}_{\lambda+i}$. There is a surjective map from U to the ring of $n \times n$ lower triangular matrices having kernel $\mathfrak{m}_0\mathfrak{m}_1 \ldots \mathfrak{m}_{n-1}$.*

Proof. Define $\varphi : U \to M_n(k)$ by

$$\varphi(y) = \begin{pmatrix} \lambda+n-1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & \lambda+n-2 & 0 & \ldots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \ldots & \lambda+1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & \lambda \end{pmatrix} \tag{1-2}$$

and

$$\varphi(x) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & & & & & \vdots & \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} \tag{1-3}$$

It is easy to check that $\varphi(x)\varphi(y) - \varphi(y)\varphi(x)$ equals $\varphi(x)$, so φ extends to an algebra homomorphism as claimed. The image of φ is contained in the lower triangular matrices. The image of the restriction of φ to $k[y]$ consists of all diagonal matrices; this can be seen by viewing the standard representation of $M_n(k)$ as a $k[y]$-module via φ. It now follows rather easily that the images of x and y generate the full lower triangular matrix ring.

$\boxed{\text{Paul}}$ Check the kernel. By Lemma 3.4.15, the fact that $\mathrm{Ext}^1_U(\mathcal{O}_\lambda, \mathcal{O}_{\lambda+1}) \neq 0$ is equivalent to the fact that $\mathfrak{m}_\lambda \mathfrak{m}_{\lambda+1} \neq \mathfrak{m}_\lambda \cap \mathfrak{m}_{\lambda+1}$. □

Lines. There are too few closed points in $\mathsf{Mod}U$ to provide much insight into this two dimensional space. After points, the most natural geometric objects to consider are lines. We have not defined a line yet. Since our basic geometric objects are categories, a line should be a certain subcategory of $\mathsf{Mod}U$. Just what kind of subcategory is not clear. Therefore we begin by considering a single module that should play the role of the "structure module" of a line. Lines in the Euclidean plane are defined by equations of the form $\alpha x + \beta y + \gamma = 0$ where $\alpha, \beta, \gamma \in k$ and at least one of α and β is non-zero, so we make the following definition.

Definition 1.5 A line module over U is one of the form $L = U/(\alpha x + \beta y + \gamma)U$ where $\alpha, \beta, \gamma \in k$ and $(\alpha, \beta) \neq (0, 0)$. For each $i \in \mathbb{Z}$, we define

$$L(i) = U/(\alpha x + \beta(y + i) + \gamma)U.$$

We call these shifts of L. ◊

Lemma 1.6 *Let $L = U/(\alpha x + \beta y + \gamma)U$ be a line module. In parts (2), (3), and (4) of the Lemma, suppose that $\beta \neq 0$, and set $\lambda = -\gamma\beta^{-1}$. Then*

1. *if $\beta = 0$ and $\gamma \neq 0$, then L is a simple module;*
2. *L has a unique simple quotient, namely $\mathcal{O}_\lambda$, and $\dim_k \mathrm{Hom}_U(L, \mathcal{O}_\lambda) = 1$;*
3. *there are exact sequences $0 \to L(i-1) \to L(i) \to \mathcal{O}_{\lambda - i} \to 0$ for all integers i;*
4. *there is a unique descending chain of submodules of L, namely*

$$L \supset L(-1) \supset L(-2) \supset \dots .$$

Proof. (1) In this case $L \cong U/(x-\mu)U$ for some non-zero $\mu \in k$. Since $\{x^i y^j \mid i,j \geq 0\}$ is a basis for U, there is decomposition $U = (x-\mu)U \oplus k[y]$ of right $k[y]$-modules. Hence L is isomorphic to $k[y]$ as a right $k[y]$-module. The right action of x on L is given by

$$y^i.x = x(y-1)^i \equiv \mu(y-1)^i.$$

Therefore the U-submodules of L are the ideals (f) that are stable under the automorphism $y \mapsto y-1$. The only such ideals are zero and $k[y]$.

(2) Let M be a simple quotient of L. Using the basis for U, one sees that $U = (\alpha x+\beta y+\gamma)U \oplus k[x]$, so L is isomorphic to $k[x]$ as a $k[x]$-module. Therefore any proper quotient of L is finite dimensional. Thus $Mx = 0$, and M is a quotient of

$$U/xU + (\alpha x + \beta y + \gamma)U = U/(x, \beta y + \gamma) \cong \mathcal{O}_\lambda.$$

(3) It suffices to prove the result when $i = 0$ because the general case is obtained from that by changing γ. The kernel of the map $L \to \mathcal{O}_\lambda$ is generated by the image of x in L, so it follows from the calculation

$$x(\alpha x + \beta(y-1) + \gamma) = (\alpha x + \beta y + \gamma)x$$

that there is a surjective map from $L(-1)$ to the kernel. However, because L and $L(-1)$ are both isomorphic to $k[x]$ as right $k[x]$-modules, any non-zero map $L(-1) \to L$ of U-modules is injective. Hence there is a short exact sequence $0 \to L(-1) \to L \to \mathcal{O}_\lambda \to 0$.

(4) It follows from (3) that L has a unique maximal submodule, namely the kernel of the surjection $L \to \mathcal{O}_\lambda$, which is isomorphic to $L(-1)$. Similarly, each $L(i)$ has a unique maximal submodule, and that submodule is isomorphic to $L(i-1)$. The result follows. $\square$

When there is a surjective map $L \to \mathcal{O}_\lambda$, we say that the point λ lies on the line L. This terminology is consistent with the commutative case because, if Y and Z are closed subschemes of a scheme X, then there is a surjective $\mathcal{O}_X$-module map $\mathcal{O}_Z \to \mathcal{O}_Y$ if and only if Y is contained in Z. On the other hand, if $\beta = 0$, then there are no points on L.

Definition 1.7 We call a line module $U/(\alpha x + \beta y + \gamma)U$ strange if $\beta = 0$, and skew if $\beta \neq 0$ and $\gamma \neq 0$. $\diamondsuit$

Thus, there are three kinds of line modules—the strange ones, which have no points, the skew ones, which have a single point, and U/xU which is the affine line $x = 0$.

We now consider how to associate to a line module L a line in the space $\mathsf{Mod}U$. In particular, we seek a non-commutative space, to be called $\mathsf{Mod}L$, that is determined in some way by L. First, $\mathsf{Mod}L$ must be a Grothendieck category. Second, the line should be a "subspace" of its ambient space, so L-modules should be U-modules. Thus $\mathsf{Mod}L$ should be a subcategory of $\mathsf{Mod}U$, and it it is reasonable to insist that it is a full subcategory. Third, the inclusion of the line in the ambient non-commutative plane should be something like an affine map. Thus, we will ask that $\mathsf{Mod}L$ be a weakly closed subspace of $\mathsf{Mod}U$. It should also contain L, and will therefore contain all submodules and all quotients of L. When L is a skew line module, $\mathsf{Mod}L$ will therefore contain $L(-i)$ and and $\mathcal{O}_{\lambda+i}$ for all $i \geq 0$. But compare this with $\mathsf{Mod}L(1)$. There are infinitely many points that lie in both $\mathsf{Mod}L$ and $\mathsf{Mod}L(1)$, and in commutative algebraic geometry, two irreducible curves with infinitely many common points are equal. So, we should define $\mathsf{Mod}L$ so that it equals $\mathsf{Mod}L(1)$. This leads us to make the following definition.

Definition 1.8 Fix a line module L.

- If L is strange, define $\mathsf{Mod}L$ to be the full subcategory of $\mathsf{Mod}U$ cosisting of the modules that are isomorphic to a direct sum of copies of L.
- If L is skew, define $\mathsf{Mod}L$ to be the smallest weakly closed subspace of $\mathsf{Mod}U$ containing all the $L(i)$.
- If $L = U/xU$, define $\mathsf{Mod}L$ to be the full subcategory of $\mathsf{Mod}U$ consisting of the modules annihilated by x.

In all cases we call $\mathsf{Mod}L$ the line with structure module L. $\Diamond$

In the picture of $\mathsf{Mod}U$ on page 188, the dashed lines parallel to $x = 0$ represent the strange lines $\mathsf{Mod}U/(x-\mu)U$, and the dashed lines at angle to $x = 0$ represent skew lines $\mathsf{Mod}U/(\alpha x + \beta y + \gamma)U$ with $\beta \neq 0$.

Because of Morita equivalence, a non-commutative affine scheme can have several different coordinate rings. Similarly, if L is a line module, any one of the $L(i)$ can play the role of a structure module for the line $\mathsf{Mod}L$.

Lemma 1.9 *If L and N are line modules, then $\mathsf{Mod}L = \mathsf{Mod}N$ if and only if $N \cong L(i)$ for some i.*

Proof. $\boxed{\text{Paul}}$ $\square$

Lemma 1.10 *Let L be a strange line module. Then $\mathsf{Mod}L \cong \mathsf{Mod}k$, and $\mathsf{Mod}L$ is weakly closed in $\mathsf{Mod}U$.*

Proof. $\operatorname{End}_U L \cong k$. $\boxed{\text{Paul}}$ Finish $\square$

If $L = U/xU$, then $\mathsf{Mod}L$ is the category of modules over the ring $U/(x) \cong k[y]$, so $\mathsf{Mod}L \cong \mathbb{A}^1$. It remains to determine the structure of the skew lines.

It follows from Lemma 1.6 that the submodule structure of a skew line module L is like the submodule behavior of $k[x]$ in $\mathsf{GrMod}k[x]$ (cf. Section 3.12). Explicitly, $L(-i)$ behaves like $k[x](-i)$ and $\mathcal{O}_{\lambda-i}$ is like $k(i)$. Further, the non-split extensions between the $\mathcal{O}_{\lambda-i}$ are like the non-split extensions between the various $k(i)$s. Therefore, the next result is not a complete surprise.

Theorem 1.11 *[228] If* $\mathsf{Mod}L$ *is a skew line, then* $\mathsf{Mod}L$ *is isomorphic to the graded line* $\mathsf{GrMod}k[x]$.

The fact that skew lines are isomorphic to the graded line is an a posteriori justification of the definition of $\mathsf{Mod}L$.

Intersection. An essential feature of geometry is the notion of incidence. Since $\mathsf{Mod}U$ has so few closed points a notion of incidence based on points is unlikely to be very useful. For example, there are no points on the strange lines, so they certainly will have no points in common with the skew lines. However, we will re-interpret the notion of intersection in terms of an Ext-group and show that this leads to a theory of incidence relations among the lines that closely mimics the commutative case.

Recall the commutative case. Let $A = k[x, y]$ be the polynomial ring. We view A as the coordinate ring of the affine plane. Let C and D be curves in $\mathbb{A}^2$, which are the zero loci of the polynomials f and g respectively. For simplicity, we will assume that f and g are irreducible, and that C and D are distinct. A point p lies on both C and D if and only if $f(p) = g(p) = 0$. Hence the intersection points of C and D are the points in $\operatorname{Spec} A/(f, g)$. Indeed, the scheme-theoretic intersection of C and D is defined to be $\operatorname{Spec} A/(f, g)$. If the coordinate rings of C and D are viewed as A-modules, then the module $A/(f, g)$ is constructed from these as their tensor product. That is,

$$A/(f, g) \cong A/fA \otimes_A A/gA.$$

We can not adopt this approach for non-commutative spaces because we cannot tensor together two right modules. There is another way to construct $A/(f, g)$ that can be copied for non-commutative rings.

Lemma 1.12 *Let* $A = k[x, y]$ *be the polynomial ring. If* f *and* g *are non-zero elements of* A*, then*

$$A/(f, g) \cong \operatorname{Ext}^1_A(A/fA, A/gA).$$

Proof. The sequence

$$0 \longrightarrow A \xrightarrow{\ f\ } A \longrightarrow A/fA \longrightarrow 0$$

is a projective resolution of A/fA. Therefore, $\operatorname{Ext}^1_A(A/fA, A/gA)$ can be computed as the appropriate homology group of the complex obtained by applying $\operatorname{Hom}_A(-, A/gA)$ to this resolution. The resulting complex is

$$0 \longleftarrow A/gA \xleftarrow{\ f\ } A/gA \longleftarrow 0$$

and the appropriate homology group is the cokernel of the middle map, namely $A/gA + fA$. $\square$

Definition 1.13 We say that two lines $\mathsf{Mod}L$ and $\mathsf{Mod}N$ meet if

$$\mathrm{Ext}^1_U(L(i), N(j)) \neq 0$$

for some $i, j \in \mathbb{Z}$. If they do not meet we say the lines are parallel. $\Diamond$

It is not apparent that the definition of *meet* is symmetric in L and N. However, Theorem 1.15 shows that it is. Although Theorem 1.15 shows that our definition of "meet" is sensible, it makes no sense to speak of two lines meeting at a point. Similarly, we have no analogue of the notion of three lines meeting simultaneously.

Definition 1.14 The slope of the line $\mathsf{Mod}L$ is infinity if $L \cong U/(x-\mu)U$, and is $-\alpha\beta^{-1}$ if $L \cong U/(\alpha x + \beta y + \gamma)U$ with $\beta \neq 0$. $\Diamond$

Linear Automorphisms. Let $\alpha, \beta, \gamma \in k$, and suppose that $\beta \neq 0$. Then there is an algebra automorphism σ of U such that

$$x^\sigma = x \qquad \text{and} \qquad y^\sigma = \beta^{-1}(\alpha x + \beta y + \gamma).$$

Each σ determines an auto-equivalence of the category $\mathsf{Mod}U$, or, equivalently, an automorphism of the space $\mathsf{Mod}U$. We call this a linear automorphism of $\mathsf{Mod}U$. It sends skew lines to skew lines, and fixes the lines with infinite slope. It sends lines with the same slope to lines with the same slope. Also two lines meet if and only if their images under the automorphism meet.

It is useful to observe that any skew line is conjugate under a linear automorphism to the line U/yU.

Theorem 1.15 *1. Every line meets itself.*

2. Two distinct lines are parallel if and only if they have the same slope.

Proof. Let $L = U/fU$ and $N = U/gU$ be line modules. Then $\mathrm{Ext}^1_U(L,N)$ is the appropriate homology group of the complex obtained by applying $\mathrm{Hom}_U(-, U/gU)$ to a projective resolution of L. Such a resolution is

$$0 \longrightarrow U \xrightarrow{f} U \longrightarrow L \longrightarrow 0$$

so $\mathrm{Ext}^1_U(L,N)$ is the cokernel of the middle map in the complex

$$0 \longleftarrow U/gU \xleftarrow{f} U/gU \longleftarrow 0.$$

The map is *right* multiplication by f, so

$$\mathrm{Ext}^1_U(L,N) \cong U/gU + Uf \cong N/Nf.$$

The statements in the theorem are invariant under linear automorphisms.

(1) First consider a line with infinite slope, say $\mathsf{Mod}N$ where $N = U/(x - \mu)U$. We will compute $\mathrm{Ext}^1_U(U/(x-\tau)U, N)$, which is isomorphic to $N/N(x -$

$\tau)$. As in the proof of Lemma 1.6, we identify N with $k[y]$. Doing so, the action of x on N becomes $y^i.x = \mu(y-1)^i$. Hence $k[y].(x-\tau)$ is spanned by $\{(\mu-\tau)(y-1)^i \mid i \geq 0\}$. It follows that

$$\operatorname{Ext}^1_U(U/(x-\tau)U, N) \cong \begin{cases} 0 & \text{if } \mu \neq \tau, \\ N & \text{if } \mu = \tau. \end{cases} \tag{1-4}$$

In particular, $\dim_k \operatorname{Ext}^1_U(N,N) = \infty$, so N meets itself.

Now consider a skew line, $\mathsf{Mod}N$. By applying a linear automorphism to $\mathsf{Mod}U$, we can assume that $N = U/yU$. Since $U = yU \oplus k[x]$, N is isomorphic to $k[x]$ as a right $k[x]$-module. The right action of x is then given by multiplication, and the right action of y on N is given by

$$x^i.y = (y+i)x^i \equiv ix^i.$$

If $L = U/(\alpha x + \beta y + \gamma)U$, then $\operatorname{Ext}^1_U(L,N) \cong N/N(\alpha x + \beta y + \gamma)$, and $N(\alpha x + \beta y + \gamma)$ is spanned by

$$\{x^i(\alpha x + \beta i + \gamma) \mid i \geq 0\}.$$

Taking the case $L = N$, we see that

$$\operatorname{Ext}^1_U(U/yU, U/yU) \cong k.$$

The unique non-split extension is $0 \to U/yU \to U/y^2U \to U/yU \to 0$.

(2) ($\Leftarrow$) We must show that $\operatorname{Ext}^1_U(L,N) = 0$ if $\mathsf{Mod}L$ and $\mathsf{Mod}N$ are distinct lines with the same slope.

First, suppose the lines have infinite slope. Therefore $N \cong U/(x-\mu)U$ and $L \cong U/(x-\tau)U$ with $\mu \neq \tau$. By (1-4), $\operatorname{Ext}^1_U(L,N) = 0$.

Now suppose that the lines have finite slope. Up to a linear automorphism of $\mathsf{Mod}U$ we can assume that $N \cong U/yU$, whence $L \cong U/(y+\gamma)U$ for some $\gamma \in k$. Since the lines are distinct, $N \ncong L(i)$ for any $i \in \mathbb{Z}$, so $\gamma \notin \mathbb{Z}$. We use the calculation in part (1). If we identify N with $k[x]$, then $\operatorname{Ext}^1_U(L,N) \cong k[x]/k[x](y+\gamma)$, and $k[x].(y+\gamma)$ is spanned by

$$\{x^i(i+\gamma) \mid i \geq 0\}.$$

Since $\gamma \notin \mathbb{Z}$, $k[x].(y+\gamma) = k[x]$ and $\operatorname{Ext}^1_U(L,N) = 0$, as required.

($\Rightarrow$) We must show that if $\mathsf{Mod}L$ and $\mathsf{Mod}N$ are lines with different slopes, then $\operatorname{Ext}^1_U(L,N) \neq 0$.

Suppose that $\mathsf{Mod}N$ has infinite slope. Thus $N \cong U/(x-\mu)U$ for some $\mu \in k$. Since $\mathsf{Mod}L$ is a skew line, we can apply a linear automorphism and assume that $L \cong U/yU$; this linear automorphism fixes $\mathsf{Mod}N$. If we identify N with $k[y]$ as in the proof of part (1), we see that $\operatorname{Ext}^1_U(L,N) \cong k[y]/k[y].y \cong k$.

Now suppose that $\mathsf{Mod}N$ has finite slope. Applying a linear automorphism we can assume that $N = U/yU$. Once more, we identify N with $k[x]$ with y-action given by $x^i.y = ix^i$. If $\mathsf{Mod}L$ has infinite slope, then $L \cong U/(x-\mu)U$ and

$$\operatorname{Ext}^1_U(L,N) \cong k[x]/k[x].(x-\mu) \cong k,$$

so the lines meet. On the other hand, if $\mathsf{Mod}L$ is a skew line and $L = U/(x + \beta y + \gamma)$, then

$$\operatorname{Ext}^1_U(L, N) \cong k[x]/k[x].(x + \beta y + \gamma).$$

But $x^i.(x + \beta y + \gamma) = x^{i+1} + (\beta i + \gamma)x^i$, and because $\deg(f.(x + \beta y + \gamma)) = \deg f + 1$, we conclude that $\operatorname{Ext}^1_U(L, N) \neq 0$. $\square$

The formulation of Theorem 1.15 is quite satisfying, but the situation is even better than it suggests. Taking a little more care in its proof, one sees that if $\mathsf{Mod}L$ and $\mathsf{Mod}N$ are distinct lines that meet, then $\dim \operatorname{Ext}^1_U(L, N) = 1$ so, if we define the multiplicity of meeting as the dimension of this Ext-group, then the lines meet with multiplicity one. By the discussion prior to Lemma 1.12, such a definition of intersection multiplicity is compatible with the commutative definition. Further, there is a non-commutative analogue of a projective plane that contains $\mathsf{Mod}U$ as an affine open subspace (as the complement to the line at infinity, and that line is isomorphic to $\mathbb{P}^1$). In that projectiove plane, two parallel lines in $\mathsf{Mod}U$ meet at infinity.

Notice that if L and N are line modules such that $\mathcal{O}_\lambda$ lies on both, but $\mathsf{Mod}L \neq \mathsf{Mod}N$, then $\mathcal{O}_{\lambda+i}$ is in both $\mathsf{Mod}L$ and $\mathsf{Mod}N$ for all $i \in \mathbb{Z}$. So, we have distinct lines having infinitely many points in common, but meeting with multiplicity one. This is slightly disturbing at first. We shall see later that there is a notion of divisor for non-commutative spaces, and $\mathsf{Mod}\mathcal{O}_\lambda$ is not a divisor on $\mathsf{Mod}L$, but all the $\mathcal{O}_{\lambda+i}$ can be bundled together to give a divisor on $\mathsf{Mod}L$.

Example 1.16 Consider the Weyl algebra in characteristic zero. This is the ring $D = k[t, \partial]$ with defining relation $\partial t - t\partial = 1$. It has no points at all. It turnd out that if we define the line modules to be $D/(\alpha\partial + \beta t + \gamma)$ with $(\alpha, \beta) \neq (0, 0)$, then all line modules are simple. There is a close relation between $\mathsf{Mod}D$ and $\mathsf{Mod}U$. They have isomorphic open subspaces. This is because $U[x^{-1}] \cong D[t^{-1}]$. $\Diamond$

EXERCISES

1.1 Verify the claim that $\operatorname{Ext}^1_U(\mathcal{O}_\mu, \mathcal{O}_\lambda) = 0$ if $\mu \notin \{\lambda, \lambda + 1\}$.

1.2 Classify the simple U-modules when the field k is algebraically closed of characteristic $p > 0$.

1.3 Continuing the previous exercise, let L be a line module, and classify the points on the non-commutative affine curve $U/\operatorname{Ann}L$. What are the extensions between the various point modules $\mathcal{O}_p$?

1.4 The ring $k[x, x^{-1}]$ is a left module over the ring $D = k[x, x\partial, x^2\partial]$, where x acts by multiplication and $\partial = d/dx$ acts by differentiation. Show that $k[x, x^{-1}]$ is a uniserial module of length three with its top isomorphic to its socle.

Thus the space $\mathsf{Mod}D$ contains a (weakly closed?) subspace that is isomorphic to the closed subspace that appeared in the previous example. This category is the simplest example of a category O, an important family of catagories occuring in the representation theory of semisimple Lie algebras. The ring D is a quotient of the enveloping algebra of the Lie algebra $\mathfrak{sl}_2$ of traceless 2×2 matrices.

This is the representation category for the quiver

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with the relation $\beta\alpha = 0$.

1.5 Let X be the affine space with coordinate ring the path algebra of the quiver Q in Proposition 5.23. Using the picture of that space in Example 14-5 for inspiration, determine what the automorphism group of the enriched space $(X, \mathcal{O}_X)$, where $\mathcal{O}_X = kQ$. An automorphism of an enriched space is an auto-equivalence of $\mathsf{Mod}X$ sending $\mathcal{O}_X$ to $\mathcal{O}_X$. (Check this is the same as the automorphism group of the k-algebra $\mathcal{O}_X$ when X is affine.)

4.2 Rings finite over their centers

The center of a ring is denoted by $C(R)$. If R is a finitely generated $C(R)$-module we say that R is finite over its center.

The following two results are fundamental to the study of such rings.

Theorem 2.1 (Artin-Tate Lemma) *Let R be a finitely generated k-algebra. If R is finite over its center, then*

1. *its center is a finitely generated k-algebra, hence noetherian;*
2. *R is noetherian.*

Proof. Write C for the center of R. Suppose that $R = k[x_1, \dots, x_m]$ and $R = Ca_1 + \ldots + Ca_n$. There is a finite set

$$\{\alpha_{pqr}, \beta_{st}\} \subset C$$

such that

$$a_p a_q = \sum_{r=1}^{n} \alpha_{pqr} a_r \qquad \text{and} \qquad x_s = \sum_{t=1}^{n} \beta_{st} a_t.$$

Since $C' := k[\alpha_{pqr}, \beta_{st}]$ is a finitely generated, commutative k-algebra, it is a noetherian ring. Since any product of the x_is is in $C'a_1 + \ldots + C'a_n$, R is generated as a C'-module by $a_1, \dots, a_n$. Hence R is a noetherian C'-module, and is therefore a noetherian ring. Since $C' \subset C \subset R$, C is a finitely generated C'-module, hence a finitely generated k-algebra, and thus a noetherian ring. $\square$

Theorem 2.2 *Let R be a finitely generated k-algebra that is finite over its center. Let M be a simple R-module. Then*

1. $\dim_k M < \infty$, *so M is tiny;*
2. *M is annihilated by a maximal ideal of the center of R;*
3. *if k is algebraically closed,* $\operatorname{End}_R M = k$.

Proof. Let M be a simple R-module. The hypotheses on R also apply to $R/\operatorname{Ann} M$, so it is enough to prove the result for $R/\operatorname{Ann} M$. Therefore we can, and do, assume that $\operatorname{Ann} M = 0$.

Write $C = C(R)$.

Because R is prime, C is a domain. By the Artin-Tate Lemma, C is noetherian. The map $C \to \operatorname{End}_R M$ is injective. Because R is a finitely generated C-module, so is M. Therefore $\operatorname{End}_C M$, and hence $\operatorname{End}_R M$, is a finitely generated C-module. By Schur's Lemma, $\operatorname{End}_R M$ is a division ring. Therefore each non-zero element x in element of C has an inverse in $\operatorname{End}_R M$, and the chain $C \subset Cx^{-1} \subset Cx^{-2} \subset \dots$ is eventually constant. Hence, for some large n, $x^{-n-1} \in Cx^{-n}$. Therefore $x^{-1} \in C$. Hence C is a field.

By the Artin-Tate Lemma, C is a finitely generated k-algebra, so $\dim_k C < \infty$. It follows that $\dim_k R < \infty$ and that $\dim_k M < \infty$. In particular, R is artinian, so M is tiny.

It follows that $\operatorname{End}_R M$ is also finite dimensional over k so, if k is algebraically closed, $\operatorname{End}_R M = k$. $\square$

Corollary 2.3 *Let R be a finitely generated k-algebra that is finite over its center. Then the closed points in the affine space $\mathsf{Mod}R$ are in bijection with the simple R-modules, and in bijection with the maximal ideals of R.*

We continue to suppose that R is a finitely generated algebra over an algebraically closed field k, and that R is finite over its center C. Write $X = \mathsf{Mod}R$ and $Z = \operatorname{Spec} C$. The inclusion $C \to R$ induces a map of spaces

$$f : X \to Z.$$

By Theorem 2.2(2), f sends closed points to closed points; that is, if q is a closed point of X, then $f_*\mathcal{O}_q$ is a module over $C/\mathfrak{m}$ for some maximal ideal of C, so $f_*\mathcal{O}_q$ is a fnite direct sum of copies of a single simple C-module $\mathcal{O}_p$. We write $p = f(q)$. The scheme theoretic fiber of f over a closed point $p \in Z$ is defined to be

$$X_p := \mathsf{Mod}R/\mathfrak{m}_p R,$$

where $\mathfrak{m}_p$ is the maximal ideal of C corresponding to p. Because C is finitely generated over k, the Nullstellensatz ensures that $C/\mathfrak{m}_p \cong k$. Hence $R/\mathfrak{m}_p R$ is a finite dimensional vector space, and X_p is a finite affine space. Thus R provides a family of finite dimensional algebras parametrized by Z.

The behavior of R along one irreducible component of Z can be independent of its behavior along another component, so we now suppose in addition that R is prime. We will show that over a dense open set, say U, of Z there is a uniformity to the behavior of the closed points of X. If p is a closed point in U, then there is a unique closed point q in the fiber over p, and the dimension of the corresponding simple module $\mathcal{O}_q$ is constant as p ranges over U. Further, all other simple R-modules have smaller dimension. The algebra in the next section illustrates this uniformity result.

The first step towards this uniformity is a criterion for deciding when a finite dimensional algebra is a matrix algebra. Suppose that A is an algebra over a commutative ring k. Then A is an A-A-bimodule, hence a left module over $A \otimes_k A^{\mathrm{op}}$, via the action $(a \otimes b).c = acb$. The module action yields a k-algebra homomorphism $\varphi : A \otimes_k A^{\mathrm{op}} \to \mathrm{End}_k A$.

Lemma 2.4 *Let A be a finite dimensional algebra over an algebraically closed field k. Then A is isomorphic to $M_n(k)$ for some n if and only if the map $\varphi : A \otimes_k A^{\mathrm{op}} \to \mathrm{End}_k A$ is an isomorphism.*

Proof. ($\Rightarrow$) In this case, A^{op} is also isomorphic to $M_n(k)$, so $A \otimes_k A^{\mathrm{op}} \cong M_{n^2}(k)$. This is a simple ring, so φ is injective. But $\dim_k \mathrm{End}_k A = (\dim_k A)^2 = \dim_k A \otimes_k A^{\mathrm{op}}$, so φ is also surjective.

($\Leftarrow$) Because A is a simple ring it is a simple module over $\mathrm{End}_k A$ so, since φ is an isomorphism, it is a simple module over $A \otimes_k A^{\mathrm{op}}$. But the $A \otimes_k A^{\mathrm{op}}$-submodules of A are precisely its two-sided ideals, so A is a simple ring. By the Artin-Wedderburn Theorem it is therefore a matrix algebra over a division ring. Since that division ring has finite dimension over k, it is isomorphic to k. $\square$

Proposition 2.5 *Let R be a prime ring that is finite over its center C, and is a finitely generated algebra over an algebraically closed field k. Then*

1. *C is a domain;*
2. *if F denotes the field of fractions of C, then the center of $A := R \otimes_C F$ is F;*
3. *$A \cong M_n(D)$ for some division algebra D, and integer n;*
4. *the induced F-algebra homomorphism $\varphi : A \otimes_F A^{\mathrm{op}} \to \mathrm{End}_F A$ is an isomorphism.*

Proof. (1) If $x, y \in C$ are such that $xy = 0$, then $(xR)(yR) = 0$, so either x or y is zero.

(2) Set $A = R \otimes_C F$. Elements of A are of the form ax^{-1} with $a \in R$ and $x \in C$, so if ax^{-1} commutes with all elements of R, so does $ax^{-1}x = a$.

(3) Clearly A is a finite module over F, so is artinian. It is a prime ring, because R is. But a prime artinian ring is simple, so the Artin-Wedderburn Theorem gives the result.

(4) The theory of central simple algebras shows that $A \otimes_F A^{\mathrm{op}}$ is a simple ring, so φ is injective. However, the F-vector-space dimension of both rings equals $(\dim_F A)^2$, so φ must be an isomorphism. $\square$

Lemma 2.6 *Let C be a commutative noetherian domain with field of fractions F. Let $f : M \to N$ be a homomorphism of finitely generated C-modules. If the induced map $M \otimes_C F \to N \otimes_C F$ is an isomorphism, then there is a non-zero element $x \in C$ such that the induced map $M[x^{-1}] \to N[x^{-1}]$ is an isomorphism.*

Proof. Let K and L be the kernel and cokernel of F. Both are finitely generated, and $K \otimes_C F \cong L \otimes_C F \cong 0$. If $a_1, \dots, a_m$ is a set of generators for K, then there are non-zero elements $x_1, \dots, x_m$ in C such that $a_i x_i = 0$ for all i. Similarly, we can choose elements $b_1, \dots, b_n$ generating L, and $y_1, \dots, y_n$ in C such that $b_i y_i = 0$ for all i. If $x = x_1 \dots x_n y_1 \dots y_n$, then $Kx = Lx = 0$, whence $K \otimes_C C[x^{-1}] \cong L \otimes_C C[x^{-1}] \cong 0$. □

Lemma 2.6 and its proof can be phrased more geometrically. If $Z = \operatorname{Spec} C$ and $Y = \operatorname{Supp} K \cup \operatorname{Supp} L$, then $f : M \to N$ is an isomorphism on $Z \backslash Y$. Furthermore, $M \otimes_C F$ is free, of rank d say, so there is a proper closed subscheme $Y' \subset Z$ such that M and N are both free of rank d on $Z \backslash Y'$.

Theorem 2.7 *Let R be a prime ring that is finite over its center C, and is a finitely generated algebra over an algebraically closed field k. Set $X = \mathsf{Mod}R$ and $Z = \operatorname{Spec} C$, and let $f : X \to Z$ be the induced map. Then there is a dense open set $U \subset Z$, and an integer n, such that if $p \in U$ is a closed point, then there is a unique closed point $q \in X$ such that $f(q) = p$, and $\dim_k \mathcal{O}_q = n$. Furthermore, if $\mathfrak{m}_p$ is the maximal ideal of C vanishing at p, then $R/\operatorname{Ann}\mathcal{O}_q = R/R\mathfrak{m}_p \cong M_n(k)$.*

Proof. Let $\varphi : R \otimes_C R^{\mathrm{op}} \to \operatorname{End}_C R$ be the natural map. Let F denote the field of fractions of C, and set $A := R \otimes_C F$. By Proposition 2.5, $\varphi \otimes F$ is an isomorphism, so by Lemma 2.6, $\varphi \otimes C[x^{-1}]$ is an isomorphism for some non-zero x in C. By the remarks after that lemma, we can also assume, perhaps after replacing x by some xx', that $R[x^{-1}]$ is a free $C[x^{-1}]$-module of rank d. Let $\mathfrak{m}$ be a maximal ideal of C that does not contain x. Then $\varphi \otimes C[x^{-1}] \otimes_C C/\mathfrak{m}$ is an isomorphism. But $C[x^{-1}] \otimes_C C/\mathfrak{m} \cong C/\mathfrak{m} \cong k$, so this is the map

$$\varphi \otimes C/\mathfrak{m} : R/R\mathfrak{m} \otimes_k R/R\mathfrak{m} \to \operatorname{End}_k R/R\mathfrak{m}.$$

Therefore by Lemma 2.4, $R/R\mathfrak{m}$ is a matrix algebra over k. Since $R[x^{-1}]$ is free of rank d, $\dim_k R/R\mathfrak{m} = d$. Hence for all $\mathfrak{m}$ not containing x, $R/R\mathfrak{m} \cong M_n(k)$ where $n^2 = d$. □

The largest open set U in the theorem is called the Azumaya locus.

Example 2.8 Let k be an algebraically closed field of characteristic two, and let $R = k[x, y, z]$ be the k-algebra with defining relations

$$xy - yx = z, \quad xz = zx, \quad yz = zy.$$

Thus R is isomorphic to the enveloping algebra of the Heisenberg Lie algebra, and to the enveloping algebra of the Lie algebra $\mathfrak{sl}_2$ of traceless 2×2 matrices. The elements $x^i y^j z^k$, $(i, j, k) \in \mathbb{N}^3$ form a basis for R. The center of R is $k[x^2, y^2, z]$, the coordinate ring of $\mathbb{A}^3$. It is clear that R is a finite module over its center. Since $R/(z)$ is commutative, if p is a closed point of $\mathsf{Mod}R$ lying on the hypersurface $z = 0$, $\dim \mathcal{O}_p = 1$. On the other hand, if p is a closed point of $\mathsf{Mod}R$ such that $\mathcal{O}_p$ is annihilated by $z - \lambda$ for some non-zero scalar λ, then $\dim \mathcal{O}_p = 2$. To see this, we argue as follows.

Since $\mathcal{O}_p$ is a finite dimensional simple it is annihilated by a maximal ideal of $k[x^2, y^2, z]$, so $R/\operatorname{Ann}\mathcal{O}_p$ is spanned by the images of 1, x, y, and xy, so has dimension at most four. Thus the dimension of $\mathcal{O}_p$ is at most two. On the other hand, if its dimension were one, then both x and y would act on it as scalars, so $z = xy - yx$ must annihilate it, contrary to our hypothesis.

Thus the Azumaya locus of R is the complement to the coordinate plane $z = 0$. $\Diamond$

4.3 Quantum planes

Let k be an algebraically closed field. Fix $0 \neq q \in k$, and let

$$A = k_q[x, y]$$

be the algebra generated by x and y with defining relation

$$yx = qxy. \tag{3-1}$$

This ring is a noetherian domain with basis $\{x^i y^j \mid i, j \geq 0\}$, and it has global dimension two. Thus, A is a reasonable non-commutative analogue of the polynomial ring in two variables. We will write

$$\mathbb{A}_q^2 = \mathsf{Mod} A$$

and call it a quantum affine plane.

The structure of A, and hence that of $\mathbb{A}_q^2$, depends delicately on whether or not q is a root of unity.

Proposition 3.1 *1. If q is not a root of unity, then the center of A is k.*

2. If q is a primitive n^{th} root of unity, then the center of A is $k[x^n, y^n]$, and A is a free module over this with basis $\{x^i y^j \mid 0 \leq i, j \leq n, \ (i, j) \neq (n, n)\}$.

Proof. Let z be a central element, and write

$$z = \sum_{i,j} \alpha_{ij} x^i y^j.$$

Then

$$xz = \sum_{i,j} \alpha_{ij} x^{i+1} y^j \qquad \text{and} \qquad zx = \sum_{i,j} \alpha_{ij} q^j x^{i+1} y^j.$$

Also

$$yz = \sum_{i,j} \alpha_{ij} q^i x^i y^{j+1} \qquad \text{and} \qquad zy = \sum_{i,j} \alpha_{ij} x^i y^{j+1}.$$

Therefore, if $\alpha_{ij} \neq 0$, then $q^i = q^j = 0$. Hence, if q is not a root of unity, then only α_{00} could be non-zero, whence $z \in k$.

If q is an n^{th} root of unity, then n divides both i and j whenever $\alpha_{ij} \neq 0$. Certainly x^n and y^n are central in that case. Therefore the center is $k[x^n, y^n]$.

Because the elements x^iy^j give a k-basis for A, $k[x^n, y^n]$ is a polynomial ring, and the claimed basis is indeed a basis. □

There are two commutative affine lines in $\mathbb{A}^2_q$, namely $x = 0$ and $y = 0$. That is, $A/(x) \cong k[y]$, and $A/(y) \cong k[x]$ are both polynomial rings, so we get two lines of points in $\mathsf{Mod}A$, meeting at the point $\operatorname{Spec} A/(x, y)$. We call this the origin, and we call the closed subspaces $\mathcal{Z}(y)$ and $\mathcal{Z}(x)$ the x-axis and y-axis respectively. The closed points on the two axes all have degree one. The corresponding simple modules are

$$\mathcal{O}_{(\lambda,0)} = A/(x - \lambda, y), \quad \mathcal{O}_{(0,\lambda)} = A/(x, y - \lambda), \qquad \lambda \in k. \tag{3-2}$$

Proposition 3.2 *If q is not a root of unity, the only closed points of $\mathbb{A}^2_q$ are the points on the two axes $x = 0$ and $y = 0$.*

Proof. Suppose that M is a finite dimensional simple module. Since y is a normal element, it either kills M or acts faithfully on it. If it kills M, then M is an $A/(y)$-module. Now suppose that y acts faithfully on M. Let $m \in M$ be an x-eigenvector, with eigenvalue λ say. Then $my^n.x = q^n\lambda my^n$. Since all $my^n \neq 0$, and since If $\lambda \neq 0$, then since q is not a root of unity, the elements my^n have distinct eigenvalues, so are linearly independent, contradicting the finite dimensionality of M. Therefore $\lambda = 0$. But x is normal, so it follows that $Mx = 0$, and M is an $A/(x)$-module. In either case, we have shown that M is one-dimensional. □

If q is a root of unity there are many other closed points in $\mathbb{A}^2_q$.

Lemma 3.3 *Suppose that q is a primitive n^{th} root of unity. If $(\zeta, \xi) \in k^2$ and $\zeta\xi \neq 0$, define the A-module $V(\zeta, \xi)$ to have k-basis v_i, $i \in \mathbb{Z}_n$, and*

$$v_i.x = \zeta q^i v_i, \qquad v_i.y = \xi v_{i+1}.$$

Then

1. *$V(\zeta, \xi)$ is a simple A-module;*
2. *the annihilator of $V(\zeta, \xi)$ is the maximal ideal $(x^n - \zeta^n, y^n - \xi^n)$;*
3. *$V(\zeta_1, \xi_1) \cong V(\zeta_2, \xi_2)$ if and only if $(\zeta_1^n, \xi_1^n) = (\zeta_2^n, \xi_2^n)$.*

Proof. (1) First we check that V is an A-module. Because $v_i.xy = \zeta\xi q^i v_{i+1}$ and $v_i.yx = \xi\zeta q^{i+1}v_{i+1}$, $yx - qyx$ acts as zero on V. Hence it is an A-module.

Let U be a non-zero submodule of V. Because the x-eigenvalues are distinct and have multiplicity one, an $k[x]$-submodule of V has a basis consisting of various v_i's. But, once a single v_j belongs to U, the y-action shows that every v_i is in U. Hence $U = V$.

(2) The two indicated elements annihilate V because $q^n = 1$. Hence V is a module over $A/(x^n - \zeta^n, y^n - \xi^n)$. But this is isomorphic to $A \otimes_C C/\mathfrak{m}$ where $C = k[x^n, y^n]$ and $\mathfrak{m}$ is the maximal ideal of C generated by $x^n - \zeta^n$ and $y^n - \xi^n$.

Since A is a free C-module of rank n^2, $A \otimes_C C/\mathfrak{m}$ is an n^2-dimensional vector space. The action of A on V gives an algebra homomorphism

$$A/(x^n - \zeta^n, y^n - \xi^n) \to \operatorname{End}_k V \cong M_n(k).$$

Because V is a simple A-module, it follows from the Jacobson Density Theorem that this map is surjective. Since both sides have dimension n^2 it is also injective. Hence the two rings are isomorphic, showing in particular that $(x^n - \zeta^n, y^n - \xi^n)$ is a maximal ideal. Thus $\operatorname{Ann} V$ is as claimed.

(3) The two modules are tiny simples, so are isomorphic if and only if they have the same annihilators (Corollary 3.4.9). But it follows from (2) that $V(\zeta_1, \xi_1)$ and $V(\zeta_2, \xi_2)$ have the same annihilator if and only if $(\zeta_1^n, \xi_1^n) = (\zeta_2^n, \xi_2^n)$. □

If ζ and ξ are non-zero scalars, define $(\lambda, \mu) = (\zeta^n, \xi^n)$. We will denote the simple module $V(\zeta, \xi)$ by

$$\mathcal{O}_{(\lambda,\mu)}.$$

and denote the corresponding closed point of $\mathbb{A}_q^2$ by (λ, μ). By part (3) of the lemma, this notation is unambiguous.

Proposition 3.4 *Suppose that q is a primitive n^{th} root of unity. The closed points in $\mathbb{A}_q^2$ consist of the points on the two axes and the points (λ, μ). The points on the axes have degree one and the others have degree n.*

Proof. We must show that up to isomorphism the simple A-modules consist of the one-dimensional simples $A/(x, y-\lambda)$ and $A/(x-\lambda, y)$, and the n-dimensional simples $V(\zeta, \xi)$ in Lemma 3.3. It is clear that $\mathcal{O}_{(\lambda,0)}$ and $\mathcal{O}_{(0,\lambda)}$, $\lambda \in k$, give all the one-dimensional simples.

Now let M be a simple A-module of dimension > 1. Then $\operatorname{End}_A M \cong k$. The action of the center on M gives a homomorphism $k[x^n, y^n] \to \operatorname{End}_R M \cong k$, so M is annihilated by $(x^n - \lambda, y^n - \mu)$ for some λ and μ. If $\lambda\mu \neq 0$ then $A/(x^n - \lambda, y^n - \mu) \cong M_n(k)$ by Lemma 3.3, so M is isomorphic to the unique simple module over this ring, namely $V(\zeta, \xi)$, where $(\zeta^n, \xi^n) = (\lambda, \mu)$.

On the other hand, if $\lambda\mu = 0$, we may assume by symmetry that $\lambda = 0$, so $Mx^n = 0$. Because M is simple and x is normal, $Mx = 0$ also, whence M is an $A/(x)$-module. □

We continue to suppose that q is a primitive n^{th} root of unity.

The inclusion $k[x^n, y^n] \to A$ gives a map $f : \mathbb{A}_q^2 \to \mathbb{A}^2$, a covering of the commutative plane by a non-commutative plane. If p is a closed point in $\mathbb{A}_q^2$, then $\mathcal{O}_p$ is annihilated by a maximal ideal of $k[x^n, y^n]$ so $f_* \mathcal{O}_p$ is a direct sum of copies of a single simple module. We denote that simple by $\mathcal{O}_{f(p)}$. We have

$$\begin{aligned} f(\lambda, 0) &= (\lambda, 0), \\ f(0, \mu) &= (0, \mu), \quad \text{and} \\ f(\lambda, \mu) &= (\lambda, \mu) \;\; \text{if } \lambda\mu \neq 0. \end{aligned}$$

The fiber of f over the open set that is the complement of the axes consists of the closed points in $\mathbb{A}_q^2$ having degree n. The fiber over the origin in $\mathbb{A}^2$ is the origin in $\mathbb{A}_q^2$ with some non-commutative scheme structure. The fiber over any other point on the axes $x^n = 0$ or $y^n = 0$ consists of n closed points of degree one.

We define lines as we did for the space $\mathsf{Mod}U$ in section 4.1.

Definition 3.5 A line module over A is one of the form $L = A/(\alpha x + \beta y + \gamma)A$ where $\alpha, \beta, \gamma \in k$ and $(\alpha, \beta) \neq (0,0)$. The slope of L is infinity if $\beta = 0$, and is $-\alpha\beta^{-1}$ otherwise. For each $(i,j) \in \mathbb{Z}^2$, we define

$$L(i,j) = A/(\alpha q^i x + \beta q^{-j} y + \gamma)A.$$

We call these shifts of L. $\diamondsuit$

Consider the closed points on the line $L = A/(\alpha x + \beta y + \gamma)A$. If $\beta \neq 0$, then there is an exact sequence

$$0 \to L(0,-1) \to L \to \mathcal{O}_{(0,-\beta^{-1}\gamma)} \to 0.$$

If $\alpha \neq 0$, then there is an exact sequence

$$0 \to L(-1,0) \to L \to \mathcal{O}_{(-\alpha^{-1}\gamma,0)} \to 0.$$

If $\alpha\beta \neq 0$, then both exact sequences occur. By repeating the argument with $L(-1,0)$ and $L(0,-1)$ in place of L, and so on, we eventually obtain submodules $L(i,j)$ of L for all integers $i,j \leq 0$. Indeed, we obtain a lattice of submodules L that is isomorphic to the lattice of ideals (X^iY^j) in the commutative polynomial ring $k[X,Y]$. These ideals, together with the ideals (X^i, Y^j) of finite codimension, are the only non-zero graded ideals of $k[X,Y]$ when $k[X,Y]$ is given the $\mathbb{Z}^2$-grading defined by $\deg X = (1,0)$ and $\deg Y = (0,1)$.

We will return to this shortly. First, we will prove in Theorem 3.9 that when q is a primitive n^{th} root of unity, then the lines "parallel" to the axes are isomorphic to the broken line $\mathsf{Mod}Q$ where Q is the quiver pictured in Theorem 3.9.

Linear Automorphisms. Let $\lambda, \mu \in k$, and suppose that $\lambda\mu \neq 0$. Then there is an algebra automorphism σ of A such that

$$x^\sigma = \lambda x \qquad \text{and} \qquad y^\sigma = \mu y.$$

Each σ determines an auto-equivalence of the category $\mathsf{Mod}A$, or, equivalently, an automorphism of the space $\mathsf{Mod}A$. We call this a linear automorphism of $\mathsf{Mod}A$.

Lemma 3.6 *Suppose that k is a field having a primitive n^{th} root of 1, say q. Let $R = k[x]/(x^n - 1)$. Then the elements*

$$e_i \;:=\; \frac{q^i}{n} \cdot \frac{x^n - 1}{x - q^i} \tag{3-3}$$

are a complete set of orthogonal idempotents for R.

Proof. Write $\omega = q^i$ and $e = e^i$. To show that $e^2 = e$, it suffices to show that there exists $f \in k[x]$ such that

$$\left(\frac{x^n - 1}{x - \omega}\right)^2 - \frac{n}{\omega}\left(\frac{x^n - 1}{x - \omega}\right) = (x^n - 1)f.$$

Multiplying this by $(x - \omega)$, and dividing it by $(x^n - 1)$, we must show that

$$x^{n-1} + \omega x^{n-2} + \ldots + \omega^{n-2}x + \omega^{n-1} - \tfrac{n}{\omega} = (x - \omega)f. \tag{3-4}$$

However, the left-hand side of this evaluated at $x = \omega$ is zero, whence $e = e^2$.

It is clear that $e_i e_j = 0$ if $i \neq j$, so it only remains to check that the sum of all the e_is is one. We have

$$\begin{aligned}\sum_{i=0}^{n-1} e_i &= \frac{1}{n}\sum_{i=0}^{n-1} q^i\left(x^{n-1} + q^i x^{n-2} + q^{2i}x^{n-3} + \cdots + q^{(n-1)i}\right)\\ &= \frac{1}{n}\left((\sum_{i=0}^{n-1} q^i)x^{n-1} + (\sum_{i=0}^{n-1} q^{2i})x^{n-2} + \cdots + (\sum_{i=0}^{n-1} q^{ni})\right)\end{aligned}$$

All the sums are zero except the last one, which is n, so $\sum_{i=0}^{n-1} e_i = 1$. □

Lemma 3.7 *Let R be a graded ring, and M a graded R-module. Then* $\mathrm{Ann}M$ *is a graded ideal.*

Proof. Since every element of M is a sum of homogeneous elements, $\mathrm{Ann}M$ is the intersection of the annihilators $\mathrm{Ann}\, m$ taken over all the homogeneous elements $m \in M$. If $r \in R$, and m is a homogeneous element of M, then mr is zero if and only if $mr_i = 0$ for all i, where $r = \sum r_i$ is a decomposition of r into its homogeneous components. Hence every element of $\mathrm{Ann}\, m$ is a sum of homogeneous elements, each of which is in $\mathrm{Ann}\, m$. Therefore $\mathrm{Ann}\, m$ is a graded right ideal. Hence $\mathrm{Ann}M$ is graded. □

Lemma 3.8 *Suppose that q is a primitive n^{th} root of unity. The annihilator of $A/(x-1)A$ is $(x^n - 1)$.*

Proof. Write $L = A/(x-1)A$. We view A as a graded k-algebra with $\deg x = 0$ and $\deg y = 1$. Then L is a graded module, so $\mathrm{Ann}L$ is graded. It is clear that $x^n - 1 \in \mathrm{Ann}L$ because it is central and in $(x-1)A$.

The degree i homogeneous component of A is $k[x]y^i$.

There is a right $k[y]$-module decomposition $A = (x-1)A \oplus k[y]$, so $L \cong k[y]$ as a right $k[y]$-module. Identifying these, the action of x on L becomes

$$y^i.x = q^i y^i,$$

so, if $f \in k[x]$, $y^i.f = f(q^i)y^i$. Therefore, a homogeneous element $fy^j \in k[x]y^j$ annihilates L if and only if $f(q^i) = 0$ for all i. But this is equivalent to the condition that $x^n - 1$ divides f. Hence the intersection of $\mathrm{Ann}L$ with the degree i component of A is $(x^n - 1)k[x]y^i$. The result follows. □

Theorem 3.9 *Suppose that q is a primitive n^{th} root of unity. Let L be a line module that is parallel to one of the axes. Then* $\mathsf{Mod}A/\operatorname{Ann}L \cong \mathsf{Mod}Q$, *where Q is the quiver*

$$\begin{tikzcd} \underset{0}{\bullet} \arrow[r] & \underset{1}{\bullet} \arrow[r] & \bullet & \cdots & \bullet \arrow[r] & \underset{n-2}{\bullet} \arrow[r] & \underset{n-1}{\bullet} \arrow[llllll, bend right] \end{tikzcd}$$

Proof. By the symmetry, and by using linear automorphisms of $\mathsf{Mod}A$, we can assume that $L = A/(x-1)A$. Then $\operatorname{Ann}L = (x^n - 1)$. Set $\bar{A} = A/(x^n-1)$. Define the idempotents e_i as in Lemma 3.6. A simple calculation shows that

$$e_{i-1}y = ye_i \tag{3-5}$$

for all i

We prove the equivalence of categories by exhibiting the two functors that implement it.

Let M be a right $\bar{A}$-module. Assign to the vertex of Q labelled i the vector space Me_i. If α_i denotes the arrow from vertex i to vertex $i+1$, we associate to α_i the linear map $Me_i \to Me_{i+1}$ given by right multiplication by y. This makes sense because of (3-5). Therefore M becomes a representation of Q.

Conversely, suppose we are given a representation of Q. If M_i is the component at the vertex i, we define $M = \oplus_{i=0}^{n-1} M_i$. We make this into a right $\bar{A}$-module by declaring that x act as $\varepsilon_0 + q\varepsilon_1 + q^2\varepsilon_2 + \ldots + q^{n-1}\varepsilon_{n-1}$, where ε_i is the trivial path at vertex i, and that y act as $\alpha_0 + \alpha_1 + \ldots + \alpha_{n-1}$. To show that this extends unambiguously to an $\bar{A}$-module action we must check that the relations $x^n = 1$ and $yx = qxy$ are satisfied by these two linear operators on M.

Since the ε_i are a complete set of orthogonal idempotents, $(\varepsilon_0 + q\varepsilon_1 + \ldots + q^{n-1}\varepsilon_{n-1})^n = 1$. On the other hand, since $\varepsilon_i\alpha_i\varepsilon_{i+1} = \alpha_i$ and any other product $\varepsilon_j\alpha_i\varepsilon_k$ is zero, yx acts as

$$\begin{aligned}(\alpha_0 + \alpha_1 + \ldots + \alpha_{n-1})&(\varepsilon_0 + q\varepsilon_1 + \ldots + q^{n-1}\varepsilon_{n-1})\\ &= q\alpha_0 + q^2\alpha_1 + \ldots + q^{n-1}\alpha_{n-2} + \alpha_{n-1}\end{aligned}$$

and xy acts as

$$\begin{aligned}(\varepsilon_0 + q\varepsilon_1 + \ldots + q^{n-1}\varepsilon_{n-1})&(\alpha_0 + \alpha_1 + \ldots + \alpha_{n-1})\\ &= \alpha_0 + q\alpha_1 + \ldots + q^{n-1}\alpha_{n-1}.\end{aligned}$$

Hence the relation $yx = qxy$ is satisfied. □

It is natural to ask which closed points lie on which lines.

Lemma 3.10 *Suppose that q is a primitive n^{th} root of unity. Let $L = A/(\alpha x + \beta y - \gamma)A$, and let p be a closed point of degree n in $\mathbb{A}^2_q$. Then there is a surjective map $L \to \mathcal{O}_p$ if and only if $f(p)$ lies on the line $\alpha^n x^n + \beta^n y^n - \gamma^n = 0$ in* $\operatorname{Spec} k[x^n, y^n]$.

Proof. Let p be the point with $\mathcal{O}_p \cong V(\zeta,\xi)$. Then $f(p) = (\lambda,\mu) = (\zeta^n,\mu^n)$.

There is a surjective map from L to $V(\zeta,\xi)$ if and only if there are some scalars λ_i, not all zero such that $\alpha x + \beta y - \gamma$ annihilates $v = \sum_{i=0}^{n-1} \lambda_i v_i$. Since

$$v.(\alpha x + \beta y - \gamma) = \sum_{i=0}^{n-1} (\alpha\zeta q^i \lambda_i + \beta\xi\lambda_{i-1} - \gamma\lambda_i)v_i,$$

there is a surjective map from $L \to V(\zeta,\xi)$ if and only if there is a non-trivial solution $(\lambda_0,\dots,\lambda_{n-1})$ to the system of equations

$$(\alpha\zeta q^i - \gamma)\lambda_i + \beta\xi\lambda_{i-1} = 0.$$

Writing this as a matrix equation, and taking the determinant, we see that there is a non-trivial solution if and only if

$$\prod_{i=0}^{n-1} (\alpha\zeta q^i - \gamma) - (-\beta\xi)^n = 0.$$

This can be rewritten as the condition that

$$(\alpha\zeta)^n + (\beta\xi)^n - \gamma^n = 0.$$

The result now follows at once. $\square$

We continue to suppose that q is a primitive n^{th} root of unity. Lemma 3.10 provides answers to several natural questions concerning the points and lines in $\mathbb{A}_q^2$. Given a closed point in $\mathbb{A}_q^2$, there is a family of lines parametrized by $\mathbb{P}^1$ passing through it. Given any two closed points in $\mathbb{A}_q^2$, there is a single line passing through both of them. Each line L in $\mathbb{A}_q^2$ passes through infinitely many closed points, so there is an injective map $L \to \prod \mathcal{O}_p$, where the product is taken over all the closed points lying on L; it follows that the common annihilator of all those points also annihilates L, so $\alpha^n x^n + \beta^n y^n - \gamma^n$ annihilates $A/(\alpha x + \beta y + \gamma)A$. If $\alpha\beta\gamma \neq 0$, then $\alpha^n x^n + \beta^n y^n - \gamma^n$ annihilates n points on each axis, but $L = A/(\alpha x + \beta y + \gamma)A$ passes through only one point on each axis. The explanation is that L contains submodules isomorphic to $L(-i,0)$ and $L(0,-i)$, $0 \leq i \leq n-1$, and these pass through the other degree one points annihilated by $\alpha^n x^n + \beta^n y^n - \gamma^n$.

Let p and q be points of $\mathbb{A}_q^2$. Is there a map $g : \mathbb{A}^1 = \operatorname{Spec} k[t] \to \mathbb{A}_q^2$ such that $g(0) = p$, $g(1) = q$, and $g_*(k[t]) \cong A/(\alpha x + \beta y + \gamma)A$. Each point $s \in \mathbb{A}^1$ appears in an exact sequence

$$0 \to k[t] \to k[t] \to \mathcal{O}_s \to 0,$$

so there will be an exact sequence

$$0 \to g_* k[t] \to g_* k[t] \to g_* \mathcal{O}_s.$$

Suppose, for argument's sake, that $\alpha \neq 0$. Then $B = A/A(\alpha x + \beta y - \gamma)$ can be identified with $k[y]$, and if we set $t = y^n$, then L becomes an A-$k[t]$-bimodule, so gives a map $g : \operatorname{Spec} k[t] \to A$. Examine this map.

The quotient of A by the annihilator of $A/(\alpha x+\beta y+\gamma)A$ is of the form $A/(z)$ where $z \in kx^n + ky^n + k$. Moreover, this ring is isomorphic to a doubly broken line in general. We return to this in section...???

One would expect the lines through the origin to behave differently. This is indeed the case. Up to a linear automorphism, any line module that passes through the origin, and is not either axis, is of the form $L = A/(x-y)A$. Notice that $L(i,j) \cong L(i',j')$ if n divides $(i+j)-(i'+j')$, so up to isomorphism there are only n distinct $L(i,j)$. We denote them by $L(i)$ for $i \in \mathbb{Z}_n$; explicitly,

$$L(i) = A/(q^i x - y)A.$$

There are short exact sequences

$$0 \to L(i-1) \to L(i) \to \mathcal{O}_{(0,0)} \to 0$$

because the kernel of the map $A/(q^i x - y)A \to A/(x,y)$ is generated by $\bar{x}$, and because $x(q^{i-1}x - y) = q^{i-1}x^2 - q^{-1}yx = q^{-1}(q^i x - y)x$, $\bar{x}$ is annihilated by $q^i x - y$.

We define $\mathsf{Mod}L = \mathsf{Mod}A/\operatorname{Ann}L$. To compute $\operatorname{Ann}L$ the reader should warm-up by computing $(x+y)^m$. It will follow that if q is a primitive n^{th} root of unity, then

$$(x - q^i y)^n = x^n + (-1)^n y.$$

Then one can show that $\operatorname{Ann}L = (x^n + (-1)^n y^n)$. We need to understand this ring.

Now suppose that q is not a root of unity.

We state a special case of a result due to Smith and Zhang [228]. Let $L = A/(\alpha x + \beta y - \gamma)$ with $\alpha\beta\gamma \neq 0$. If we define $\mathsf{Mod}L$ to be the smallest full subcategory of $\mathsf{Mod}X$ that contains $\varinjlim L(i,j)$, and is closed under direct sums, submodules and quotient modules, then

$$\mathsf{Mod}L \cong \mathsf{GrMod}_{\mathbb{Z}^2} k[x,y]/Fdim.$$

If $\alpha = 0$, then $\mathsf{Mod}L \cong \mathsf{GrMod}k[x]$. It follows from its definition that $\mathsf{Mod}L$ is a weakly closed subspace of X, so there is a weak map embedding the graded line as a weakly closed subspace of $\mathbb{A}^2_q$.

Questions. Given a single line in $X = \mathsf{Mod}A$, the analogy with the commutative case suggests that X is the "union" of all the lines parallel to the given line. What might this mean in the non-commutative setting? We have not defined the union of subspaces. Set $L = A/(\alpha x+\beta y)A$, and define $L_\gamma = A/(\alpha x + \beta y - \gamma)A$, $\gamma \in k$. Perhaps one can show that if M is any finitely generated A-module of GK-dimension one, then $\operatorname{Ext}^1(M, L_\gamma) \neq 0$ for some $\gamma \in k$. A better result would be to take a fixed $a \in A$, set $N_\gamma = A/(a-\gamma)A$, and show that $\operatorname{Ext}^1_A(M, N_\gamma) \neq 0$ for some $\gamma \in k$.

One can ask the same sort of question for an Ore extension $S = R[x;\sigma,\delta]$. The "union" of the modules $S/(x-\gamma)S$ should cover the space $\mathsf{Mod}S$.

A special case of a union is an open cover (see Definition 3.7.6).

EXERCISES

3.1 Consider $A = k_q[x,y]$ with q not a root of unity. Show that $A/(xy-1)A$ is an infinite dimensional simple A-module. We should probably think of it as a "hyperbola module", and if we were to draw a picture of it, it would be a dashed version of the curve $xy = 1$ that is drawn in the commutative case.

3.2 Consider $A = k_q[x,y]$ with q not a root of unity. Define a line module to be one of the form $L = A/(\alpha x + \beta y + \gamma)A$ with $(\alpha, \beta) \neq 0$. Show that there are four kinds of line modules, the axes $A/(x)$ and $A/(y)$, the lines "parallel" to one of the axes where $\alpha\beta = 0$, the skew lines where $\alpha\beta \neq 0$, and the lines through the origin ($\gamma = 0$). Discuss the closed points that lie on the various line modules.

3.3 Consider $A = k_q[x,y]$ with q not a root of unity. Develop a theory of intersection that is like that developed for $\mathsf{Mod}U$. Discuss the intersection of the line modules.

3.4 Consider $A = k_q[x,y]$ with q not a root of unity. What is the intersection multiplicity of a hyperbola module with the various line modules?

3.5 Consider $A = k_q[x,y]$ with q not a root of unity. Define $\mathsf{Mod}L$ when L is a line module. Show that the line $\mathsf{Mod}L$ is isomorphic to the graded line when L is parallel to one of the axes.

3.6 Consider $A = k_q[x,y]$ with q not a root of unity. I do not know a good description of $\mathsf{Mod}L$ when L is a line module through the origin.

3.7 Consider $A = k_q[x,y]$ when q is a primitive n^{th} root of unity. Show that there is a ring isomorphism
$$A/(x^n y^n - 1) \cong M_n(k[t,t^{-1}]).$$

3.8 When q is a primitive n^{th} root of unity prove that there is an algebra isomorphism $A/(x^n - 1) \cong kQ$, where Q is the quiver in Theorem 3.9.

3.9 Show that $x = e_0 + qe_1 + \ldots + q^{n-1}e_{n-1}$ in Lemma 3.6.

3.10 Suppose that q is a primitive n^{th} root of unity (where $n \neq 1$). Let R be the ring $k[x,y]$ with defining relation $yx - qxy = 1$. Find the finite dimensional simple R-modules. (Hint: First show that the center of R is $k_q[x,y]$.) See

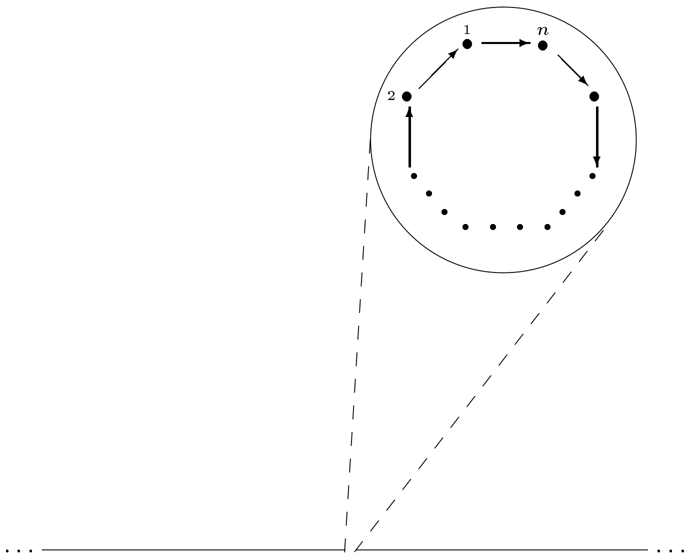
`http://www.math.tamu.edu/~edward.letzter/summer.1998/paper.ps.`

3.11 Describe the lines for the space $\mathsf{Mod}R$ in the previous exercise, and the points lying on them. Let $L = A/(x-\alpha)A$ with $\alpha \neq 0$, and define $\mathsf{Mod}L = A/\operatorname{Ann}L$. Is $\mathsf{Mod}L$ isomorphic to $\mathsf{Mod}Q$ where Q is the quiver in Theorem 3.9? What does $\mathsf{Mod}L$ look like when $L = A/(x-y)A$?

3.12 Show that the algebra $k[x,y]$ with relations $xy + yx = x^2 - y^2 = 0$ is isomorphic to the algebra $k[u,v]$ with relations $u^2 = v^2 = 0$.

4.4 A product of two broken lines

This section exhibits some non-commutative curves that are natural generalizations of the the broken lines discussed in Chapter 3. Recall that broken lines look like $\mathbb{A}^1$ except that 0 is replaced by, or broken into, several points linked into a cycle. The picture is



(4-6)

This curve is the path algebra of the quiver

$$\begin{tikzcd} \underset{n}{\bullet} \arrow[r] & \underset{n-1}{\bullet} \arrow[r] & \bullet & \cdots & \bullet \arrow[r] & \underset{2}{\bullet} \arrow[r] & \underset{1}{\bullet} \arrow[llllll, bend right] \end{tikzcd}$$

The picture suggests that there should be similar examples where instead of breaking apart just one point of the affine line, one breaks apart several points. Here is a picture of such a curve.

(4-7)

Example 4.1 Let $R = k[u,v]/(u^2 - u, v^2 - v)$. Let $A = k_q[x,y]$ with $q = -1$. There is a surjective map $A \to R$ defined by

$$x \mapsto u + v - 1, \qquad y \mapsto u - v.$$

A computation shows that $R \cong A/(x^2+y^2-1)$. Under this isomorphism, we have the following correspondences:

$$\begin{aligned} R/uR &\leftrightarrow A/(x+y+1)A, \\ R/(u-1)R &\leftrightarrow A/(x+y-1)A, \\ R/vR &\leftrightarrow A/(x-y+1)A, \\ R/(v-1)R &\leftrightarrow A/(x-y-1)A. \end{aligned}$$

It is not difficult to show that R has four one-dimensional modules, and a family of two-dimensional simple modules parametrized by the affine line less two points. Thus $\mathsf{Mod}R$ is an affine line broken at two points, and each of those two points has been broken into two points. The breaking occurs where the line $\mathsf{Mod}A/(x^2+y^2-1)$ meets the two axes $x=0$ and $y=0$ in the quantum plane.

$\boxed{\text{Paul}}$ Explain the connection between $\mathsf{Mod}R$ and the representations of the quiver

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \bullet \arrow[l] \arrow[u] \end{tikzcd}$$

$\Diamond$

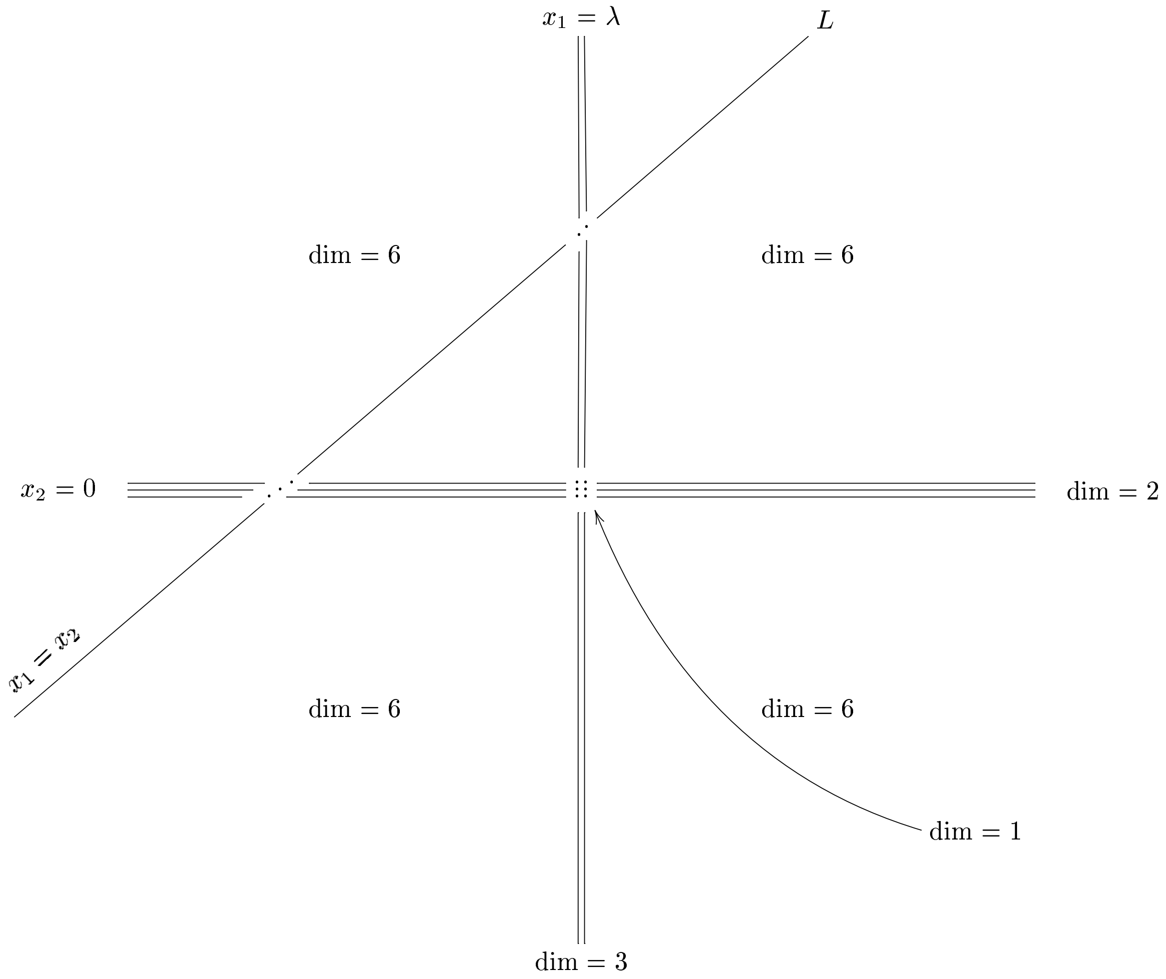
We will motivate our construction of broken lines by describing a surface which we would expect to contain lines that look like the previous picture.

The surface we have in mind is a product $X = X_1 \times X_2$ where each X_i is a broken line over an algebraically closed field k. Write $X_i = \mathsf{Mod}R_i$. Since R_1 and R_2 are finite modules over their centers, so is $R_1 \otimes_k R_2$. By Proposition 3.4.16, the simple $R_1 \otimes_k R_2$-modules are all of the form $V_1 \otimes_k V_2$ where V_i is a simple R_i-module. Hence the closed points in $\mathsf{Mod}R_1 \otimes_k R_2$ are in bijection with pairs (p_1, p_2) where p_i is a closed point in $\mathsf{Mod}R_i$. The dimension of the module $V_1 \otimes_k V_2$ is the product of the dimensions of the individual V_i. Hence the picture we draw of $\mathsf{Mod}R_1 \otimes_k R_2$ should be the cartesian product of the pictures for $\mathsf{Mod}R_1$ and $\mathsf{Mod}R_2$.

To be explicit, suppose that R_1 is the affine line with coordinate function x_1 broken at the point $x_1 = \lambda$, and this point is broken into two points. Suppose that R_2 is the affine line with coordinate function x_2 broken at the point $x_2 = 0$, and this point is broken into three points. Thus,

$$R_1 = \begin{pmatrix} k[x_1] & (x_1-\lambda) \\ k[x_1] & k[x_1] \end{pmatrix} \quad \text{and} \quad R_2 = \begin{pmatrix} k[x_2] & (x_2) & (x_2) \\ k[x_2] & k[x_2] & (x_2) \\ k[x_2] & k[x_2] & k[x_2] \end{pmatrix}.$$

A picture of $\mathsf{Mod} R_1 \otimes R_2$ is



There are six one-dimensional simples, two families of three-dimensional simples parametrized by $\mathbb{A}^1 \backslash \{0\}$ lying on $x_1 = \lambda$, three families of two-dimensional simples parametrized by $\mathbb{A}^1 \backslash \{0\}$ lying on $x_2 = 0$, and a family of six-dimensional simples parametrized by $\mathbb{A}^2 \backslash \{\text{the two axes}\}$.

Primitive intuition suggests that a skew line, such as the one labelled L in the picture above, would be like an affine line broken at two points: one point would break into two three-dimensional simples, the other into three two-dimensional simples, and the other points on L would correspond to six-dimensional simples parametrized by the affine line less two points.

An explicit candidate for L is the line $x_1 = x_2$. Thus we define L by

$$\mathsf{Mod} L = \mathsf{Mod} R_1 \otimes_k R_2 / (x_1 \otimes 1 - 1 \otimes x_2).$$

If we denote both x_1 and x_2 by x, then $x_1 \otimes 1 - 1 \otimes x_2 = x \otimes 1 - 1 \otimes x$, and it is easy to see that

$$R_1 \otimes_k R_2/(x_1 \otimes 1 - 1 \otimes x_2) \cong R_1 \otimes_{k[x]} R_2.$$

We have made the trivial observation that if

$$R_1 \cong \begin{pmatrix} k[x] & (x-\lambda) \\ k[x] & k[x] \end{pmatrix} \quad \text{and} \quad R_2 \cong \begin{pmatrix} k[x] & (x) & (x) \\ k[x] & k[x] & (x) \\ k[x] & k[x] & k[x] \end{pmatrix},$$

then

$$R_1 \otimes_{k[x]} R_2 \cong \begin{pmatrix} k[x] & (x) & (x) & (x-\lambda) & (x^2-\lambda x) & (x^2-\lambda x) \\ k[x] & k[x] & (x) & (x-\lambda) & (x-\lambda) & (x^2-\lambda x) \\ k[x] & k[x] & k[x] & (x-\lambda) & (x-\lambda) & (x-\lambda) \\ k[x] & k[x] & k[x] & k[x] & (x) & (x) \\ k[x] & k[x] & k[x] & k[x] & k[x] & (x) \\ k[x] & k[x] & k[x] & k[x] & k[x] & k[x] \end{pmatrix}.$$

One should think of the formation of $R_1 \otimes_{k[x]} R_2$ as analogous to the formation of a fiber product: There is a commutative diagram of rings

$$\begin{array}{ccc} R_1 \otimes_{k[x]} R_2 & \longleftarrow & R_2 \\ \uparrow & & \uparrow \\ R_1 & \longleftarrow & k[x] \end{array}$$

and a corresponding diagram of spaces

$$\begin{array}{ccc} X \times_{\mathbb{A}^1} Y & \longrightarrow & Y \\ \uparrow & & \uparrow \\ X & \longleftarrow & \mathbb{A}^1 \end{array}.$$

There is an obvious generalization. We can break the affine line at any set of points we choose, and each of the chosen points can be broken into as many points as we wish. Fix distinct points $\lambda_1, \lambda_2, \dots, \lambda_s$ and positive integers $n_1, \dots, n_s$. We define

$$\mathbb{A}^1(\lambda_1, \dots, \lambda_s; n_1, \dots, n_s)$$

to be the non-commutative curve with coordinate ring $R_1 \otimes_{k[x]} \otimes \dots \otimes_{k[x]} R_s$, where R_i is the subalgebra

$$\begin{pmatrix} k[x] & (x-\lambda_1) & (x-\lambda_1) & \dots & (x-\lambda_1) \\ k[x] & k[x] & (x-\lambda_1) & \dots & (x-\lambda_1) \\ \dots & & & & \vdots \\ k[x] & k[x] & k[x] & \dots & (x-\lambda_1) \\ k[x] & k[x] & k[x] & \dots & k[x] \end{pmatrix}.$$

of $M_{n_i}(k[x])$. This curve is an affine line broken as described.

There is a further obvious generalization. We could begin with any commutative curve in place of the affine line, and break it at any collection of points.

Show that one can find lots of these broken curves in the quantum plane $\mathsf{Mod}k_q[x,y]$ when q is a root of unity. For example, consider some skew lines, and parabolas, and analogues of cubic curves.

EXERCISES

4.1 Show that the ring R in Example 4.1 is isomorphic to

$$\begin{pmatrix} k[t] & (t) \\ (t-1) & k[t] \end{pmatrix}.$$

4.5 Quadrics related to the Lie algebra $\mathfrak{sl}_2$

In the section k denotes an algebraically closed field of characteristic zero.

The Lie algebra $\mathfrak{sl}_2$ of 2×2 trace zero matrices has basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and relations

$$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f. \tag{5-1}$$

Its enveloping algebra, $U(\mathfrak{sl}_2)$, is the associative k-algebra $k[e,f,h]$ with defining relations

$$ef - fe = h, \quad he - eh = 2e, \quad hf - fh = -2f.$$

It follows from the PBW Theorem that $U(\mathfrak{sl}_2)$ has basis

$$\{e^i f^j h^k \mid i,j,k \geq 0\},$$

and that it is a noetherian domain. This ring has played an influential role in the development of non-commutative ring theory. In this section we study the space $\mathsf{Mod}U(\mathfrak{sl}_2)$. We view it as a non-commutative analogue of $\mathbb{A}^3$.

Convention. It is standard practice when studying the representation theory of Lie algebras to deal with *left* modules, and we will do that in this section, thus breaking with our usual convention of dealing with right modules.

Computations. Various computations inside $U(\mathfrak{sl}_2)$ are needed for some of the proofs in this section. One such is that

$$ef^j = f^j e + j f^{j-1}(h - j + 1). \tag{5-2}$$

This is proved by induction on j.

Weights. If M is a left $\mathfrak{sl}_2$-module, an h-eigenvector $m \in M$ is called a weight vector, and if $h.m = \lambda m$, we say that m has weight λ. The set of elements of weight λ is a subspace called the λ-weight space, and is denoted by M_λ. It is easy to see that $e.M_\lambda \subset M_{\lambda+2}$ and $f.M_\lambda \subset M_{\lambda-2}$. If M is generated by M_λ, and $M = \sum_{i\geq 0} M_{\lambda-2i}$, we call λ the highest weight, and call M a highest weight module.

The adjoint action. We make $U(\mathfrak{sl}_2)$ a left module over itself by defining

$$x.u = xu - ux$$

for $x \in \mathfrak{sl}_2$ and $u \in U(\mathfrak{sl}_2)$. One needs to check that this action of $\mathfrak{sl}_2$ can be extended to $U(\mathfrak{sl}_2)$. It does. We call this the adjoint representation. Looking in particular at the adjoint action of h, one sees that e has weight 2, and f has weight -2. It is easy to show that $U_\lambda U_\mu \subset U_{\lambda+\mu}$, so each $e^i f^j h^k$ is a weight vector of weight $2(i-j)$. Thus

$$U(\mathfrak{sl}_2) = \oplus_{n\in\mathbb{Z}} U(\mathfrak{sl}_2)_{2n}. \tag{5-3}$$

Thus $U(\mathfrak{sl}_2)$ is a $\mathbb{Z}$-graded algebra with the homogeneous components being the weight spaces.

The Casimir element is

$$\Omega = 2ef + 2fe + h^2.$$

Since Ω has weight zero it commutes with h. More is true.

Lemma 5.1 *The center of $U(\mathfrak{sl}_2)$ is $k[\Omega]$, the polynomial ring generated by the Casimir element.*

Proof. Let Z denote the center of $U(\mathfrak{sl}_2)$, and write U_n for the n-weight space under the adjoint action of h. Clearly $Z \subset U_0$. A calculation shows that Ω commutes with e, f, and h, so $k[\Omega] \subset Z$.

It is clear that U_0 is spanned by all $e^i f^i h^j$, and that $k[ef, h] \subset U_0$. An induction argument shows that $e^i f^i \in k[ef, h]$, so $U_0 = k[ef, h]$. But this equals $k[\Omega, h]$ which is a commutative polynomial ring in two variables. If we take an arbitrary element of U_0 and write it as a sum of elements according to the decomposition $U_0 = \oplus k[\Omega]h^j$, it is clear that the only elements of U_0 commuting with e are those in $k[\Omega]$. □

Proposition 5.2 *Consider the polynomial ring $k[X, Y]$ with its standard grading.*

1. *For each integer $n \geq 0$ the action of*

$$e = X\partial_Y, \quad h = X\partial_X - Y\partial_Y, \quad f = Y\partial_X \tag{5-4}$$

makes the degree n component $k[X, Y]_n$ a simple $\mathfrak{sl}_2$-module of dimension $n + 1$.

2. $k[X,Y]_n$ *is the unique simple* $\mathfrak{sl}_2$*-module of dimension* n *up to isomorphism.*

3. $k[X,Y]_n$ *is annihilated by* $\Omega - (n+1)^2 + 1$.

Proof. (1) It is straightfoward to check that the relations (5-1) are satisfied by the given differential operators. Thus $k[X,Y]$ becomes an $\mathfrak{sl}_2$-module. It is clear that each homogeneous component is preserved by the action, so it remains to check that each component is simple. Since X^iY^{n-i} has weight $2i-n$, and e acts by lowering the Y-degree by one, and f acts by lowering the X-degree by 1, it is easy to see that $k[X,Y]_n$ is simple.

(2) Suppose that V is a simple $\mathfrak{sl}_2$-module of dimension $n+1$. Then V has an h-eigenvector, say v, with eigenvalue μ say. If $e^iv \neq 0$, then it is a weight vector with weight $\mu+2i$. Since $\dim_k V < \infty$, there is a weight vector, m say, such that $em = 0$. Let λ be the weight of m. Since V is simple, $V = Um$. Therefore, by the PBW basis, V is spanned by the elements $f^jh^ke^im$; hence V is spanned by elements f^jm. These are weight vectors of weight $\lambda - 2j$. Therefore, $m, fm, \ldots, f^nm$ is a basis for V, and $f^{n+1}m = 0$.

It follows from (5-2) that

$$0 = ef^{n+1}m = (n+1)f^n(\lambda - n)m.$$

Therefore $\lambda = n$. Now define a linear map $V \to k[X,Y]_n$ by

$$f^jv \mapsto \binom{n}{j} X^{n-j}Y^j.$$

It is easy to check that this is a $U(\mathfrak{sl}_2)$-module map, so $V \cong k[X,Y]_n$.

(3) If the action of e, f, and h, are represented by matrices with respect to the basis X^iY^{n-i}, a simple calculation shows that Ω is sent to diagonal matrix $(n+1)^2-1$. Alternatively, each $X^iY^{(n-i)}$ is an eigenvalue for ef and fe, so one can directly compute the action of $2ef + 2fe + h^2$ on each X^iY^{n-i}. $\square$

We will write V_n for the $(n+1)$-dimensional simple module. The one-dimensional representation V_0 is called the trivial module. Notice that $x.V_0 = 0$ for all $x \in \mathfrak{sl}_2$.

The action of $\mathfrak{sl}_2$ on $k[X,Y]$ makes each X^iY^j an eigenvector for the action of h, with eigenvalue $i-j$. Thus $k[X,Y]_n$ has weights $n, n-2, \ldots, 2-n, -n$. Hence V_n is a highest weight module with highest weight n. We call X^n a highest weight vector.

Lemma 5.3 *Let* $\mathfrak{g}$ *be a Lie algebra. If* M *and* N *are* $\mathfrak{g}$*-modules, then* $\operatorname{Hom}_k(M,N)$ *becomes a* $\mathfrak{g}$*-module via*

$$(x.\theta)(m) := x.\theta(m) - \theta(x.m) \tag{5-5}$$

for $x \in \mathfrak{g}$ *and* $m \in M$.

Proof. We must check that $x.(y.\theta) - y.(x.\theta) = [x,y].\theta$ for all $x, y \in \mathfrak{g}$. This is done by evaluating each side at $m \in M$ and calculating. □

Proposition 5.4 *Every finite dimensional module over $\mathfrak{sl}_2$ is a direct sum of simple modules.*

Proof. It suffices to show that if L and N are finite dimensional simple modules, then every exact sequence $0 \to L \to M \to N \to 0$ splits. We will follow the proof in [114, Section 6.3].

First suppose that $\dim_k N = 1$. If the central annihilators of L and N are different, then M splits by the remarks after Lemma 3.5.19. Therefore, by part (3) of Proposition 5.2, it suffices to show that this splits when $L \cong N$. So, suppose that $L \cong N$. Because $xN = xL = 0$ for all $x \in \mathfrak{sl}_2$, $xyM = 0$ for all $x, y \in \mathfrak{sl}_2$. It follows that $[x,y]M = 0$. However, $\mathfrak{sl}_2$ is spanned by the elements $[x,y]$ as x and y run over $\mathfrak{sl}_2$. Therefore $xM = 0$ for all $x \in \mathfrak{sl}_2$. Hence any vector space decomposition $M = L \oplus L'$ is an $\mathfrak{sl}_2$-module decomposition, whence $M \cong L \oplus N$ as $\mathfrak{sl}_2$-modules.

Now let L and N be arbitrary. By Lemma 5.3, $\operatorname{Hom}_k(M, L)$ is an $\mathfrak{sl}_2$-module under the action defined by (5-5). A simple calculation shows that the subspaces $S \subset T \subset \operatorname{Hom}_k(M, L)$ defined by

$$
\begin{aligned}
S &:= \{\theta \mid \theta|_L = 0\} \\
T &:= \{\theta \mid \theta|_L \text{ is multiplication by some } \lambda \in k\}
\end{aligned}
$$

are $\mathfrak{sl}_2$-modules. However $\dim_k T/S = 1$, so the previous paragraph shows that the sequence $0 \to S \to T \to T/S \to 0$ splits. We write $T = S \oplus k\theta$ with $k\theta$ an $\mathfrak{sl}_2$-module. Thus $x.\theta = 0$ for all $x \in \mathfrak{sl}_2$. Therefore $x.\theta(m) = \theta(xm)$ for all $m \in M$ and $x \in \mathfrak{sl}_2$. This says that $\theta : M \to L$ is a $\mathfrak{sl}_2$-module homomorphism. By definition of θ, it does not vanish on L, so $\ker\theta \oplus L = M$, as required. □

Thus, in sharp contrast to the analogues of $\mathbb{A}^2$ in sections one and three, the set of closed points in $\mathsf{Mod}U(\mathfrak{sl}_2)$ is discrete, and there are no links between them.

We now introduce the universal highest weight modules, $M(\lambda)$. Because they are defined by two linear relations, they should be thought of as structure modules of lines in $\mathsf{Mod}U(\mathfrak{sl}_2)$.

Definition 5.5 If $\lambda \in k$, the Verma module with highest weight λ is

$$M(\lambda) := U/Ue + U(h - \lambda).$$

◊

It follows from the PBW basis that

$$U = (Ue + U(h - \lambda)) \oplus k[f],$$

so $M(\lambda)$ has basis given by the images of f^n for $n \geq 0$. Each f^n is a weight vector of weight $\lambda - 2n$, so $M(\lambda)$ is a highest weight module, with highest weight λ. It is universal in the sense that if M is any module generated by a highest weight vector of weight λ, then there is a surjection $M(\lambda) \to M$ because the highest weight vector is annihilated by e and $h - \lambda$.

Our labelling of Verma modules differs from that in Dixmier [75]. What we call $M(\lambda)$ corresponds to the Verma module he labels $M(\lambda - 1)$. His labelling is the standard one; the difference is that we do not shift by the half-sum of the positive roots. We have chosen our labelling because we will only discuss the special case of $\mathfrak{sl}_2$, and the extra baggage is not necessary for that.

Proposition 5.6 *Consider the Verma module $M(\lambda)$.*

1. *$M(\lambda)$ is annihilated by $\Omega - \lambda^2 - 2\lambda$.*

2. *If $n \in \mathbb{N}$, then there is a non-split exact sequence*

$$0 \to M(-n-2) \to M(n) \to V_n \to 0.$$

3. *$M(\lambda)$ is simple if and only if $\lambda \notin \mathbb{N}$.*

Proof. (1) Let v be a highest weight vector. A simple calculation shows that $\Omega.v = (2\lambda + \lambda^2)v$.

(2) By Proposition 5.2, $X^n \in k[X,Y]_n = V_n$ is a highest weight vector, with weight n. Since V_n is simple there is a surjective module map $\varphi : M(n) \to V_n$. Let $m \in M(n)$ be a highest weight vector. Since X^n is killed by f^i for $i \geq n+1$, $\ker\varphi$ has basis $\{f^i m \quad i \geq n+1\}$. Hence $\ker\varphi$ is generated by $f^{n+1}m$. This element has weight $n - 2(n+1) = -n-2$. And, by (5-2),

$$e.f^{n+1}m = f^{n+1}em + (n+1)f^n(h-n)m = 0.$$

Hence $f^{n+1}m$ is a highest weight vector in $\ker\varphi$, so there is a surjective module map $M(-n-2) \to \ker\varphi$. Since both these are isomorphic to $k[f]$ as $k[f]$-modules, φ is an isomorphism. This gives the short exact sequence in the statement of the proposition. This sequence is non-split because $M(\lambda)$ is indecomposable as a $k[f]$-module.

(3) We have just seen that $M(\lambda)$ is not simple if $\lambda \in \mathbb{N}$, so it remains to show that if $\lambda \notin \mathbb{N}$, then every non-zero element m' in $M(\lambda)$ generates it. Let m be a highest weight vector in $M(\lambda)$. It suffices to show that $e^n m'$ is a non-zero scalar multiple of m for some n. Choose n maximal so that m' has a non-zero component of weight $\lambda - 2n$, say m''. Replacing m' by a scalar multiple, we can assume that $m'' = f^n m$. Then $e^n m' = e^n m''$, so we need only show that $e^n f^n m \neq 0$. If $n = 0$ we are done, so suppose that $n > 0$. By (5-2), $ef^n m = nf^{n-1}(\lambda - n + 1)m$. Since $\lambda \notin \mathbb{N}$, this is a non-zero scalar multiple of $f^{n-1}m$. By induction on n, $e^n f^n m \neq 0$. Hence $M(\lambda)$ is simple, as claimed. $\square$

For each $\lambda \in k$, we will write

$$\begin{aligned} J_\lambda &:= (\Omega - \lambda^2 - 2\lambda), \\ U_\lambda &:= U(\mathfrak{sl}_2)/J_\lambda, \\ Q_\lambda &:= \mathsf{Mod} U_\lambda. \end{aligned}$$

Because we view $U(\mathfrak{sl}_2)$ as the coordinate ring of a non-commutative analogue of $\mathbb{A}^3$, we think of the closed subspaces Q_λ as quadric surfaces. Because $M(\lambda)$ is annihilated by $\Omega - \lambda^2 - 2\lambda$, we should think of it as the structure module of a line lying on Q_λ.

Later we will see that Q_{-1} behaves like a quadric cone, and that the other Q_λs behave like smooth quadrics in $\mathbb{A}^3$.

Commutative quadrics in $\mathbb{A}^3$. Let's review the commutative quadrics $2xz - y^2 + \lambda^2 = 0$ in $\mathbb{A}^3$. The quadric is smooth if $\lambda \neq 0$. There are two families of lines on it, namely

$$\begin{aligned} L_{(\alpha,\beta)} \quad &\text{given by} \quad 2\alpha x - \beta(y-\lambda) = \beta z - \alpha(y+\lambda) = 0, \\ \ell_{(\alpha,\beta)} \quad &\text{given by} \quad 2\alpha x - \beta(y+\lambda) = \beta z - \alpha(y-\lambda) = 0, \end{aligned}$$

indexed by $(\alpha, \beta) \in \mathbb{P}^1$. It is easy to show that each family of lines provides a ruling, making the quadric a ruled surface in two different ways. We say that a family of lines provides a ruling of a subvariety of $\mathbb{A}^n$ if the subvariety is the disjoint union of the lines.

The degenerate quadric $2xz = y^2$ has a singular point at the origin. There is a single family of lines on it, namely

$$L_{(\alpha,\beta)} \quad \text{given by} \quad 2\alpha x - \beta y = \beta z - \alpha y = 0,$$

parametrized by $(\alpha, \beta) \in \mathbb{P}^1$. The union of these lines is the whole quadric, but they all pass through the singular point $(0,0,0)$, so do not provide a ruling.

We will show that there is a non-commutative analogue of this. So far, we only have two lines on each Q_λ: Proposition 5.6 says that the line modules $M(\lambda)$ and $M(-\lambda-2)$ lie on Q_λ. (These two modules coincide when $\lambda = -1$.) However, the fact that we have only produced two line modules on each quadric is due to the fact that we worked with a fixed basis of $\mathfrak{sl}_2$. We now make amends for this.

Linear Automorphisms. Our analysis of $U(\mathfrak{sl}_2)$ has been carried out using the basis e, f, h. To carry out a basis-free analysis we should observe that Verma modules can be defined with respect to any basis satisfying the relations (5-1).

If we conjugate a matrix with an element of the group $GL(2)$, the trace of the resulting matrix is same as that of the original matrix. Therefore the action of $GL(2)$ by conjugation on $M_2(k)$ sends $\mathfrak{sl}_2$ to itself. Since conjugation is an algebra automorphism of $M_2(k)$, and since the Lie bracket is defined in terms of the multiplication, conjugation is a Lie algebra automorphism of $\mathfrak{sl}_2$. Therefore the action of $GL(2)$ extends to an action on $U(\mathfrak{sl}_2)$ as algebra automorphisms.

An elementary calculation shows that the action of every $g \in GL(2)$ fixes the Casimir element Ω. A more fundamental explanation of this fact is that Ω can be defined in a basis-free fashion as follows. The adjoint action of $\mathfrak{sl}_2$ on itself gives a map $\rho : \mathfrak{sl}_2 \to M_3(k)$ sending x to $[x, -]$. We now define a pairing $(x, y) = \mathrm{Tr}(\rho(x)\rho(y))$, where Tr denotes the trace. Using the basis elements e, f, and h, one sees that $(-,-)$ is a non-degenerate symmetric bilinear form on $\mathfrak{sl}_2$. This is called the Killing form. Let x_1, x_2, x_3 be a basis for $\mathfrak{sl}_2$, and write x_1^*, x_2^*, x_3^* for the dual basis. The element $x_1x_1^* + x_2x_2^* + x_3x_3^*$ of $U(\mathfrak{sl}_2)$ is then independent of the choice of basis. This element is a non-zero scalar multiple of Ω (see Exercise 8). It is therefore clear that every Lie algebra automorphism of $\mathfrak{sl}_2$ fixes the Casimir element.

Because the Casimir element is fixed by the action of $GL(2)$, each automorphism of $\mathfrak{sl}_2$ induces an algebra automorphism of $U(\mathfrak{sl}_2)/J_\lambda$. We view this as an automorphism of the non-commutative quadric Q_λ.

Explicitly, if σ is an automorphism of a ring R, then there is an adjoint pair (f^*, f_*) of functors $\mathsf{Mod}R \to \mathsf{Mod}R$ which are mutual quasi-inverses. Thus f^* and f_* are auto-equivalences of the category $\mathsf{Mod}R$. Recall that $f^* = - \otimes_R B$, where B denotes the bimodule ${}_RR_R$ with the actions $x.b = \sigma(x)b$ and $b.x = bx$ for $b \in B$ and $x \in R$. We denote f_*M by M^σ. The R-action on M^σ is given by

$$m * x = m\sigma(x) \tag{5-6}$$

for $m \in M$. Thus, as an abelian group $M^\sigma = M$, but the action of R is now given by the operation $*$ in (5-6).

We are working with *left* $U(\mathfrak{sl}_2)$-modules, but the principle remains the same. If $g \in GL(2)$, then the conjugation $x \mapsto gxg^{-1}$ for $x \in \mathfrak{sl}_2$, induces an automorphism γ of $U(\mathfrak{sl}_2)$, and hence new $U(\mathfrak{sl}_2)$-modules $M(\lambda)^\gamma$. These are Verma modules with respect to different Borel subalgebras. Explicitly,

$$M(\lambda)^\gamma \cong U/U\gamma^{-1}(e) + U(\gamma^{-1}(h) - \lambda).$$

We should think of this as giving an action of $GL(2)$ that moves the lines on each Q_λ.

Since conjugation by a diagonal matrix is the identity, we prefer to work with $SL(2)$, the subgroup of matrices of determinant one.

The stabilizer subgroup in $SL(2)$ of $\mathfrak{b}$ consists of the subgroup of upper triangular matrices. We denote this by B. Hence the set of all Borel subalgebras can be identified with $SL(2)/B$. Indeed, with a little more care and technology, we can make this natural, and say that the set of Borel subalgebras is a variety in a natural way (the two-dimensional subspaces of $\mathfrak{sl}_2$ can be given the structure of $\mathbb{P}^2$). However, $SL(2)/B \cong \mathbb{P}^1$. To see this, look at the natural action of $SL(2)$ on k^2. Each line is sent to another line by an element of $SL(2)$, and all lines lie in a single orbit. The stabilizer of the line $\begin{pmatrix} * \\ 0 \end{pmatrix}$ is B. Therefore, the space of lines in k^2 is naturally isomorphic to $SL(2)/B$. But that space of lines is also isomorphic to $\mathbb{P}^1$ by definition.

Lemma 5.7 *For each $\alpha \in k$, the elements*

$$e' = \alpha^2 e - f + \alpha h = \begin{pmatrix} \alpha & \alpha^2 \\ -1 & -\alpha \end{pmatrix},$$
$$h' = -2\alpha e - h = \begin{pmatrix} -1 & -2\alpha \\ 0 & 1 \end{pmatrix},$$
$$f' = -e = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},$$

satisfy the standard relations (5-1).

Proof. These elements are obtained as geg^{-1}, ghg^{-1}, and gfg^{-1}, where

$$g = \begin{pmatrix} \alpha & 1 \\ -1 & 0 \end{pmatrix}$$

□

A basis of h'-eigenvectors is given by $\begin{pmatrix} 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} -\alpha & 1 \end{pmatrix}$. The second of these eigenvectors is annihilated by e'. This data can be interpreted as giving a family of lines parametrized by $\mathbb{P}^1$.

Line modules on Q_λ. We return to the question of the lines on the quadrics Q_λ. Using the notation in Lemma 5.7, we define new Verma modules $M_\alpha(\lambda) = U/Ue' + U(h' - \lambda)$, and $M_\alpha(-\lambda - 2) = U/Ue' + U(h' + \lambda + 2)$ indexed by $\alpha \in k$. These provide two families of line modules on Q_λ.

$\boxed{\text{Paul}}$ Are there maps $f : \mathsf{Mod}U(\mathfrak{sl}_2) \to \mathbb{P}^1$ and $g : \mathsf{Mod}U(\mathfrak{sl}_2) \to \mathbb{P}^1$ so that $f^*\mathcal{O}_\alpha \cong M_\alpha(\lambda)$ and $g^*\mathcal{O}_\alpha \cong M_\alpha(-\lambda - 2)$?

I do not know how to make sense of a non-commutative analogue of the statement that the surface $2xz - y^2 + \lambda^2 = 0$ is the disjoint union of the lines in one of the rulings. One of the problems is that we must phrase such a statement in terms of the category $\mathsf{Mod}Q_\lambda$. For a general λ, the Verma modules $M(\lambda)$ are all simple, and although every Q_λ-module embeds in a suitably large direct product of these simples, that is not relevant. Again for a general λ, U_λ has no finite dimensional simple modules, so it is not helpful to say that every point of Q_λ lies on some line.

We now consider another analogy with the commutative case.

The map $\mathbb{A}^2 \to Q$. Fix a non-zero $\lambda \in k$. Let Q denote the commutative quadric surface $2xz - y^2 + \lambda^2 = 0$. There is a morphism of varieties, $g : \mathbb{A}^2 \to Q$, defined by

$$g : (u, v) \mapsto (v, 2uv - \lambda, 2u(uv - \lambda)). \tag{5-7}$$

This corresponds to an injective map of commutative rings, namely

$$k[x, y, z]/(2xz - y^2 + \lambda^2) \to k[u, v], \tag{5-8}$$

defined by

$$x \mapsto v, \quad y \mapsto 2uv - \lambda, \quad z \mapsto 2u(uv - \lambda).$$

Since g sends the line $u = \alpha$ (isomorphically) to the line $\ell_{(\alpha,1)}$, the image of $\mathbb{A}^2$ is the open complement in Q to the line $\ell_{(1,0)}$. A computation shows that the map g has an inverse on this open subvariety, so $Q \backslash \ell_{(1,0)} \cong \mathbb{A}^2$.

The analogue for U_λ of the map (5-8) is the map in the next result.

Proposition 5.8 *Let $R = k[x, \partial]$ denote the ring of differential operators on the affine line over a field k of characteristic zero.*

1. *There is a ring homomorphism $U(\mathfrak{sl}(2)) \to R$ defined by*

$$e \mapsto \partial, \quad h \mapsto -2x\partial + \lambda, \quad f \mapsto -x(x\partial - \lambda). \tag{5-9}$$

2. *The kernel of this map is the ideal $(\Omega - \lambda^2 - 2\lambda)$.*

3. *If $\lambda = n \in \mathbb{N}$, then $k[x]$ has length two as an $\mathfrak{sl}_2$-module, and its socle is isomorphic to V_n.*

4. *If $\lambda \notin \mathbb{N}$, then $k[x]$ is simple.*

Proof. (1) To check that the map defined on e, f, and h, extends to a k-algebra homomorphism we need only check that the defining relations (5-1) hold for the images in R. This is straightforward. We will call the map φ.

(2) A calculation in R gives $h^2 + 2(ef + fe) = \lambda^2 + 2\lambda$.

(3) First, let λ be arbitrary. It is clear that $x^i \in k[x]$ has weight $\lambda - 2i$. The linear span of $\{1, x, \dots, x^n\}$ is stable under the action of e and h. However, it is stable under the f-action if and only if $f.x^n = 0$; that is, if and only if $\lambda = n$. In that case this gives a submodule which is necessarily simple, hence isomorphic to V_n. □

We will write R_λ for the image of the map $U(\mathfrak{sl}_2) \to k[x, \partial]$ described in Proposition 5.8. Thus $R_\lambda \cong U_\lambda$.

Proposition 5.9 *If $\lambda \notin \{-1, -2, \dots\}$, then the inclusion of U_λ in $R = k[x, \partial]$ defined by (5-9) has the following properties:*

1. *R is flat as a right U_λ-module;*

2. *If M is a simple left U_λ-module, then $R \otimes_U M = 0$ if and only if $M \cong M(-\lambda - 2)$;*

3. *If M and N are non-isomorphic simple left U_λ-modules, then $R \otimes_U M$ and $R \otimes_U N$ are non-isomorphic R-modules.*

4. *Every simple R-module is isomorphic to $R \otimes_U M$ for some simple left U_λ-module M.*

5. *If $\lambda = n \in \mathbb{N}$, then $R \otimes_U V_n \cong k[x]$ with its usual action of $k[x, \partial]$.*

Proof. $\boxed{\text{Paul}}$ □

The geometric interpretation of this result is the following. Because R is a noetherian domain with basis $x^i\partial^j$, we think of $W := \mathsf{Mod}R$ as a non-commutative analogue of $\mathbb{A}^2$. The inclusion $U_\lambda \to R$ induces an affine map $f : W \to Q_\lambda$ of affine spaces. Because R is flat as a right U_λ-module, the functor $f^* = R \otimes_{U_\lambda} -$ is exact, so f is the inclusion of W an open subspace of Q_λ.

Let L be the weakly closed subspace of Q_λ defined by declaring the L-modules to be all direct sums of $M(-\lambda-2)$. The fact that f^* is the localization functor along L says that W is the open complement to the line L in Q_λ.

There is a non-commutative analogue of the fact that the map $g : \mathbb{A}^2 \to Q$ sends each line $u = \alpha$ in $\mathbb{A}^2$ to the line $\ell_{(\alpha,1)}$ on Q. To see this we need to use the basis e', f', h', given in Lemma 5.7.

The analogue of the line $\ell_{(\alpha,1)}$ is the module $R/R(x-\alpha)$. Now f_* applied to this is simply $R/R(x-\alpha)$ viewed as a U-module. I claim that it is isomorphic to the Verma module $M_\alpha(-\lambda-2) = U/Ue' + U(h' + \lambda + 2)$, where e', f', h' is the basis for $\mathfrak{sl}_2$ defined in Lemma 5.7. Thus, f_* sends this line module for W to a line module in Q_λ. Here is one way to verify the claim. Realize the module $R/R(x-\alpha)$ as $k[t]e^{\alpha t}$, where x acts as the derivative d/dt, and ∂ acts as multiplication by $-t$. The action of $f' = -e$ on $k[t]e^{\alpha t}$ is simply multiplication by t, so such that $k[t]e^{\alpha t} = k[f'].e^{\alpha t}$. Therefore $k[t]e^{\alpha t}$ is a cyclic U-module generated by $e^{\alpha t}$, and is isomorphic to $k[f']$ as a left $k[f']$-module. A computation shows that the elements e' and $h'+\lambda+2$ both annihilate $e^{\alpha t}$, so $k[t]e^{\alpha t}$ is a quotient of $M_\alpha(-\lambda-2)$. However, as left $k[f']$-modules both $k[t]e^{\alpha t}$ and $M_\alpha(-\lambda-2)$ are isomorphic to $k[f']$. Hence, as U-modules, $R/R(x-\alpha) \cong M_\alpha(-\lambda-2)$.

$\boxed{\text{Paul}}$ Do the other lines in W, namely $R/R(\partial-\gamma)$ get sent to Whittaker modules by f_*?

The fact that $R \otimes V_n \cong k[x]$ says that we should think of V_n as giving the same geometric object as $k[x]$; in other words that geometric object is a curve, not a point.

Example 5.10 (The case $\lambda = 0$.) When $\lambda = 0$, there is a ring between U_0 and R that has played a role in the development of non-commutative ring theory. That ring, which we denote by S is the idealizer of a maximal ideal in the Weyl algebra. Explicitly, we have

$$U_0 = k[\partial, x\partial, x^2\partial] \subset S = k + R\partial \subset R.$$

and the associated geometry of the non-commutative affine surfaces

$$W = \mathsf{Mod}R, \qquad X = \mathsf{Mod}S, \qquad Q_0 = \mathsf{Mod}U_0,$$

where Mod denotes the category of *left* modules. There are maps

$$W \xrightarrow{f} X \xrightarrow{g} Q_0. \tag{5-10}$$

We will write $h = gf$. We think of W as the open subspace of Q_0 gotten by removing the line with "structure module" $M(-\lambda-2)$. Since $M(-\lambda-2)$ is simple, $\mathsf{Mod}S$ cannot be a localization of U_0, so X cannot be gotten by removing any part of Q.

There is no reasonable commutative analogue of this. The problem is that if we take the "corresponding" subrings of $k[u,v]$, namely

$$k[v, uv, u^2v] \subset S' = k + v.k[u,v] \subset k[u,v],$$

then the ring in the middle is not noetherian. Thus $\operatorname{Spec} S'$ is not a noetherian scheme. The scheme $\operatorname{Spec} S'$ has a very bad singularity at the origin. The ring S has infinite global dimension, and if $\mathfrak{m} = v.k[u,v]$, then $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \infty$. I think of this as a black hole. In contrast, the non-commutative ring S is left noetherian and has global dimension one. $\lozenge$

We consider one more analogy between the commutative and non-commutative quadrics.

Maps $Q \to \mathbb{P}^1$ and $Q_\lambda \to \mathbb{P}^1$. Fix one of the rulings on Q, say that provided by the lines $L_{(\alpha,\beta)}$. There is an associated map $Q \to \mathbb{P}^1$ sending the points of Q that lie on $L_{(\alpha,\beta)}$ to (α,β). A reasonable analogue of this for Q_λ would be a map $f : Q_\lambda \to \mathbb{P}^1$ such that $f^*\mathcal{O}_p$ is a line module for all $p \in \mathbb{P}^1$. The construction of f is a little technical. However, those technicalities are natural, and to be expected because in the commutative case all one has really done is say what $f_*\mathcal{O}_q$ is for each point $q \in Q$. And in the non-commutative case we need to say what f_* is for every Q_λ-module. One constructs f by realizing U_λ as the global sections of a ring of twisted differential operators on $\mathbb{P}^1$. The idea is this........

$\boxed{\text{Paul}}$ **To do.** Twisted diffl ops. Show lines in the various rulings on Q_λ meet in the expected way where "meet" is defined via Ext^1. Whittaker modules. Morita equivalence, translation principle. Hodges non-isomorphism proof. Krull dimension, Stafford's global dimension results. Projectivization of this; pencil of quadrics in $\mathbb{P}^3$ generated by x^2 and $yz + w^2$.

EXERCISES

5.1 A k-linear derivation on a k-algebra R is a k-linear map $\delta : R \to R$ such that $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in R$. Fix $x \in R$. Show that the linear map $[x, -]$ that sends a to $xa - ax$ is a derivation.

5.2 Let δ be a k-linear derivation of R. For each $\lambda \in k$, define $R_\lambda = \{r \in R \mid \delta(r) = \lambda r\}$. Show that the direct sum of all the R_λ is a graded algebra.

5.3 If $\mathfrak{g}$ is a Lie algebra show that there is an algebra homomorphism $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes_k U(\mathfrak{g})$ defined by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in \mathfrak{g}$.

5.4 Show that the map $x \mapsto -x$ for $x \in \mathfrak{g}$ extends to an algebra ani-automorphism of $U(\mathfrak{g})$. Hence the map $x \mapsto x \otimes 1 - 1 \otimes x$ from $U(\mathfrak{g})$ to $U(\mathfrak{g}) \otimes U(\mathfrak{g})^{\mathrm{op}}$ is an algebra homomorphism. Since $U(\mathfrak{g})$ is a left module over $U(\mathfrak{g}) \otimes U(\mathfrak{g})^{\mathrm{op}}$ in a natural way, this explains why $U(\mathfrak{g})$ becomes a left $U(\mathfrak{g})$-module under the adjoint action.

5.5 Show that the embedding of $U(\mathfrak{sl}_2)$ in the ring of differential operators $k[X,Y][\partial_X,\partial_Y]$ given in (5-4) sends the Casimir element to
$$(X\partial_X + Y\partial_Y)(X\partial_X + Y\partial_Y + 2).$$

5.6 Give $D = k[X,Y][\partial_X,\partial_Y]$ the grading with
$$\deg X = \deg Y = 1, \qquad \deg \partial_X = \deg \partial_Y = -1.$$
Observe that $U(\mathfrak{sl}_2)$ is contained in the degree zero component. Examine D as a graded ring, and describe the quotient category of $\mathsf{GrMod}D$ modulo the right bounded modules. How is this related to $\mathsf{Mod}U(\mathfrak{sl}_2)$? Does this data provide a map from $\mathsf{Mod}U(\mathfrak{sl}_2)$ to $\mathsf{Mod}\mathbb{P}^1$? Show that D_n is the n-eigenspace for $[X\partial_X + Y\partial_Y, -]$. Thus D_0 is a polynomial extension of $U(\mathfrak{sl}_2)$. The inclusion $k[X,Y] \to D$ must provide something like a map $\mathsf{GrMod}D \to \mathsf{Mod}\mathbb{P}^1$ with the modules $D/(\beta X - \alpha Y)D$ being the fiber modules over the points of $\mathbb{P}^1$. Are these modules important as $U(\mathfrak{sl}_2)$-modules?

5.7 Make $U(\mathfrak{sl}_2)$ a graded k-algebra as in (5-3). Explore $\mathsf{Tails}U(\mathfrak{sl}_2)$. The degree zero component is $k[h,\Omega]$. Does the inclusion of this give an interesting map $\mathsf{Mod}U(\mathfrak{sl}_2) \to \mathbb{A}^2$? What are the fiber modules, $U/(h-\alpha)U + (\Omega - \lambda)$? This is a conic module lying on Q_ν where $\nu^2 + 2\nu = \lambda$. This is the point of view for hyperbolic extensions.

5.8 Carry out the computation of the dual basis for $\mathfrak{sl}_2$ with respect to the Killing form (see page 221). Show that the element $\sum_{i=1}^{3} x_i x_i^*$ is a non-zero multiple of the Casimir element.

5.9 Let V be a finite dimensional vector space. Let $\Phi : V \otimes_k V^* \to \operatorname{End}_k V$ be the k-linear map defined by
$$\Phi(v \otimes \alpha)(u) = \alpha(u)v$$

(a) Show that Φ is an isomorphism of vector spaces.

(b) Show that Φ is a k-algebra isomorphism if $V \otimes_k V^*$ is endowed with the product rule $(u \otimes \alpha).(v \otimes \beta) = \alpha(v)u \otimes \beta$.

(c) If $v_1, \dots, v_n$ is a basis for V, and $v_1^*, \dots, v_n^*$ is its duals basis, show that $\Phi(\sum v_i \otimes v_i^*) = 1$. Hence show that $\sum v_i \otimes v_i^*$ is independent of the choice of basis. That is, it is invariant under the action of $GL(V)$.

This explains why the Casimir element is invariant under the action of the automorphism group.

5.10 Consider the map $g : \mathbb{A}^2 \to Q$ described in (eq.A2.to.Q). Describe the images of the lines $v = \gamma$ in $\mathbb{A}^2$, and the preimages of the lines $L_{(\alpha,\beta)}$ on Q.

5.11 Let x and y be commuting indeterminates. Fix $\lambda \in k$. Make $N = k[x/y, y/x]x^\lambda$ a $U(\mathfrak{sl}_2)$-module through $e = x\partial_y$, $f = y\partial_x$, and $h = x\partial_x - y\partial_y$.

(a) Show that $\Omega - \lambda^2 - 2\lambda$ annihilates this module.

(b) Show that $k[y/x]x^\lambda$ is a submodule isomorphic to $M(\lambda)$.

(c) Show that the quotient $N/M(\lambda)$ is isomorphic to the Verma module $M'(-\lambda-2)$ with respect to the basic $e' = -f$, $h' = -h$, and $f' = -e$.

(d) Show that the extension $0 \to M(\lambda) \to N \to M'(-l-2) \to 0$ is non-split, and hence that $\operatorname{Ext}^1_{U_\lambda}(M'(-\lambda-2), M(\lambda)) \neq 0$.

We interpret this as saying showing that one particular line in one ruling on Q_λ meets a particular line in the other ruling.

5.12 The quantized enveloping algebra of $\mathfrak{sl}_2$, denoted $U_q(\mathfrak{sl}_2)$, where $q \in k$ is a non-zro element that is not a fourth root of unity, is the algebra $k[E,F,K,K^{-1}]$ with defining relations
$$KE = q^2EK, \quad KF = q^{-2}FK, \quad EF - FE = \frac{K^2 - K^{-2}}{q^2 - q^{-2}}.$$
Study this in the same spirit as the analysis of $U(\mathfrak{sl}_2)$ in this section.

Chapter 5

Non-commutative projective spaces

A scheme is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X, and a sheaf of rings $\mathcal{O}_X$ on X, this data being subject to certain axioms. The scheme X can be recovered from the pair $(\mathsf{Qcoh}\mathcal{O}_X, \mathcal{O}_X)$, so in a sense the objects of algebraic geometry are pairs $(\mathcal{C}, \mathcal{O})$ consisting of a category together with a distinguished object.

Let A be a connected graded algebra over a base field k. For simplicity suppose that it is right noetherian and locally finite dimensional. The non-commutative projective space with homogeneous coordinate ring A the pair

$$\operatorname{Proj}(A) := (\mathsf{Tails}A, \mathcal{A}),$$

where $\mathsf{Tails}A$ is the quotient category of $\mathsf{GrMod}A$, the category of graded right A-modules, modulo its full subcategory of direct limits of finite dimensional modules, and $\mathcal{A}$ is the image of the distinguished module ${}_AA$ in $\mathsf{Tails}A$.

If A is a quotient of a commutative polynomial ring generated in degree 1, Serre [204] proved that $\operatorname{Proj} A$ is isomorphic (in an obvious sense) to the pair $(\mathsf{Qcoh}\mathcal{O}_X, \mathcal{O}_X)$, where X is the projective scheme determined by A, $\mathcal{O}_X$ is the sheaf of regular functions on X, and $\mathsf{Qcoh}\mathcal{O}_X$ is the category of quasi-coherent $\mathcal{O}_X$-modules. Thus $\mathsf{Tails}A$ is the non-commutative analogue of $\mathsf{Qcoh}\mathcal{O}_X$, and the objects in $\mathsf{Tails}A$ are the non-commutative geometric objects analogous to sheaves of $\mathcal{O}_X$-modules.

We write X for the space with $\mathsf{Mod}X = \mathsf{Tails}A$, and $\mathcal{O}_X$ for $\mathcal{A}$. We define cohomology functors $H^q(X, -)$ as the derived functors of $\operatorname{Hom}_X(\mathcal{O}_X, -)$. The cohomology groups $H^q(X, \mathcal{F})$ generalize the Čech cohomology groups—one has the usual long exact sequence, they agree with the with usual cohomology groups when X is commutative, and there is a version of Serre's Finiteness Theorem (13.4) provided a certain technical condition χ holds (see Definition 12.3). Every commutative algebra satisfies χ, but there exist rather nice non-commutative algebras which do not (Example 12.8). We compute the cohomology groups $H^q(X, \mathcal{A}(d))$, $d \in \mathbb{Z}$, when A is an Artin-Schelter regular algebra. This family of algebras includes the commutative polynomial ring, and in that case $H^q(X, \mathcal{A}(d)) = H^q(\mathbb{P}^n, \mathcal{O}(d))$. Artin-Schelter regular algebras are non-commutative algebras which enjoy many of the properties of polynomial rings;

amongst the non-commutative Artin-Schelter regular algebras are most graded iterated Ore extensions, homogenizations of enveloping algebras, and Sklyanin algebras. Artin-Schelter regular algebras always satisfy the condition χ.

Maps between $\operatorname{Proj} A$ and $\operatorname{Proj} B$ are discussed in section **??**.

The polarized projective space associated to A is the triple $(\mathsf{Tails}A, \mathcal{A}, (1))$, where (1) is the twisting functor induced by the degree shift functor (1) on $\mathsf{GrMod}(M)$, namely $M(1)_i = M_{i+1}$. In particular, twisting is an auto-equivalence of $\mathsf{Tails}A$. The extra data inherent in this auto-equivalence is the analogue of specifying a line bundle on a scheme X; it is natural to ask whether that line bundle is very ample, i.e., whether it determines an embedding of X in some projective space (or, equivalently, whether it arises from an embedding of X in some $\mathbb{P}^n$). This leads to the notion of ampleness for (1) on $\operatorname{proj}(A)$ (see Definition **??**). Whether or not (1) is ample in $\operatorname{proj}(A)$ is closely related to the condition χ.

Polarized projective schemes are objects in a category of triples $(\mathcal{C}, \mathcal{O}, s)$ where $\mathcal{C}$ is a k-linear category, $\mathcal{O}$ is a distinguished object in $\mathcal{C}$, and s is an auto-equivalence of $\mathcal{C}$. The notion of ampleness is defined in this larger context. If s is ample, and $(\mathcal{C}, \mathcal{O}, s)$ satisfies some modest finiteness conditions, then $(\mathcal{C}, \mathcal{O}, s) \cong (\mathsf{Tails}A, \mathcal{A}, (1))$ for some right noetherian, locally finite, $\mathbb{N}$-graded algebra A which satisfies χ_1. This result gives some idea of the scope of non-commutative algebraic geometry because it says (roughly) which $\mathcal{C}$ can be non-commutative schemes. The result may also be used to exhibit some non-commutative homogeneous coordinate rings of commutative schemes. For example, if A is a twisted homogeneous coordinate ring (see Example **??**), usually written $A = B(X, \sigma, \mathcal{L})$, where X is a projective scheme, $\sigma \in \operatorname{Aut} X$ and $\mathcal{L}$ is a σ-ample line bundle on X, then $(\mathsf{Tails}A, \mathcal{A}, (1)) \cong (\mathsf{Qcoh}\mathcal{O}_X, \mathcal{O}_X, s)$ for a suitable s (the hypothesis that $\mathcal{L}$ is σ-ample guarantees that s is ample). Since $\mathsf{Tails}A$ is equivalent to $\mathsf{Qcoh}\mathcal{O}_X$ the representation theory of A can be studied via the methods of algebraic geometry. The utility of this result is due to the fact that twisted homogeneous coordinate rings turn up rather often in the theory of non-commutative graded algebras.

Under Serre's equivalence of categories we have the correspondence

$$\begin{array}{rcl} \mathcal{A} & \leftrightarrow & \mathcal{O}_X \\ \mathcal{A}(d) & \leftrightarrow & \mathcal{O}_X(d), \end{array}$$

where $\mathcal{O}_X(d)$ is the line bundle on X induced from the degree d line bundle on $\mathbb{P}^n$ (by definition $\mathcal{O}_X(d)(X_f)$ is the degree d component of $k[X_0, \dots, X_n][f^{-1}]$, where $X_f = \{p \in \mathbb{P}^n \mid f(p) \neq 0\}$).

5.1 Projective space $\mathbb{P}^n$

Fix a base field k. Projective n-space over k, denoted by $\mathbb{P}^n$ or $\mathbb{P}^n_k$, is, by definition, the set of lines through the origin in $\mathbb{A}^{n+1}$ or, equivalently, the one-dimensional subspaces of k^{n+1}. If $(\alpha_0, \alpha_1, \dots, \alpha_n)$ is a point in $\mathbb{A}^{n+1} \backslash \{0\}$ we denote the line through it, thought of as a point in $\mathbb{P}^n$, by $(\alpha_0, \alpha_1, \dots, \alpha_n)$ also. Thus, in $\mathbb{P}^n$,

$$(\alpha_0, \alpha_1, \dots, \alpha_n) = (\lambda\alpha_0, \lambda\alpha_1, \dots, \lambda\alpha_n)$$

for all non-zero $\lambda \in k$.

One reason for introducing $\mathbb{P}^n$ is that if $f_1, \dots, f_r$ are homogeneous polynomials in $k[X_0, \dots, X_n]$, then their zero locus is a union of lines, so can be viewed as a subset of $\mathbb{P}^n$ (after removing the origin). It turns out in retrospect that projective spaces are better than affine ones. Many theorems are more elegantly stated in projective space; the classical example is that any two lines in $\mathbb{P}^2$ intersect. Thus $\mathbb{P}^n$, rather than $\mathbb{A}^n$, is the arena for algebraic geometry. We shall see that similar remarks apply to non-commutative projective spaces.

Although it makes no sense to evaluate a polynomial g at a point of $\mathbb{P}^n$, because $g(\alpha_0, \alpha_1, \dots, \alpha_n)$ usually will not be equal to $g(\lambda\alpha_0, \lambda\alpha_1, \dots, \lambda\alpha_n)$, it does make sense to speak of a homogeneous polynomial vanishing at a point in $\mathbb{P}^n$. If g is homogeneous of degree d, then

$$g(\lambda\alpha_0, \lambda\alpha_1, \dots, \lambda\alpha_n) = \lambda^d g(\alpha_0, \alpha_1, \dots, \alpha_n).$$

Thus, homogeneous polynomials are the relevant ones when dealing with projective spaces. If $f_1, \dots, f_r$ are homogeneous polynomials in $k[X_0, \dots, X_n]$, we define their zero locus in $\mathbb{P}^n$ to be the set of lines on which they vanish. Notice that

We call

$$A = k[X_0, \dots, X_n]$$

the homogeneous coordinate ring of $\mathbb{P}^n$. We write A_d for the span of the homogeneous polynomials of degree d.

The ideals generated by homogeneous polynomials are called graded ideals. An ideal is graded if and only if it is the sum of its homogeneous components. That is, an ideal I is graded if and only if $I = \sum(I \cap A_d)$. We write $I_d = I \cap A_d$. If I is graded, and $g \in I$, then each homogeneous component of g is in I. If $I \subset J$ are graded ideals, then I/J may be given the structure of a graded module by defining its degree d component to be the image of the degree d component of I. Thus graded modules are the relevant ones when considering projective spaces.

The ideal $\mathfrak{m} = (X_0, \dots, X_n)$ is not relevant to the study of $\mathbb{P}^n$ because the only point of $\mathbb{A}^{n+1}$ where it vanishes is the origin, and it therefore vanishes at no points of $\mathbb{P}^n$. Consequently the graded A-modules that are supported only at the origin are irrelevant to the study of $\mathbb{P}^n$. These are the modules in which every element is annihilated by some power of $\mathfrak{m}$. The full subcategory of $\mathsf{GrMod}A$ consisting of such modules is denoted by $\mathsf{Fdim}A$. We use this notation because every such module is a union of finite dimensional modules: if $m \in M$ is annihilated by $\mathfrak{m}^d$, then mA is a quotient of $A/\mathfrak{m}^d$, so finite dimensional. Notice that $\mathfrak{m}^d = A_d \oplus A_{d+1} \oplus \dots$. Conversely, if M is a union of finite dimensional graded A-modules, then each element of M is annihilated by $\mathfrak{m}^d$ for sufficiently large d, so is supported only at the origin.

To remove these modules from consideration we pass to the quotient category (cf. the passage from a space X to an open subspace $X\backslash Y$). Therefore the appropriate category of modules on $\mathbb{P}^n$ is the quotient category

$$\mathsf{Mod}\mathbb{P}^n := \mathsf{GrMod}A/\mathsf{Fdim}A.$$

We write $\pi : \mathsf{GrMod}A \to \mathsf{Mod}\mathbb{P}^n$ for the quotient functor. It is exact, and every object in $\mathsf{Mod}\mathbb{P}^n$ is of the form πM for some M in $\mathsf{GrMod}A$. The objects in the quotient category are, by definition, the same as the objects in $\mathsf{GrMod}A$, but we write the π in front of M to remind ourselves that we are working in the quotient category. The difference between $\mathsf{Mod}\mathbb{P}^n$ and $\mathsf{GrMod}A$ therefore lies in the morphisms. There are more morphisms in $\mathsf{Mod}\mathbb{P}^n$. The morphisms in a quotient category are described in section 2.13. If $f : M \to N$ is a map of graded A-modules, then $\pi(f) : \pi M \to \pi N$ is an isomorphism in $\mathsf{Mod}\mathbb{P}^n$ if and only if $\ker f$ and $\operatorname{coker} f$ belong to $\mathsf{Fdim}A$. Thus, all finite dimensional graded A-modules become isomorphic to zero in $\mathsf{Mod}\mathbb{P}^n$. And two graded A-modules become isomorphic in $\mathsf{Mod}\mathbb{P}^n$ if they differ by finite dimensional modules.

The following theorem of Serre says that if A is a commutative k-algebra generated by elements of degree one, then $\mathsf{GrMod}A/\mathsf{Fdim}A$ is equivalent to the category of quasi-coherent sheaves on the projective scheme $\operatorname{Proj} A$.

Theorem 1.1 (Serre) *Let I be a graded ideal in the commutative polynomial ring $A = k[X_0, \dots, X_n]$ endowed with its standard grading, viz., $\deg X_i = 1$ for all i. Let X be the subscheme of $\mathbb{P}^n$ defined by the vanishing of I. Let $\mathsf{Qcoh}X$ denote the category of quasi-coherent $\mathcal{O}_X$-modules. Then there is an equivalence of categories*

$$\mathsf{Qcoh}X \cong \mathsf{GrMod}A/\mathsf{Fdim}A.$$

The scheme X is constructed as follows. If A is an $\mathbb{N}$-graded commutative k-algebra of the form $A = k \oplus A_1 \oplus A_2 \oplus \cdots$, then one defines the topological space

$$\operatorname{Proj} A = \{\text{non-trivial graded prime ideals}\}$$

endowed with the Zariski topology. The trivial prime is $A_1 \oplus A_2 \oplus \cdots$. The structure sheaf of $\operatorname{Proj} A$ is defined by declaring that the ring of sections of $\mathcal{O}_{\operatorname{Proj} A}$ on the open set $z \neq 0$, where z is a homogeneous regular element of A, is $A[z^{-1}]_0$, the degree zero component of the localization. Thus, this open set is isomorphic to the affine scheme $\operatorname{Spec} A[z^{-1}]_0$.

If A is not generated in degree one, the equivalence of categories fails. One must therefore take care to distinguish $\mathsf{GrMod}A/\mathsf{Fdim}A$ from $\operatorname{Proj} A$.

Example 1.2 Let $X = \mathsf{GrMod}k[y]/\mathsf{Fdim}$, where $k[y]$ is the commutative polynomial ring with $\deg y = n > 0$. Then $\mathsf{GrMod}k[y]/\mathsf{Fdim} \cong \mathsf{Mod}(k^{\times n})$, where $k^{\times n}$ denotes the product of k with itself n times. Thus X is the disjoint union of n copies of the point $\operatorname{Spec} k$. $\Diamond$

5.2 Graded rings and modules

This section introduces some of the basic language and ideas for the theory of graded modules.

Definition 2.1 Let A be a graded k-algebra. We write $\mathsf{GrMod}A$ for the category of $\mathbb{Z}$-graded right A-modules, with morphisms the A-module maps that preserve degree. We write $\mathsf{grmod}A$ for the full subcategory consisting of the noetherian A-modules. $\Diamond$

An A-module homomorphism of graded A-modules $f : N \to M$ has degree d if $f(N_i) \subset M_{i+d}$ for all $i \in \mathbb{Z}$. We define

- $\operatorname{Hom}_A(N, M)_d := \{f \in \operatorname{Hom}_A(N, M) \mid \deg(f) = d\}$, and
- $\underline{\operatorname{Hom}}_A(N, M) := \bigoplus_{d \in \mathbb{Z}} \operatorname{Hom}_A(N, M)_d$.

Lemma 2.2 *If N is finitely generated, then $\underline{\operatorname{Hom}}_A(N, M) = \operatorname{Hom}_A(N, M)$.*

Example 2.3 Let V be a graded vector space such that $\dim_k V_n \geq 1$ for all $n \in \mathbb{Z}$. If $f : V \to k$ is such that $f(V_n) \neq 0$ for infinitely many n, then $f \notin \underline{\operatorname{Hom}}_k(V, k)$. Thus $\underline{\operatorname{Hom}}_k(V, k) \neq \operatorname{Hom}_k(V, k)$. $\Diamond$

Definition 2.4 Let M be a graded vector space over a field k. We say that M is locally finite if $\dim_k M_n < \infty$ for all n. We use the notation

$$M_{\geq n} = \bigoplus_{d \geq n} M_d \qquad \text{and} \qquad M_{\leq n} = \bigoplus_{d \leq n} M_d.$$

We say that M is left (respectively, right) bounded if $M_{\leq n} = 0$ (respectively, $M_{\geq n} = 0$) for some n.

Mostly we are interested in $\mathbb{N}$-graded algebras. Such an algebra, A say, is left bounded, and so are its finitely generated modules. Further, if M is a graded A-module, so is $M_{\geq n}$.

A graded k-algebra generated by a finite number of elements of positive degree is locally finite. Finitely generated modules over a locally finite algebra are locally finite.

If M and N are graded modules over a graded ring R, we define, for each integer d, $\operatorname{Hom}_R(M, N)_d$ to be the R-module homomorphisms $\theta : M \to N$ such that $\theta(M_i) \subset N_{i+d}$ for all i. We also define

$$\underline{\operatorname{Hom}}_R(M, N) = \oplus_{d \in \mathbb{Z}} \operatorname{Hom}_R(M, N)_d.$$

Thus $\underline{\operatorname{Hom}}_R(M, N)$ is a graded abelian group, and its degree zero component is $\operatorname{Hom}_{\mathsf{Gr}}(M, N)$.

To show that a homomorphism $\varphi : A \to B$ of graded rings induces a map of non-commutative spaces

$$g : \mathsf{GrMod}B \to \mathsf{GrMod}A$$

we follow the idea in Example 3.3.3, but the gradings add some complications.

If R and S are graded rings, and M is a graded R-S-bimodule in the sense that $R_i M_n S_j \subset M_{n+i+j}$ for all i and j, then $\underline{\mathrm{Hom}}_R(M,N)$ becomes a graded right S-module in the obvious way. For example, if $M = R$ is viewed as an R-R-bimodule in the obvious way, the map $\underline{\mathrm{Hom}}_R(R,N) \to N$ defined by $\theta \mapsto \theta(1)$ is an isomorphism of right R-modules. In other words, $\underline{\mathrm{Hom}}_R(R,-)$ is naturally equivalent to the identity functor.

If P is a graded R-S-bimodule, we define a grading on $M \otimes_R P$ by declaring its degree-n component to consist of the span of all elements $m \otimes p$ where m and p are homogeneous and $\deg m + \deg p = n$.

Proposition 2.5 *Let R and S be graded k-algebras. Let $M \in \mathsf{GrMod}R$, and $N \in \mathsf{GrMod}S$. If P is a graded R-S-bimodule, then*

$$\underline{\mathrm{Hom}}_S(M \otimes_R P, N) \cong \underline{\mathrm{Hom}}_R(M, \underline{\mathrm{Hom}}_S(P,N)); \tag{2-1}$$

In particular, $- \otimes_R P$ is left adjoint to $\underline{\mathrm{Hom}}_S(P,-)$.

Proof. The isomorphism is implemented by the map Φ defined by $\Phi(f)(m)(p) = f(m \otimes p)$ for $p \in P$ and $m \in M$. We leave the details to the reader. Taking the degree zero components of (2-1) gives the adjointness property. $\square$

Proposition 2.6 *Let $\varphi : A \to B$ be a map of graded k-algebras. Then φ induces an affine map of spaces $g : \mathsf{GrMod}B \to \mathsf{GrMod}A$.*

Proof. We define $g_* : \mathsf{GrMod}B \to \mathsf{GrMod}A$ to be the functor that sends a graded B-module N to N viewed as a graded A-module. Formally, this is

$$g_* N = \underline{\mathrm{Hom}}_B(B,N). \tag{2-2}$$

By Proposition 2.5, g_* has a left adjoint g^* given by

$$g^* M = M \otimes_A B \tag{2-3}$$

where B is made into a graded A-B-bimodule in the obvious way. We may also write $g_* N = N \otimes_B B$, where B is viewed as a graded B-A-bimodule in the obvious way. Therefore, the functor

$$g^! M = \underline{\mathrm{Hom}}_A(B,M). \tag{2-4}$$

is a right adjoint to g_*. Finally, it is clear that g_* is faithful. $\square$

Corollary 2.7 *If I is a graded ideal in a graded ring A, then $\mathsf{GrMod}A/I$ is a closed subspace of $\mathsf{GrMod}A$.*

The degree-shift functor, or twisting. If M is a graded A-module and n is an integer, we define $M(n)$ to the graded A-module that is equal to M with its original A-action, but is graded by

$$M(n)_i := M_{n+i}$$

for all $i \in \mathbb{Z}$. It is clear that the rule $M \mapsto M(n)$ extends to an auto-equivalence of $\mathsf{GrMod}A$; (n) acts on morphisms as the identity. We call this the degree-shift functor. In keeping with the terminology of algebraic geometry we also call this the n^{th} twisting functor.

It is an easy but worthwhile exercise to check that

$$\underline{\mathrm{Hom}}_A(N(i), M(j)) \cong \underline{\mathrm{Hom}}_A(N, M)(j-i)$$

as graded vector spaces.

Since $\mathrm{Hom}_A(A, -)$ is exact, so is $\mathrm{Hom}_{\mathsf{Gr}}(A(n), -)$ for all $n \in \mathbb{Z}$. Thus $A(n)$ is projective in $\mathsf{GrMod}A$, whence $\mathsf{GrMod}A$ has enough projectives. A module M is free if it is a direct sum of shifts of A. In general A is not a generator in $\mathsf{GrMod}A$; for example, if $M_0 = 0$, then $\mathrm{Hom}_{\mathsf{Gr}}(A, M) = 0$. However, the pair $(A, (1))$ acts somewhat like a generator in that $P = \bigoplus_{n \in \mathbb{Z}} A(n)$ is a generator.

If A has a homogeneous unit of positive degree, say n, then $\mathsf{GrMod}A$ is an affine space because $P = A \oplus A(1) \oplus \ldots \oplus A(n-1)$ is a progenerator in $\mathsf{GrMod}A$.

Definition 2.8 A graded k-algebra A is connected if

$$A = k \oplus A_1 \oplus A_2 \oplus \ldots. \tag{2-5}$$

We write $\mathfrak{m} = A_1 \oplus A_2 \oplus \ldots$ and call this the augmentation ideal. We write $k = A/\mathfrak{m}$ and call this the trivial A-module.

For a connected algebra, the only simple modules in $\mathsf{GrMod}A$ are the shifts $k(n)$ of the trivial module.

There is a useful analogue of Nakayama's Lemma for connected algebras.

Lemma 2.9 *Let A be connected. If $M \in \mathsf{GrMod}A$ is left bounded, then $M = 0$ if and only if $M \otimes_A k = 0$.*

Proof. Suppose that $M \neq 0$. Since M is bounded below, we can choose $0 \neq m \in M$, homogeneous of minimal degree. Such m cannot belong to $MA_{\geq 1}$. This is absurd, since $M \otimes_A k = 0$ implies that $MA_{\geq 1} = M$, so we conclude that $M = 0$. □

Lemma 2.10 *Let A be connected, and $M \in \mathsf{GrMod}A$. If M is bounded below, then*

1. *M is free if and only if $\mathrm{Tor}_1^A(M, k) = 0$*
2. *M is projective if and only if M is free.*

Proof. (1) ($\Leftarrow$) Choose a graded vector space V such that $V \oplus MA_{\geq 1} = M$. Then $(M/VA)\otimes_A k = 0$ so, by Nakayama's Lemma $M = VA$. Let $\psi : V \otimes_k A \to M$ be the multiplication map. Since $\operatorname{Tor}_1^A(M,k) = 0$, there is an exact sequence

$$0 \longrightarrow \ker \psi \otimes_A k \longrightarrow V \otimes_k A \otimes_A k \xrightarrow{\psi\otimes 1} M \otimes_A k \longrightarrow 0.$$

Since $\psi \otimes 1$ is an isomorphism, $\ker \psi \otimes_A k = 0$. But $\ker \psi$ is bounded below so, by Nakayama's Lemma, ψ is an isomorphism.

(2) This follows immediately from (1). □

5.3 Tails

Throughout this section k will denote a field.

One could at this stage proceed quickly to the definition of the homogeneous coordinate rings and their associated projective spaces. But we prefer to postpone that and use this section to build some of the machinery that is needed to proceed beyond the definitions.

Definition 3.1 Let A be a locally finite graded k-algebra. We define the following categories.

- $\mathsf{fdim}A$ is the full subcategory of $\mathsf{grmod}A$ consisting of the finite dimensional graded A-modules.
- $\mathsf{Fdim}A$ is the full subcategory of $\mathsf{GrMod}A$ consisting of the direct limits of finite dimensional graded A-modules.
- $\mathsf{Tails}A$ is the quotient category $\mathsf{GrMod}A/\mathsf{Fdim}A$. We write $\pi : \mathsf{GrMod}A \to \mathsf{Tails}A$ for the quotient functor.
- $\mathsf{tails}A$ is the full subcategory of $\mathsf{Tails}A$ consisting of the noetherian objects ◊

Torsion. Modules in $\mathsf{Fdim}A$ are called torsion modules, and we denote by τ the functor that sends a graded A-module to its torsion submodule

$$\tau M := \text{the sum of all finite dimensional submodules of } M.$$

In other words, $\tau M = \varinjlim \underline{\operatorname{Hom}}_A(A/A_{\geq n}, M)$ and the right derived functors of τ are

$$R^i\tau = \varinjlim \underline{\operatorname{Ext}}^i_A(A/A_{\geq n}, -).$$

An A-module M is torsion if $\tau M = M$, and is torsion-free if $\tau M = 0$. It is clear that $M/\tau M$ is torsion-free. Since every module has a largest torsion submodule, $\mathsf{Fdim}A$ is a localizing subcategory of $\mathsf{GrMod}A$.

Theorem 3.2 *The functor* $\pi : \mathsf{GrMod}A \to \mathsf{Tails}A$ *has a right adjoint,*

$$\omega : \mathsf{Tails}A \to \mathsf{GrMod}A.$$

Proof. This follows from the preceding remarks and Theorem 1.14.12 because $\mathsf{GrMod}A$ has enough injectives. $\square$

Recall how $\omega\pi M$ is constructed. Since it depends only on $M/\tau M$, we can, for simplicity, assume that M is torsion-free. Let $E(M)$ denote the injective envelope of M in $\mathsf{GrMod}A$. Then $\omega\pi M$ is the largest submodule of $E(M)$ that contains M and is torsion modulo M. If $M = \omega\pi M$ we say that M is saturated. Because $\pi\omega \cong \mathrm{id}$, the saturated modules are precisely those of the form $\omega\pi M$. They are saturated in the sense that they cannot be extended in a non-trivial way by a torsion module.

We will make frequent use of the adjoint isomorphism

$$\mathrm{Hom}_{\mathsf{Tails}}(\pi N, \mathcal{F}) \cong \mathrm{Hom}_{\mathsf{Gr}}(N, \omega\mathcal{F}). \tag{3-1}$$

This implies that $\omega\mathcal{F}$ is torsion-free since, if N is torsion then $\pi N = 0$, which ensures that both the above homomorphism groups are zero.

Definition 3.3 We call $\omega\pi M$ the saturation of M, and say that M is saturated if the natural map $M \to \omega\pi M$ is an isomorphism.

As for any quotient functor and its right adjoint, we have

$$\pi \circ \omega \simeq \mathrm{id}\,.$$

Proposition 3.4 $\omega\pi M \cong \varinjlim \underline{\mathrm{Hom}}_A(A_{\geq n}, M)$

Proof. The proof is a "finger exercise":

$$\begin{aligned}
\omega\pi M &= \underline{\mathrm{Hom}}_A(A, \omega\pi M) && \text{because } {}_AA \text{ is finitely generated,}\\
&= \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{Gr}}(A, \omega\pi M(d)) \\
&= \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{Tails}}(\pi A, \pi M(d)) && \text{by the adjoint isomorphism,}\\
&= \bigoplus_{d\in\mathbb{Z}} \varinjlim \mathrm{Hom}_{\mathsf{Gr}}(A_{\geq n}, M(d)) && \text{by Proposition 3.7,}\\
&= \varinjlim \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{Gr}}(A_{\geq n}, M(d)) \\
&= \varinjlim \underline{\mathrm{Hom}}_A(A_{\geq n}, M).
\end{aligned}$$

$\square$

Since $\mathsf{GrMod}A$ is a k-linear abelian category, so is $\mathsf{Tails}A$. By Theorem 1.13.8, π is an exact functor. The objects in the quotient category are the same as those in the original category—they are of the form πM—but there are more morphisms in the quotient category. In particular, if $f : N \to M$ is a degree 0 homomorphism of graded A-modules, such that $\ker f$ and $\operatorname{coker} f$ are finite dimensional, then πf is an isomorphism in $\mathsf{Tails}A$.

Thus, up to isomorphism, every object in $\mathsf{Tails}A$ is of the form πN for some torsion-free $N \in \mathsf{GrMod}A$. We will use script letters like $\mathcal{N}$ to denote objects in

$\mathsf{Tails}A$ so as to reinforce the idea that objects in $\mathsf{Tails}A$ are analogues of sheaves of modules. In particular, we will write

$$\mathcal{A} = \pi A$$

for the image of A in $\mathsf{Tails}A$.

The origin of the name Tails is as follows. If M is a graded A-module, then

$$M_{\geq d} := M_d \oplus M_{d+1} \oplus \dots$$

is called a tail of M. It is an A-submodule, and since every element of $M/M_{\geq d}$ is annihilated by a suitably large power of $\mathfrak{m}$, (π applied to) the inclusion $M_{\geq d} \to M$ is an isomorphism in $\mathsf{Tails}A$. Thus πM depends only on the tail of M.

Proposition 3.5 *If $M, N \in \mathsf{grmod}A$, then $\pi M \cong \pi N$ if and only if $M_{\geq n} \cong N_{\geq n}$ for some n.*

Proof. Suppose that $\pi M \cong \pi N$. By Proposition 3.7, the isomorphism is given by πf for some $f : N_{\geq n} \to M$. Thus $\ker(f)$ and $\operatorname{coker}(f)$ are torsion, and hence finite dimensional by the noetherian hypotheses. It follows that for $r \gg 0$, $f : N_{\geq r} \to M_{\geq r}$ is an isomorphism, as required. The converse is trivial. □

The noetherian hypothesis is essential to the previous result: if $A = k$ and $M = \oplus_{n \leq 0} k(n)$, then $\pi M \cong 0$, but $M_{\geq n} \not\cong 0$ for any n.

As we will see in section 5.4, in projective geometry $\mathsf{Tails}A$ is more important than $\mathsf{GrMod}A$. It is not just a case of the tail wagging the dog—the tail *is* the dog!

The morphisms in $\mathsf{Tails}A$ can be a little tricky to understand. By definition of the quotient category, if M and N are graded A-modules

$$\operatorname{Hom}_{\mathsf{Tails}}(\pi N, \pi M) = \varinjlim \operatorname{Hom}_{\mathsf{Gr}}(N', M/M')$$

where the direct limit is taken over the directed set, I say, consisting of all pairs (N', M') of submodules of N and M respectively, for which N/N' and M' are torsion modules. The quasi-ordering on I is defined by

$$(N', M') \leq (N'', M'') \qquad \text{if } N'' \subset N' \text{ and } M' \subset M''.$$

Since I is directed, every morphism $\pi N \to \pi M$ is of the form πf for some $f \in \operatorname{Hom}_{\mathsf{Gr}}(N', M/M')$ some $(N', M') \in I$; that is, every morphism in $\mathsf{Tails}A$ is the image, in the appropriate direct limit, of a morphism in $\mathsf{GrMod}A$.

Under reasonable hypotheses this description of the morphisms in $\mathsf{Tails}A$ may be simplified.

Proposition 3.6 *Let N and M be graded A-modules. Then*

$$\operatorname{Hom}_{\mathsf{Tails}}(\pi N, \pi M) = \varinjlim \operatorname{Hom}_{\mathsf{Gr}}(N', M/\tau M)$$

where the direct limit is taken over the submodules $N' \subset N$ such that N/N' is torsion.

Proof. By Lemma **??**, the set $(N', \tau M)$ is cofinal in the index set I defined above, so the result follows. □

Proposition 3.7 *Let A be an $\mathbb{N}$-graded, k-algebra. Suppose that $M \in \mathsf{GrMod}A$ and that N is a noetherian module.*

1. *If A is left noetherian or locally finite,*
$$\mathrm{Hom}_{\mathsf{Tails}}(\pi N, \pi M) = \varinjlim \mathrm{Hom}_{\mathsf{Gr}}(N_{\geq n}, M/\tau M).$$
2. *If A is left noetherian,*
$$\mathrm{Hom}_{\mathsf{Tails}}(\pi M, \pi N) = \varinjlim \mathrm{Hom}_{\mathsf{Gr}}(N_{\geq n}, M).$$

Proof. (1) If N/N' is torsion, then N/N' is right bounded so $N_{\geq n} \subset N'$ for $n \gg 0$. Hence $\{(N_{\geq n}, \tau M) \mid n \in \mathbb{Z}\}$ is cofinal in the index set I defined above, and the equality follows.

(2) First we prove this when M is finitely generated. The direct limit over n of the exact sequences

$$0 \to \mathrm{Hom}_{\mathsf{Gr}}(N_{\geq n}, \tau M) \to \mathrm{Hom}_{\mathsf{Gr}}(N_{\geq n}, M) \to \\ \mathrm{Hom}_{\mathsf{Gr}}(N_{\geq n}, M/\tau M) \to \mathrm{Ext}^1_{\mathsf{Gr}}(N_{\geq n}, \tau M)$$

remains exact. Since M is noetherian, τM is right bounded so $\mathrm{Hom}_{\mathsf{Gr}}(N_{\geq n}, \tau M) = 0$ for $n \gg 0$. Since a minimal free resolution of $N_{\geq n}$ is zero in degree $< n$, it also follows that $\mathrm{Ext}^1_{\mathsf{Gr}}(N_{\geq n}, \tau M) = 0$ for $n \gg 0$. Therefore, the direct limits of the first and last terms are zero, so the direct limits of the middle two terms are isomorphic. Hence the result follows from (1).

Now suppose that M is arbitrary, and write $M = \varinjlim M_\alpha$ as a direct limit of finitely generated graded modules. The index set is directed. Clearly $\tau M_\alpha = M_\alpha \cap \tau M$, so $\varinjlim(\tau M_\alpha) = \tau M$. Therefore, taking the direct limit of the exact sequences $0 \to \tau M_\alpha \to M_\alpha \to M_\alpha/\tau M_\alpha \to 0$, Proposition 1.5.7 implies that $\varinjlim(M_\alpha/\tau M_\alpha) = M/\tau M$. Since $N_{\geq n}$ is finitely generated, the functor $\mathrm{Hom}_{\mathsf{Gr}}(N_{\geq n}, -)$ commutes with direct limits. Thus

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{Tails}}(\pi N, \pi M) &= \varinjlim{}_n \mathrm{Hom}_{\mathsf{Gr}}(N_{\geq n}, M/\tau M) \\
&= \varinjlim{}_n \mathrm{Hom}_{\mathsf{Gr}}(N_{\geq n}, \varinjlim{}_\alpha (M_\alpha/\tau M_\alpha)) \\
&= \varinjlim{}_n \varinjlim{}_\alpha \mathrm{Hom}_{\mathsf{Gr}}(N_{\geq n}, M_\alpha/\tau M_\alpha) \\
&= \varinjlim{}_\alpha \varinjlim{}_n \mathrm{Hom}_{\mathsf{Gr}}(N_{\geq n}, M_\alpha/\tau M_\alpha) \\
&= \varinjlim{}_\alpha \varinjlim{}_n \mathrm{Hom}_{\mathsf{Gr}}(N_{\geq n}, M_\alpha) \\
&= \varinjlim{}_n \varinjlim{}_\alpha \mathrm{Hom}_{\mathsf{Gr}}(N_{\geq n}, M_\alpha) \\
&= \varinjlim{}_n \mathrm{Hom}_{\mathsf{Gr}}(N_{\geq n}, \varinjlim{}_\alpha M_\alpha) \\
&= \varinjlim{}_n \mathrm{Hom}_{\mathsf{Gr}}(N_{\geq n}, M)
\end{aligned}$$

as required. □

Twisting. The subcategory $\mathsf{Fdim}A$ is stable under the shift functor because M is torsion if and only if $M(1)$ is. Hence there is an induced automorphism on $\mathsf{Tails}A$, which we still denote by (1), and call the twisting functor; there is no ambiguity in writing $\pi M(1)$.
Notation. It is convenient to write

$$\underline{\mathrm{Hom}}_{\mathsf{Tails}}(\mathcal{F},\mathcal{G}) := \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{Tails}}(\mathcal{F},\mathcal{G}(d)).$$

With this notation, the proof of Proposition 3.4 says that $\omega\mathcal{F} \cong \underline{\mathrm{Hom}}(\mathcal{A},\mathcal{F})$; in fact, there is a natural equivalence of functors

$$\omega \simeq \underline{\mathrm{Hom}}(\mathcal{A},-).$$

We also note that there is a natural map

$$\rho : A \to \underline{\mathrm{Hom}}(\mathcal{A},\mathcal{A}) = \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}(\mathcal{A},\mathcal{A}[d])$$

sending $a \in A_d$ to $\pi\rho_a$, where $\rho_a : A \to A$ is right multiplication by a. It is easy to check that ρ is an anti-homomorphism of graded algebras, so each $\underline{\mathrm{Hom}}(\mathcal{A},\mathcal{F})$ has a natural right A-module structure. Of course,

$$\underline{\mathrm{Hom}}(\mathcal{A},\mathcal{F}) \cong \varinjlim \underline{\mathrm{Hom}}_A(A_{\geq n},\omega\mathcal{F})$$

already has a natural right A-module structure coming from the right action of A on $A_{\geq n}$. These two actions of A on $\underline{\mathrm{Hom}}(\mathcal{A},\mathcal{F})$ coincide.

Although Proposition 3.4 gives an explicit description of ω, its existence and basic properties are usually established by defining ω as follows. Given $M \in \mathsf{GrMod}A$, let E denote the injective envelope of $\overline{M} = M/\tau M$. Then $\omega\pi M$ is defined to be the largest graded submodule, H say, of E such that $\overline{M} \subset H$ and $H/\overline{M}$ is torsion. Thus $H/\overline{M} = \tau(E/\overline{M})$, and there is an exact sequence

$$0 \to \tau M \to M \to \omega\pi M \to \text{torsion} \to 0;$$

the last term in this sequence will be described in Proposition 11.5.

Example 3.8 Let $A = k[x]$. One can check directly that $E = k[x,x^{-1}]$ is an injective A-module, and hence is the injective envelope of A in $\mathsf{GrMod}A$. (Notice this shows that, in contrast to projectives, injectives in $\mathsf{GrMod}A$ need not be injective in $\mathsf{Mod}A$.) Since E/A is torsion it follows that $\omega\pi A \cong E$. (We will see later that for the polynomial ring in ≥ 2 variables, $\omega\pi A \cong A$.) In particular, $\omega\pi A$ is not a finitely generated A-module. ◊

Generalization The construction of $\mathsf{Tails}A$ can be made for graded rings that are not locally finite, and the theory in this section should be developed in that generality (as Artin and Zhang did in [228]). In that case the category $\mathsf{Fdim}A$ is replaced by $\mathsf{Tors}A$ which is defined to be the full subcategory of $\mathsf{GrMod}A$ consisting of those M such that very $m \in M$ is annihilated by $A_{\geq n}$ for some sufficiently large n.

5.4 Non-commutative projective spaces

Throughout this section k will denote a field.

The equivalence $\mathsf{Qcoh}X \cong \mathsf{GrMod}A/\mathsf{Fdim}A$ in Serre's theorem equates a topologically defined category with a purely algebraically defined category. For a non-commutative graded algebra, the construction of the topologically defined category presents insurmountable problems. There may be too few two-sided ideals to give a topological space of reasonable size, and it is usually not possible to localize in any reasonable way at prime ideals. However, there are no obstacles to forming the algebraically defined category $\mathsf{GrMod}A/\mathsf{Fdim}A$. That is therefore the approach we take. We consider this category as if it is the category of "quasi-coherent modules" on some imaginary non-commutative projective space.

Definition 4.1 Let A be a right noetherian, locally finite, connected graded k-algebra. The (non-commutative) projective space with homogeneous coordinate ring A is the space X defined by declaring

$$\mathsf{Mod}X = \mathsf{Tails}A.$$

We define $\operatorname{Proj} A$ to be the enriched projective space

$$\operatorname{Proj} A := (X, \mathcal{O}_X)$$

where $\mathcal{O}_X$ is A viewed as an object in $\mathsf{Tails}A = \mathsf{Mod}X$. That is, X is given the structure module $\mathcal{O}_X = \pi A$. ◊

Example 4.2 A non-commutative homogeneous coordinate ring of $\operatorname{Spec} k \times k$. First give $M_2(k[x]) = M_2(k) \otimes k[x]$ the grading with $\deg M_2(k) = 0$ and $\deg x = 1$. Let A be the subring

$$A = \begin{pmatrix} k[x^2] & xk[x^2] \\ xk[x^2] & k[x^2] \end{pmatrix}$$

with the inherited grading. Let X denote the projective space with homogeneous coordinate ring A. Thus $\mathsf{Mod}X = \mathsf{Tails}A$. To see that $\mathsf{Tails}A$ is equivalent to $\mathsf{Mod}k \times k$ first observe that a graded A-module is finite dimensional if and only if it is annihilated by x^2. Thus $\mathsf{Fdim}A$ consists of exactly those graded A-modules in which every element is killed by some power of x^2. Thus $\mathsf{Tails}A$ is equivalent to $\mathsf{GrMod}A[x^{-2}]$. However, $A[x^{-2}]$ has a unit of degree one, namely

$$\begin{pmatrix} 0 & x^{-1} \\ x^{-1} & 0 \end{pmatrix},$$

so $A[x^{-2}]$ is a progenerator in $\mathsf{GrMod}A[x^{-2}]$. The endomorphism ring of $A[x^{-2}]$ in $\mathsf{GrMod}A[x^{-2}]$ is

$$A[x^{-2}]_0 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}.$$

Thus $\mathsf{GrMod}A[x^{-2}]$ is equivalent to $\mathsf{Mod}k \times k$. Under the equivalence of categories $\mathcal{O}_X = \pi A$ corresponds to $k \times k$, so $\operatorname{Proj} X = (X, \mathcal{O}_X)$ is isomorphic to $\operatorname{Spec} k \times k$ with its usual structure sheaf.

The ring A is isomorphic as a graded ring to the path algebra of the quiver

$$\overset{1}{\bullet} \leftleftarrows \overset{2}{\bullet} \tag{4-1}$$

$\Diamond$

$\boxed{\text{Paul}}$ I'm a bit puzzled about this example. Example 3.14.5 showed that the affine space with coordinate ring A has two open subspaces isomorphic to the affine line. The only difference between those two lines occurs at the two points where x^2 is zero: each point belongs to just one of those lines, and the lines are the same once those two points are removed. So, after removing those two points the affine space with coordinate ring $A[x^{-2}]$ is isomorphic to $\mathbb{A}^1 \backslash \{0\}$. However, by analogy with the commutative case, the closed points in $\operatorname{Proj} A$ should be in bijection with the lines in $\mathsf{Mod}A[x^{-2}]$; obviously that does not happen!?

Using the twisting functor we define, for $\mathcal{F}, \mathcal{G} \in \mathsf{Tails}A$,

$$\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{Tails}}(\mathcal{F}, \mathcal{G}(n)).$$

In this way $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ is a graded k-vector space with

$$\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})_n := \mathrm{Hom}(\mathcal{F}, \mathcal{G}(n)).$$

By Proposition 2.6, a map $\varphi : A \to B$ of graded k-algebras induces an affine map of spaces $g : \mathsf{GrMod}B \to \mathsf{GrMod}A$. Whether g^*, g_*, and $g^!$, induce functors between the Tails categories, and hence a map between the associated projective spaces, depends on whether they send torsion modules to torsion modules. In the situation $A \to A/I$, it is easy to see that g^*, g_*, and $g^!$, all send torsion modules to torsion modules, so induce functors between $\mathsf{Tails}A$ and $\mathsf{Tails}A/I$. To show that I determines a closed subspace of $\mathsf{Tails}A$, we must show that the induced functors are adjoints to one another. We require some preparatory results.

Lemma 4.3 *Let $0 \to M \to E \to T \to 0$ be an exact sequence of R-modules. Let I be a two sided ideal in R. If $MI = 0$, then there is a surjective R-module map $T \otimes_R I \to EI$.*

Proof. The map $(E/M) \times I \to EI$ defined by

$$([e + M], x) \mapsto ex$$

is well-defined, bilinear, and agrees on $([er + M], x)$ and $([e + M], rx)$, so induces a surjective R-module homomorphism $(E/M) \otimes_R I \to EI$. $\square$

Lemma 4.4 *Let A be a left noetherian graded ring, I a graded two-sided ideal, and T a right A-module such that if $t \in T$, then $tA_n = 0$ for $n \gg 0$. If $t \otimes x \in T \otimes_A I$, then $(t \otimes x)A_{\geq n} = 0$ for $n \gg 0$.*

Proof. Since A is left noetherian, there is a surjective map of graded left A-modules
$$\oplus_{i=1}^{s} A(n_i) \to I$$
for some finite set of integers $n_1, \dots, n_s$. The induced map
$$\oplus_{i=1}^{s} T \otimes_A A(n_i) \to T \otimes_A I$$
is surjective. However, $T \otimes_A A(n_i) \cong T(n_i)$ via the map $t \otimes a \mapsto ta$, so $t \otimes a = 0$ in $T \otimes A(n)$ if $\deg a \gg 0$. The result follows. $\square$

We will use the following notation for the rest of this section
$$\begin{array}{ccc} \mathsf{GrMod}A/I & \xrightarrow{g_*} & \mathsf{GrMod}A \\ \pi_1 \Big\downarrow \quad \Big\uparrow \omega_1 & & \pi_2 \Big\downarrow \quad \Big\uparrow \omega_2 \\ \mathsf{Tails}A/I & & \mathsf{Tails}A \end{array} \tag{4-2}$$

Lemma 4.5 *Let I be a graded ideal in a graded ring A. Let $g : \mathsf{GrMod}A/I \to \mathsf{GrMod}A$ be the associated map. Then*

1. *if $P \in \mathsf{GrMod}A$ is torsion-free and saturated, then $g^! P = \omega_1 \pi_1 g^! P$;*
2. *$\omega_1 \pi_1 g^! = g^! \omega_2 \pi_2$;*
3. *$\omega_2 \pi_2 g_* = g_* \omega_1 \pi_1$.*

Proof. (1) Clearly $g^! P$ is the submodule of P annihilated by I. The injective envelope of $g^! P$ embeds in the injective envelope of P. Hence $\omega_1 \pi_1 g^! P$, which is an essential extension of $g^! P$, is a submodule of that injective envelope. Since $\omega_1 \pi_1 g^! P / g^! P$ is torsion, so is $\omega_1 \pi_1 g^! P + P/P$. But P is saturated, so $\omega_1 \pi_1 g^! P + P = P$. Thus $\omega_1 \pi_1 g^! P \subset P$. It is an A/I-module though, so it must be contained in $g^! P$. Hence the result.

(2) Let $N \in \mathsf{GrMod}A$, and set $\bar{N} = N/\tau N$. The last term in the exact sequence
$$0 \to g^!(\tau N) \to g^! N \to g^! \bar{N} \to \underline{\mathrm{Ext}}_A^1(A/I, \tau N).$$
is a subquotient of $\underline{\mathrm{Hom}}_A(A/I, J^1)$, where $0 \to \tau N \to J^0 \to J^1 \to \dots$ is a minimal injective resolution of τN. But all the J^i are torsion because $\mathsf{Fdim}A$ is closed under injectives, so $\underline{\mathrm{Hom}}_A(A/I, J^1)$ and $\underline{\mathrm{Ext}}_A^1(A/I, \tau N)$ are torsion. Therefore $\pi_1 g^! N = \pi_1 g^! \bar{N}$. Also $\pi_2 N = \pi_2 \bar{N}$, so to prove that both functors take the same value at N, we can, and will, assume that N is torsion-free.

Applying $\pi_1 g^!$ to the exact sequence $0 \to N \to \omega_2 \pi_2 N \to T \to 0$ gives an exact sequence $0 \to \pi_1 g^! N \to \pi_1 g^! \omega_2 \pi_2 N \to \pi_1 g^! T$. Since T is torsion, so is $g^! T$, whence $\pi_1 g^! N = \pi_1 g^! \omega_2 \pi_2 N$. Therefore, $\omega_1 \pi_1 g^! N = \omega_1 \pi_1 g^! \omega_2 \pi_2 N$. But part (1) applied to $P = \pi_2\omega_2 N$ gives $g^!\omega_2\pi_2 N = \omega_1\pi_1 g^!\omega_2\pi_2 N$. Combining the last two sentences gives the result.

(3) Let $M \in \mathsf{GrMod}A/I$, and set $\bar{M} = M/\tau M$. Since $\pi_1 M = \pi_1\bar{M}$ and $\pi_2 g_* M = \pi_2 g_*\bar{M}$, to prove that both functors take the same value at M, we can, and will, assume that M is torsion-free. To prove the result it suffices to show that $\omega_2\pi_2 M = \omega_1\pi_1 M$. More precisely, it suffices to show that the maximal essential extension, in $\mathsf{GrMod}A$, of M by a torsion module is actually an A/I-module.

Consider the exact sequence $0 \to M \to \omega_2\pi_2 M \to T \to 0$. It suffices to show that $\omega_2\pi_2 M$ is an A/I-module. By Lemmas 4.3 and 4.4, $(\omega_2\pi_2 M)I$ is a quotient of $T \otimes_A I$, so is torsion. However, M is an essential submodule of $\omega_2\pi_2 M$, so if $(\omega_2\pi_2 M)I$ were non-zero it would meet M in a non-zero submodule. But M is torsion-free, whence $(\omega_2\pi_2 M)I = 0$. □

Proposition 4.6 *Let A be a graded ring, and I a graded two-sided ideal. Then $\mathsf{Tails}A/I$ is a closed subspace of $\mathsf{Tails}A$.*

Proof. We will exhibit an adjoint triple $(f^*, f_*, f^!)$ so that $f_* : \mathsf{Tails}A/I \to \mathsf{Tails}A$ makes (4-2) a commutative diagram (up to natural equivalence).

Let $g : \mathsf{GrMod}A/I \to \mathsf{GrMod}A$ be the natural map. Since the functors g^*, g_*, and $g^!$ send torsion modules to torsion modules, $\pi_1 g^!$ and $\pi_1 g^*$ vanish on $\mathsf{Fdim}A$, and $\pi_2 g_*$ vanishes on $\mathsf{Fdim}A/I$. Therefore the universal property of the quotient categories, as expressed in Theorem 2.13.9, guarantees the existence of functors f^*, f_*, and $f^!$, satisfying

$$\pi_1 g^! = f^!\pi_2, \quad \pi_2 g_* = f_*\pi_1, \quad \pi_1 g^* = f^*\pi_2. \tag{4-3}$$

(Of course, these are not really equalities, but natural isomorphisms.) It follows that f^*, f_*, and $f^!$, could be defined as

$$f^! = \pi_1 g^!\omega_2, \quad f_* = \pi_2 g_*\omega_1, \quad f^* = \pi_1 g^*\omega_2. \tag{4-4}$$

To prove that $(f^*, f_*, f^!)$ is an adjoint triple, we fix modules $\mathcal{M} \in \mathsf{Tails}A/I$ and $\mathcal{N} \in \mathsf{Tails}A$. There are torsion-free saturated modules $M \in \mathsf{GrMod}A/I$ and $N \in \mathsf{GrMod}A$ such that $\mathcal{M} = \pi_1 M$ and $\mathcal{N} = \pi_2 N$. We will use the results in Lemma 4.5.

Write $Y = \mathsf{Tails}A/I$ and $X = \mathsf{Tails}A$. Then

$$\begin{aligned}
\mathrm{Hom}_Y(f^*\mathcal{N}, \mathcal{M}) &= \mathrm{Hom}_Y(f^*\pi_2 N, \pi_1 M) \\
&= \mathrm{Hom}_Y(\pi_1 g^* N, \pi_1 M) \\
&\cong \mathrm{Hom}_A(N, g_*\omega_1\pi_1 M) \\
&= \mathrm{Hom}_A(N, \omega_2\pi_2 g_* M) \\
&\cong \mathrm{Hom}_X(\pi_2 N, f_*\pi_1 M) \\
&= \mathrm{Hom}_X(\mathcal{N}, f_*\mathcal{M}).
\end{aligned}$$

Furthermore,

$$\begin{aligned}\operatorname{Hom}_X(f_*\mathcal{M},\mathcal{N}) &= \operatorname{Hom}_X(f_*\pi_1 M,\pi_2 N)\\ &= \operatorname{Hom}_X(\pi_2 g_* M,\pi_2 N)\\ &\cong \operatorname{Hom}_{A/I}(M, g^!\omega_2\pi_2 N)\\ &= \operatorname{Hom}_{A/I}(M,\omega_1\pi_1 g^! N)\\ &\cong \operatorname{Hom}_Y(\pi_1 M, f^!\pi_2 N)\\ &= \operatorname{Hom}_Y(\mathcal{M}, f^!\mathcal{N}).\end{aligned}$$

This proves the adjointness.

By Lemma 4.5, $\omega_2\pi_2 g_* \cong g_*\omega_1\pi_1$, so

$$\omega_2 f_* = \omega_2\pi_2 g_*\omega_1 \cong g_*\omega_1\pi_1\omega_1 \cong g_*\omega_1$$

and

$$f_*\pi_1 = \pi_2 g_*\omega_1\pi_1 \cong \pi_2\omega_2\pi_2 g_* \cong \pi_2 g_*.$$

Also, the natural transformation $f^*f_* \to \mathrm{id}_{\mathsf{Tails}A/I}$ is a natural equivalence because

$$f^*f_* = \pi_1 g^*\omega_2\pi_2 g_*\omega_1 \cong \pi_1 g^* g_*\omega_1\pi_1\omega_1 \cong \pi_1 g^* g_*\omega_1 \cong \pi_1\omega_1 \cong \mathrm{id}_{\mathsf{Tails}A/I}\,.$$

Hence, by Theorem 1.6.15, f_* is full and faithful. It follows from these remarks that we can view $\mathsf{Tails}A/I$ as a full subcategory of $\mathsf{Tails}A$, and simply write π and ω without any ambiguity. That is, if M is an A/I-module, then πM and $\omega\pi M$ are unambiguously defined.

It remains to show that $\mathsf{Tails}A/I$ is closed under submodules and quotients. Let $\mathcal{M} \in \mathsf{Tails}A/I$, and consider an exact sequence $0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$ in $\mathsf{Tails}A$. Write $\mathcal{M} = \pi M$, where $M = \omega\mathcal{M}$ is in $\mathsf{GrMod}A/I$. There is an exact sequence $0 \to \omega\mathcal{L} \to \omega\mathcal{M} \to \omega\mathcal{N} \to R^1\omega\mathcal{L}$. Since $\omega\mathcal{L}$ is a submodule of $\omega\mathcal{M}$, it is in $\mathsf{GrMod}A/I$. Therefore $\pi\omega\mathcal{L}$ is in $\mathsf{Tails}A/I$. But this is isomorphic to $\mathcal{L}$. Although $\omega\mathcal{N}$ is not necessarily a quotient of $\omega\mathcal{M}$, it differs from a quotient of $\omega\mathcal{M}$ by a torsion module because $R^1\omega\mathcal{L}$ is torsion (Corollary 11.4). Applying π kills the torsion, so the result follows. □

Perhaps the crucial point in the proof is the fact that $\omega_2 f_* = g_*\omega_1$, which can be interpreted as saying that if $\mathcal{M}$ is in $\mathsf{Tails}A/I$, then $\omega\mathcal{M}$ is in $\mathsf{GrMod}A/I$.

We call $\mathsf{Tails}A/I$ the closed subspace of $\mathsf{Tails}A$ **cut out** by I, or the **zero locus** of I.

Lemma 4.7 *Let A be a right noetherian graded ring, and I a graded two-sided ideal. Write $\tilde{X} = \mathsf{GrMod}A$, $X = \mathsf{Tails}A$, $\tilde{Y} = \mathsf{GrMod}A/I$, and $Y = \mathsf{Tails}A/I$. Let $\mathcal{M}$ be an X-module. Then $\mathcal{M}$ is supported on Y if and only if $\omega\mathcal{M}$ is supported on $\tilde{Y}$.*

Proof. Because the inclusion of $\mathsf{Mod}_Y X$ in $\mathsf{Mod}X$ has a right adjoint, $\mathsf{Mod}_Y X$ is closed under direct limits. Similarly, $\mathsf{Mod}_{\tilde{Y}}\tilde{X}$ is closed under direct limits. Therefore it is enough to prove the lemma for noetherian modules.

($\Rightarrow$) Suppose that $\mathcal{M}$ is noetherian. Then there is a finite chain $0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \ldots \subset \mathcal{M}_n = \mathcal{M}$ such that each $\mathcal{M}_i/\mathcal{M}_{i-1}$ is a $\tilde{Y}$-module. Since ω is left exact, this gives a chain $0 = \omega\mathcal{M}_0 \subset \omega\mathcal{M}_1 \subset \ldots \subset \omega\mathcal{M}_n = \omega\mathcal{M}$. By the remark preceeding this lemma, each $\omega(\mathcal{M}_i/\mathcal{M}_{i-1})$ is a $\tilde{Y}$-module. Since $\omega\mathcal{M}_i/\omega\mathcal{M}_{i-1}$ is a submodule of $\omega(\mathcal{M}_i/\mathcal{M}_{i-1})$, it is also a $\tilde{Y}$-module. Hence $\omega\mathcal{M}$ is supported on $\tilde{Y}$.

($\Leftarrow$) Suppose that $\mathcal{M}$ is noetherian. By Lemma 2.14.19, there is a noetherian submodule M of $\omega\mathcal{M}$ such that $\pi M \cong \mathcal{M}$. By hypothesis, M is supported on $\tilde{Y}$, so there is a finite chain $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ such that each M_i/M_{i-1} is a $\tilde{Y}$-module. Since π is exact, this gives a chain $0 = \pi M_0 \subset \pi M_1 \subset \ldots \subset \pi M_n = \mathcal{M}$, with each slice a Y-module. □

In preparation for the next section, we now specialize to the case of a closed subspace of $\mathsf{Tails}A$ cut out by a normal regular element of degree one. We will show that the open complement is an affine space.

Lemma 4.8 *If A is a connected graded algebra, and let z be a homogeneous normal regular element of degree one. Then*

$$\mathsf{GrMod}A[z^{-1}] \cong \mathsf{Mod}A[z^{-1}]_0.$$

Proof. If a graded algebra B contains a homogeneous unit of degree n, then $B \oplus B(1) \oplus \ldots \oplus B(n-1)$ is a progenerator in $\mathsf{GrMod}B$. Hence $\mathsf{GrMod}B$ is equivalent to the category of modules over the graded endomorphism ring of that module. Applying this to $B = A[z^{-1}]$, we see that B itself is a progenerator, and its endomorphism ring is its degree zero component. □

Example 4.9 If A is strongly graded, then $A.A_{\geq n} = A$, so $\mathsf{Tors}A$ consists only of the zero module, whence $\mathsf{Tails}A \cong \mathsf{GrMod}A \cong \mathsf{Mod}A_0$. ◊

Theorem 4.10 *Let A be a right noetherian graded ring, and let z be a regular normal element of positive degree. Let Y be the closed subspace of $X = \mathsf{Tails}A$ defined by $z = 0$. Let $j : X \backslash Y \to X$ be the inclusion. Let $\pi : \mathsf{GrMod}A \to \mathsf{Tails}A$ be the quotient functor, and let ω be its right adjoint.*

1. *An X-module, $\mathcal{M}$ say, is supported on Y if and only if $\omega\mathcal{M}$ is z-torsion.*
2. *$\mathsf{Mod}X\backslash Y$ is equivalent to $\mathsf{GrMod}A[z^{-1}]$, and to $\mathsf{Tails}A[z^{-1}]$.*
3. *If $\alpha : \mathsf{GrMod}A[z^{-1}] \to \mathsf{GrMod}A$ is the open immersion induced by $A \to A[z^{-1}]$, then $j^* \cong \alpha^*\omega$ and $j_* \cong \pi\alpha_*$.*
4. *If $\deg z = 1$, then $\mathsf{Mod}X\backslash Y \cong \mathsf{Mod}A[z^{-1}]_0$.*

Proof. By definition, $\mathsf{Mod}X\backslash Y$ is the quotient category $\mathsf{Mod}X/\mathsf{Mod}_Y X$. Since $\mathsf{Mod}X = \mathsf{Tails}A$ is a quotient category of $\mathsf{GrMod}A$, it follows from Proposition 2.13.11 that

$$\mathsf{Mod}X\backslash Y \cong \mathsf{GrMod}A/\mathsf{T},$$

where T is the full subcategory consisting of those M such that πM is in $\mathsf{Mod}_Y X$. By Lemma 4.7, this is precisely $\mathsf{Mod}_{\tilde{Y}}\tilde{X}$, where $\tilde{X} = \mathsf{GrMod}A$ and $\tilde{Y} = \mathsf{GrMod}A/(z)$. Therefore

$$\mathsf{Mod}X\backslash Y \cong \mathsf{Mod}\tilde{X}/\mathsf{Mod}_{\tilde{Y}}\tilde{X} = \mathsf{Mod}\tilde{X}\backslash\tilde{Y}.$$

We follow the proof of Proposition 3.10.2 to compute $\mathsf{Mod}\tilde{X}\backslash\tilde{Y}$. The map $A \to A[z^{-1}]$ induces a map of spaces $g : \mathsf{GrMod}A[z^{-1}] \to X$. If N is an $A[z^{-1}]$-module, then $N \otimes_A A[z^{-1}] \cong N$, so g^*g_* is naturally equivalent to the identity functor. The $\tilde{X}$-modules supported on $\tilde{Y}$ are those M such that every element of M is annihilated by a power of z. These are the modules M such that $M \otimes_A A[z^{-1}]$ is zero. Therefore g^* vanishes precisely on $\mathsf{Mod}_{\tilde{Y}}\tilde{X}$. It follows from Theorem 3.7.2 that $\mathsf{GrMod}A[z^{-1}] \cong \mathsf{Mod}\tilde{X}/\mathsf{Mod}_{\tilde{Y}}\tilde{X}$. But this is $\mathsf{Mod}(\tilde{X}\backslash\tilde{Y})$ by definition.

Finally, $\mathsf{GrMod}A[z^{-1}] \cong \mathsf{Mod}A[z^{-1}]_0$ by Lemma 4.8. □

Show in the previous result that $j : X\backslash Y \to X$ is such that $j^*(\pi M) = M[z^{-1}]_0$.

$\boxed{\text{Paul}}$ Do I really need the noetherian hypothesis for this? I think it might be need to show that z-torsion is a stable torsion theory i.e. that an essential extension of a z-torsion module is z-torsion. Check. If so, we would need a noetherian hypothesis to prove 5.3.

Questions. Suppose that $\mathsf{Mod}Y \subset \mathsf{GrMod}A$ is weakly closed and stable under the shift functor. Is its "image" under π weakly closed in $\mathsf{Tails}A$?

EXERCISES

4.1 Suppose that A is a finite dimensional graded k-algebra. Show that every object in $\mathsf{Tails}A$ is isomorphic to zero. Find weaker conditions on A for which the conclusion still holds.

4.2 Let $A = k[X]$ with $\deg X = 1$. Show that $\mathsf{Tails}A$ contains a unique irreducible object and that every object is isomorphic to a finite direct sum of copies of this irreducible object.

4.3 Let $A = k[X]$ with $\deg X = r > 0$. Show that $\mathsf{tails}A$ contains r irreducible objects up to isomorphism, that all these are shifts of a single one, and that every object is isomorphic to a finite direct sum of copies of these irreducible objects.

4.4 Let $A = k[x, y]/(f)$ where $k[x, y]$ is the commutative polynomial ring with its usual grading, and f is a homogeneous polynomial of degree $n \geq 1$ which is a product of n distinct linear terms. Show that $\mathsf{Tails}A$ has n non-isomorphic irreducible objects, and that every object is isomorphic to a direct sum of various irreducible objects. Are any of these irreducible objects shifts of other ones?

4.5 Let $A = k[x, y]/(y^2)$ be commutative with $\deg x = \deg y = 1$. Show that there is a unique irreducible object in $\mathsf{Tails}A$ and show that there exists an object of length two in $\mathsf{Tails}A$ which is not a direct sum of copies of this irreducible object.

4.6 If $\mathcal{F}$ is a non-zero irreducible object in $\mathsf{Tails}A$ show that $\omega\mathcal{F}$ is critical.

4.7 Show that the algebra A in Example 4.2 is isomorphic as a graded algebra to the path algebra of the quiver

$$\begin{tikzcd} \overset{1}{\bullet} & \overset{2}{\bullet} \arrow[l, shift right] \arrow[l, shift left] \end{tikzcd}. \tag{4-5}$$

5.5 Embedding an affine space in a projective space

A standard proceedure in algebraic geometry is to embed an affine variety as an open subvariety of a projective variety. In this section we show that the same proceedure can be carried out in the non-commutative setting.

Example 5.1 We examine how an affine curve is embedded in a projective one in the commutative case. We take the polynomial ring $k[X,Y,Z]$ with its standard grading, $\deg X = \deg Y = \deg Z = 1$, as the homogeneous coordinate ring of $\mathbb{P}^2$.

A plane projective curve is defined to be the zero locus of a homogeneous polynomial $g \in k[X,Y,Z]$ of degree ≥ 1. The quotient $k[X,Y,Z]/(g)$ is called the homogeneous coordinate ring of the curve. For example, the line at infinity, defined as the zero locus of Z is a projective curve. It consists of the points $(\alpha,\beta,0)$ and is isomorphic to the projective line $\mathbb{P}^1$.

The projective plane contains a copy of the affine plane. The map $(\alpha,\beta) \to (\alpha,\beta,1)$ from $\mathbb{A}^2$ to $\mathbb{P}^2$ is injective. We will identify $\mathbb{A}^2$ with this subset of $\mathbb{P}^2$. Thus $\mathbb{A}^2$ is the complement in $\mathbb{P}^2$ to the line at infinity. The coordinate functions on $\mathbb{A}^2$ are $x = X/Z$ and $y = Y/Z$. The coordinate ring $k[x,y]$ of $\mathbb{A}^2$ can be constructed directly from $k[X,Y,Z]$ as the degree zero component of the localization $k[X,Y,Z,Z^{-1}]$.

Let C be a curve in the affine plane. Suppose that C is the zero locus of a polynomial $f \in k[x,y]$. We will now exhibit a projective curve $\tilde{C}$ such that $C = \tilde{C} \cap \mathbb{A}^2$. If $\deg f = d$, where $\deg x = \deg y = 1$, and

$$f = \sum \lambda_{ij} x^i y^j,$$

we define

$$\tilde{f} := \sum \lambda_{ij} X^i Y^j Z^{d-i-j}.$$

Thus $\tilde{f}$ is a homogeneous polynomial of degree d, and therefore defines a projective curve $\tilde{C}$. Evaluating $\tilde{f}$ at a point in $\mathbb{A}^2 \subset \mathbb{P}^2$, we have

$$\tilde{f}(\alpha,\beta,1) = \sum \lambda_{ij}\alpha^i\beta^j = f(\alpha,\beta),$$

so $C = \mathbb{A}^2 \cap \tilde{C}$. The diffence between C and $\tilde{C}$ lies outside $\mathbb{A}^2$, so on the line at infinity. We think of $\tilde{C}$ as obtained from C by putting in some points at infinity.

Some simple examples are provided by the conics. Suppose C is the parabola $y = x^2$. Then $f = y - x^2$ and $\tilde{f} = YZ - X^2$. There is only one point at infinity on $\tilde{C}$, namely $(1,0,0)$. If C is the hyperbola defined by $f = xy - 1$, then $\tilde{f} = XY - Z^2$, and there are two points at infinity, $(1,0,0)$ and $(0,1,0)$. If C is a line, say $x + y + 1 = 0$ for example, then $\tilde{C}$ is the zero locus of $X + Y + Z$. This meets the line at infinity at one point.

The polynomial f can be recovered directly from $\tilde{f}$ as

$$f = \frac{1}{Z^d}\tilde{f}.$$

We now want to consider the relation between the coordinate rings

$$R = k[x,y]/(f)$$

and

$$\tilde{R} = k[X,Y,Z]/(\tilde{f}).$$

Recall that $k[x,y] = k[X/Z, Y/Z] = k[X,Y,Z,Z^{-1}]_0$. The degree zero component of $\tilde{R}[Z^{-1}]$ is generated by X/Z and Y/Z. Since $Z^{-d}\tilde{f} = 0$, we obtain $f(X/Z, Y/Z) = 0$ in $\tilde{R}[Z^{-1}]$. Therefore

$$R \cong \tilde{R}[Z^{-1}]_0, \tag{5-1}$$

the degree zero component of $\tilde{R}[Z^{-1}]$. Inverting Z amounts to working in the complement of $Z = 0$. $\Diamond$

If we are given an affine scheme $\operatorname{Spec} R$ in isolation, with no particular embedding in any $\mathbb{A}^n$, how can we construct a projective scheme that contains a copy of $\operatorname{Spec} R$? For example, how can we construct $\tilde{R}$ directly from R?

Let us first observe, that there is more than one way to embed an affine variety in a projective one. For example, although the affine line embeds in the projective line, it also embeds in the cuspidal cubic $X^3 = Y^2 Z$. To see this observe that the line $Y = 0$ meets the curve only at its singular point $(0,0,1)$, so the complement to that point is the affine curve with coordinate ring

$$\frac{k[X,Y,Z]}{(X^3 - Y^2 Z)}[Y^{-1}]_0 = k[X/Y, Z/Y].$$

However, $(X/Y)^3 = Z/Y$, so this is the polynomial ring in one variable, and the curve is the affine line.

Our construction of $\tilde{R}$ in Example 5.1 began not with $k[x,y]/(f)$, but with f as an element in $k[x,y]$. Thus we started with a preferred set of generators for R which we declared to be of degree one, and then "homogenized" f to obtain $\tilde{f}$, a homogeneous element in $k[X,Y,Z]$. Here is the process in the abstract.

Let R be a k-algebra generated by elements $x_1, \dots, x_n$. Set

$$R_1 = k + kx_1 + \ldots + kx_n$$

and define R_d to be the subspace of R spanned by all products $v_1 v_2 \ldots v_d$ where each $v_i \in R_1$. That is, $R_d = (R_1)^d$. We set $R_0 = k$. This gives subspaces

$$R_0 \subset R_1 \subset R_2 \subset \ldots,$$

and R is the union of them. These subspaces also satisfy

$$R_i R_j \subset R_{i+j}.$$

Such an ascending chain of subspaces is called a filtration on R, and R is called a filtered ring. We say that the elements in R_d have degree $\leq d$.

There are several graded rings one can associate to a filtered ring.

The associated graded ring of a filtered ring R is defined to be

$$\operatorname{gr} R = R_0 \oplus \frac{R_1}{R_0} \oplus \frac{R_2}{R_1} \oplus \cdots,$$

with multiplication defined by

$$[a + R_{i-1}].[b + R_{j-1}] := [ab + R_{i+j-1}]$$

whenever $a \in R_i \backslash R_{i-1}$ and $b \in R_j \backslash R_{j-1}$.

Make the polynomial extension $R[Z]$ a graded ring by setting $\deg R = 0$, and $\deg Z = 1$. The Rees ring of a filtered ring R is defined to be the subring

$$\tilde{R} = R_0 \oplus R_1 Z \oplus R_2 Z^2 \oplus \cdots$$

of $R[Z]$. Check this is a ring.

Proposition 5.2 *Let R be a filtered ring, and write $\tilde{R}$ for its Rees ring. Then*

1. *$\tilde{R}$ is connected, and is generated as an algebra by its degree one component;*
2. *Z is a central, regular element in $\tilde{R}$, so the natural map $\tilde{R} \to \tilde{R}[Z^{-1}]$ is injective;*
3. *$\tilde{R}[Z^{-1}]_0 \cong R$;*
4. *$\tilde{R}/(Z) \cong \operatorname{gr} R$.*

Proof. (1) The statement concerns two subrings of $R[Z, Z^{-1}]$. We will show they are equal. The degree zero component of $\tilde{R}[Z^{-1}]$ equals

$$R_0 + (R_1 Z)Z^{-1} + (R_2 Z^2)Z^{-2} + \cdots.$$

However, $R_0 \subset R_1 = (R_1 Z)Z^{-1} \subset R_2 = (R_2 Z^2)Z^{-2} \subset \cdots$, so $\tilde{R}[Z^{-1}] = R$.

(2) We have

$$\frac{\tilde{R}}{(Z)} = \frac{\oplus(\tilde{R})_d}{\oplus(Z\tilde{R})_d} = \frac{\oplus R_d Z^d}{\oplus Z R_{d-1} Z^{d-1}} = \oplus \frac{R_d Z^d}{R_{d-1} Z^d} \cong \oplus \frac{R_d}{R_{d-1}}.$$

It is now a matter of checking that the multiplications match up. □

Theorem 5.3 *Let $U = \mathsf{Mod} R$ be an affine space where R is a finitely generated k-algebra. Then there is a non-commutative projective space X with a closed hypersurface Y, such that $U \cong X \backslash Y$. Furthermore, $Y \cong \mathsf{Tails}(\operatorname{gr} R)$.*

Proof. Fix a finite set of algebra generators for R, and let $R_0 \subset R_1 \subset \dots$ be the associated filtration. Let A denote the associated Rees ring, and $z \in A_1$ be the homogenizing element. Set $X = \mathsf{Tails}A$, and let Y be the zero locus of z. Then $X \backslash Y$ is isomorphic to $\mathsf{Mod}A[z^{-1}]_0$ by Theorem 4.10, and $A[z^{-1}]_0 \cong R$ by Proposition 5.2. This proves the result. □

Theorem 5.3 says that the affine space $\mathsf{Mod}R$ embeds as an open subspace of the non-commutative projective space $\mathsf{Tails}\tilde{R}$. It is the open complement to a closed subspace that is isomorphic to $\mathsf{Tails}\,\mathrm{gr}\,R$.

Let $j : U \to X$ and $i : Y \to X$ be the inclusions in Theorem 5.3. We now proceed to describe the inverse and direct image functors.

Let M be an R-module. We will construct a graded $\tilde{R}$-module $\tilde{M}$ such that $M \cong j^*(\pi\tilde{M}) = \tilde{M}[Z^{-1}]_0$. The construction of $\tilde{M}$ depends on a choice of filtration on M and, if $i : Y \to X$ is the inclusion, then $i^*(\pi\tilde{M}) = \pi(\tilde{M}/MZ) = \pi(\mathrm{gr}\,M)$, where $\mathrm{gr}\,M$ is the associated graded module.

$\boxed{\text{Paul}}$ Show that $j^*(\pi\tilde{M})$ doesn't depend on the choice of filtration.

Fix a set of generators m_i for M, and define

$$M_n = \sum_i m_i R_n.$$

Then

$$M_0 \subset M_1 \subset M_2 \subset \dots$$

is an ascending chain of subspaces of M such that $M = \cup_{i=0}^{\infty} M_i$ and $M_i R_j \subset M_{i+j}$. Such data gives M the structure of a $\mathsf{filtered}$ R-module. There are many choices of generators for M, and therefore many different ways to make M a filtered R-module.

We write $M[z]$ for the $R[z]$-module $M \otimes_R R[z]$. It can be made into a graded $R[z]$-module by defining $\deg M = 0$ and $\deg z = 1$. We define

$$\tilde{M} = M_0 \oplus M_1 z \oplus M_2 z^2 \oplus \dots .$$

It is a graded module over the Rees ring $\tilde{R}$. Of course, $\tilde{M}$ depends on the choice of filtration on M.

We define $\tilde{M}[z^{-1}] = \tilde{M} \otimes_{\tilde{R}} \tilde{R}[z^{-1}]$.

If M is a filtered R-module, its $\mathsf{associated\ graded\ module}$ is defined to be

$$\mathrm{gr}\,M = M_0 \oplus \frac{M_1}{M_0} \oplus \frac{M_2}{M_1} \oplus \cdots .$$

It is made into a graded module over $\mathrm{gr}\,R$ by defining

$$[m + M_{i-1}].[b + R_{j-1}] := [mb + M_{i+j-1}]$$

whenever $m \in M_i \backslash M_{i-1}$ and $b \in R_j \backslash R_{j-1}$.

Proposition 5.4 *Let R be a filtered ring, and write $\tilde{R}$ for its Rees ring. Let M be a filtered R-module, and let $\tilde{M}$ be the module constructed above. Then*

1. $\tilde{M}[Z^{-1}]_0 \cong M$, *and*
2. $\tilde{M}/MZ \cong \operatorname{gr} M$.

Proof. (1) □

Notice that $\tilde{M}[Z^{-1}]_0$ does not depend on the choice of filtration.

Proposition 5.5 *Let $U = \mathsf{Mod}R$ be an affine space where R is a finitely generated k-algebra. Let U, X, and Y, be the spaces constructed in Theorem 5.3. Let $j : U \to X$ and $i : Y \to X$ be the inclusions. If M is an R-module, and $\tilde{M}$ an associated Rees module, then $j^*\pi\tilde{M} \cong M$ and $i^*\pi\tilde{M} \cong \pi(\operatorname{gr} M)$.*

Proof. □

EXERCISES

5.1 View the polynomial ring $R = k[x]$ as the coordinate ring of the affine line. Consider the three embeddings of AA^1 into the projective curves that are obtained by forming the Rees rings of R with respect to the filtrations induced by the three generating sets $\{x\}$, $\{x, x^2\}$, and $\{x, x^3\}$. Show that the ambient projective curves are $\mathbb{P}^1$, a smooth plane conic, and the cuspidal cubic respectively.

5.2 Let R be an arbitrary k-algebra, σ a k-linear algebra automorphism of R and δ a k-linear σ-derivation of R. Consider the Ore extension $A = R[t; \sigma, \delta]$ with defining relations

$$tr = r^\sigma t + \delta(r)$$

for $r \in R$. Give A the ascending filtration defined by $A_n = R + Rt + \ldots + Rt^n$ for $n \geq 0$. Show that the Rees ring $\tilde{A}$ is isomorphic as a graded algebra to the extension $R[u, v]$ with defining relations

$$ur = r^\sigma u + \delta(r)v, \qquad uv = vu, \qquad rv = vr,$$

for $r \in R$. Hence show that the affine space with coordinate ring $R[t; \sigma, \delta]$ is isomorphic to $X \backslash Y$, where $X = \mathsf{Tails}R[u, v]$, and Y is the zero locus of v.

5.6 Closed points in projective spaces

Throughout this section k denotes an algebraically closed field of characteristic zero.

Let $A = k_0 \oplus A_1 \oplus \ldots$ be a noetherian graded k-algebra, and set $X = \mathsf{Tails}A$. It is usual to impose some technical conditions on A (the condition χ of Artin and Zhang [22]) to ensure that X behaves like a commutative projective scheme. The condition ensures that $\dim_k \operatorname{Ext}^i_X(M, N) < \infty$ for all i and all noetherian X-modules M and N. Rather than discussing this condition, we will make the following assumption. It is mild, and is sufficient for our present needs.

Hypothesis. Throughout this section we suppose that

$$\dim_k \operatorname{Hom}_X(M, N) < \infty$$

for all noetherian X-modules M and N.

The hypothesis implies the following.

Proposition 6.1 *Let A be a connected graded noetherian k-algebra. Let $X = \mathsf{Tails}A$. Every simple X-module is tiny.*

Proof. If S is a simple X-module, and M is a noetherian X-module, then $\operatorname{Hom}_X(M, S)$ is finite dimensional over k, and therefore finite dimensional over $D = \operatorname{End}_X S$. $\square$

Therefore every simple X-module gives a closed point in X.

Definition 6.2 A graded A-module M is 1-critical if $M_n \neq 0$ for all $n \gg 0$, but $(M/N)_n = 0$ for all $n \gg 0$ whenever N is a non-zero submodule of M. $\Diamond$

If M is 1-critical, then πM is simple, so there is a closed point $p \in X$ such that $\mathcal{O}_p = \pi M$.

Corollary 6.3 *If $\mathcal{M}$ is a noetherian X-module, then there is an epimorphism $\mathcal{M} \to \mathcal{O}_p$ for some closed point $p \in X$.*

Hence there is a reasonably rich supply of points. (Perhaps if X is a surface there will be at least a curve of them.)

I do not think that every simple X-module really deserves to be called a closed point. The next example provides a simple module that probably does not deserve to be considered as a closed point. Perhaps we could call it a strange point.

Example 6.4 Let $B = B(E, \sigma, \mathcal{L})$ be the twisted homogeneous coordinate ring with respect to a degree three line bundle $\mathcal{L}$ on an elliptic curve E, where σ is an automorphism of E having infinite order. If I is a non-zero two-sided ideal of B, then $\dim_k B/I < \infty$. Hence, if B is viewed as a right module over $A = B^{\mathrm{op}} \otimes_k B$, then πB is a simple module over $X = \mathsf{Tails}A$. However, since $\operatorname{GKdim} B = 2$, πB will be a strange simple module. $\Diamond$

The next result provides reassurance by showing that our notion of closed point agrees with the usual one for projective varieties.

Theorem 6.5 *Assume that k is algebraically closed. Let A denote the homogeneous coordinate ring of a projective algebraic variety $X \subset \mathbb{P}^n_k$. Then there is a bijection*

$$\{\textit{closed points in } X\} \leftrightarrow \{\textit{closed points in } \mathsf{Tails}A\}$$

given by

$$p \leftrightarrow \pi(A/I_p)$$

where I_p is the ideal generated by the homogeneous $f \in A$ such that $f(p) = 0$.

Proof. Choose homogeneous coordinate functions $X_0, \dots, X_n$ on $\mathbb{P}^n$. Let $p = (\alpha_0, \dots, \alpha_n) \in X$. Then I_p is generated by $\{\alpha_i X_j - \alpha_j X_i \mid 0 \le i, j \le n\}$. We may assume that $\alpha_0 = 1$, whence X_0 generates A/I_p as a k-algebra. Thus $A/I_p \cong k[T]$, the polynomial ring in one variable. If we give T degree one, then this is an isomorphism of graded algebras. Since every proper quotient of $k[T]$ is finite dimensional, A/I_p is 1-critical, whence $\pi(A/I_p)$ is a simple module.

Before proving that every closed point in $\mathsf{Tails}A$ is of this form we make an observation. The exact sequence $0 \to (T) \to k[T] \to k \to 0$ translates to an exact sequence $0 \to (A/I_p)(-1) \to A/I_p \to k \to 0$. Therefore $\pi(A/I_p) \cong \pi(A/I_p)(-1)$. It follows that $\pi(A/I_p) \cong \pi(A/I_p)(r)$ for all $r \in \mathbb{Z}$.

Now let πM be a closed point of $\mathsf{Tails}A$. We may assume that M is finitely generated and 1-critical. Choose a graded ideal which is maximal amongst those which annihilate some non-zero homogeneous element of M; say $I = \operatorname{Ann}(m)$ with $0 \neq m \in M_r$. It follows that I is a prime ideal: if not, there exist homogeneous elements x and y such that $mxy = 0$ but neither mx nor my is zero—hence $\operatorname{Ann}(mx)$ is strictly larger that I, contradicting the choice of I.

Since M has no non-zero finite dimensional submodules, I is not equal to the augmentation ideal of A. Hence $\mathcal{V}(I) \subset X$ is non-empty. If $p \in \mathcal{V}(I)$ then $I \subset I_p$, so A/I_p is a quotient of $A/I \cong mA(r)$. But $\pi(mA) \cong \pi M$ since M is 1-critical, so $\pi(A/I_p)(r)$ is isomorphic to a quotient of πM; but πM is irreducible so $\pi M \cong \pi(A/I_p)(r) \cong \pi(A/I_p)$ as required. $\square$

$\boxed{\text{Paul}}$ Possibly Example 6.4 should be interpreted as saying we need a different definition of a closed point. What is needed is a new notion of dimension; to avoid confusion with other notions of dimension, like Krull dimension, let's call it size. The size of a module M should be defined in terms the behavior of $\operatorname{Ext}^i_X(M, N)$ and/or $\operatorname{Ext}^i_X(N, M)$. The bigger M is the more likely that these Ext groups are non-zero for a wide range of i and N. A somewhat simpler possibility is to use the Euler form on $K_0(X)$.

Let $U = \mathsf{Mod}R$ be an affine space. Let $X = \mathsf{Tails}\tilde{R}$ be a projective space containing U as the open complement of a hypersurface $Y = \mathsf{Tails}(\operatorname{gr} R)$. Show that $j : U \to X$ sends closed points to closed points, and that the union of the closed points on U together with the closed points on Y gives all the closed points in X.

5.7 A non-commutative projective plane

Throughout this section k denotes an algebraically closed field of characteristic zero.

In this section we study the non-commutative projective plane corresponding to the two-dimensional non-abelian Lie algebra.

Let $U = k[x, y]$ be the algebra studied in section 4.1 of chapter 4. Its defining relation is $xy - yx = x$. The Rees ring, or homogenization, of U with respect to the generators x and y is the graded algebra $A = k[x, y, z]$ with defining relations

$$zx = xz, \quad zy = yz, \quad xy - yx = xz. \tag{7-1}$$

The ring A is a noetherian domain with basis $\{x^i y^j z^k \mid i, j, k \geq 0\}$. It is a reasonable non-commutative analogue of the projective plane, and we shall therefore think of

$$X := \mathsf{Tails} A$$

as a non-commutative analogue of $\mathbb{P}^2$. The locus of $z = 0$ is $\mathsf{Tails} A/(z)$. Since $A/(z)$ is a commutative polynomial ring in two variables, $\mathsf{Tails} A/(z) \cong \mathsf{Mod}\mathbb{P}^1$. We will call this the line at infinity and denote it by L_∞. By Theorem 5.3, $X \backslash L_\infty$ is isomorphic to the affine space $\mathsf{Mod} U$.

Now consider a line module in $\mathsf{Mod} U$, say $L = U/(\alpha x + \beta y + \gamma)U$. Then the corresponding line module in $\mathsf{Tails} A$ is $A/(\alpha x + \beta y + \gamma)A$. It meets L_∞ at the point $A/(z, \alpha x + \beta y)$. This is the point $(-\beta, \alpha, 0)$. Recall that the slope of L is $-\alpha\beta^{-1}$, or infinity if $\beta = 0$. Thus, we see that $L \cap L_\infty$ is the slope of L, so parallel lines in $\mathsf{Mod} U$ meet at infinity.

In particular, the strange line modules $U/(x-\lambda)U$, or rather their projective completions $M = A/(x - \lambda z)A$ all pass through the point $(0, 1, 0)$. Write $\mathcal{L} = \pi(A/(x - \lambda z)A)$. Applying π to the exact sequence $0 \to M(-1) \to M \to A/(x, z) \to 0$ gives a sequence

$$0 \to \mathcal{L}(-1) \to \mathcal{L} \to \mathcal{O}_p \to 0.$$

There are exact sequences

$$0 \to \mathcal{L}(n-1) \to \mathcal{L}(n) \to \mathcal{O}_p \to 0$$

for all n.

We define a line L with structure module $\mathcal{L}$ by defining $\mathsf{Mod} L$ to be the smallest weakly closed subcategory of $\mathsf{Mod} X$ that contains all the $\mathcal{L}(i)$, and the obvious direct limit of the $\mathcal{L}(i)$s. I do not understand $\mathsf{Mod} L$.

5.8 Another projective plane

Example 8.1 Let $X = \mathsf{Tails} A$, where $A = k[x, y, z]$ be the ring with defining relations

$$yx = \alpha xy, \quad zy = \beta yz, \quad xz = \gamma zx$$

where $\alpha, \beta, \gamma \in k$ are such that $\alpha\beta\gamma \neq 1$. If $\alpha\beta\gamma = 1$, then A is a Zhang-twist of the commutative polynomial ring in three variables, so $X \cong \mathbb{P}^2$, meaning that $\mathsf{Mod} X$ is equivalent to $\mathsf{Mod}\mathbb{P}^2$.

The three normal elements x, y, and z, cut out three closed subspaces. Each of these is isomorphic to the projective line $\mathbb{P}^1$ because each of the three homogeneous coordinate rings is a Zhang-twist of the commutative polynomial ring in two variables.

The open complements to these three lines are all isomorphic to $\mathsf{Mod}k_q[x,y]$, where $q=\alpha\beta\gamma$. We will only check this for the open subspace $x\neq 0$. One has $A[x^{-1}]_0=k[yx^{-1},zx^{-1}]$, and elementary computations give

$$(yx^{-1})(zx^{-1})=\gamma^{-1}yzx^{-2} \quad\text{and}\quad (zx^{-1})(yx^{-1})=\alpha\beta yzx^{-2},$$

so if we set $u=zx^{-1}$ and $v=yx^{-1}$, then $vu=quv$. One needs to check that there are no other relations but this will follow from a suitable Hilbert series argument.

This shows that X is a projective space covered by three open affine spaces, each of which is isomorphic to $\mathbb{A}^2_q$, and that any one of these copies of $\mathbb{A}^2_q$ is the open complement to a closed subspace that is isomorphic to $\mathbb{P}^1$. We will call the line $z=0$ the line at infinity.

$\Diamond$

Example 8.2 Let $R=k[x,y]$ with defining relation $xy-yx=1$. Thus R is the first Weyl algebra. Consider the graded algebra $A=k[x,y,z]$ with defining relations

$$zx=xz, \quad zy=yz, \quad xy-yx=z^2.$$

It is not hard to show that $\{x^iy^jz^k \mid i,j,k\geq 0\}$ is a basis for A. Another important feature of A is that it is a noetherian domain. Thus, we consider $\mathsf{Tails}A$ as a non-commutative analogue of the usual projective plane.

Then $A/(z)$ is the commutative polynomial ring on two variables, so the locus $z=0$ is isomorphic to the usual projective line. We call this the line at infinity.

The open complement to the line at infinity is isomorphic to $\mathsf{Mod}R$ because $A[z^{-1}]_0\cong R$. Thus, the non-commutative affine plane $\mathsf{Mod}R$ has been embedded in this non-commutative projective plane.

Although $\mathsf{Mod}R$ has no closed points (Example 4.14) when $\operatorname{char}k=0$, $\mathsf{Tails}A$ has a projective line of closed points. Every known example of a non-commutative projective surface contains a commutative curve. We might say that although $\mathsf{Mod}R$ has no closed points, it has some closed points at infinity.

Notice that the "lines" $R/(x-\lambda)R$ all pass through a common point at infinity, namely $(0,1,0)=\mathsf{Tails}A/(x,z)$.

$\Diamond$

EXERCISES

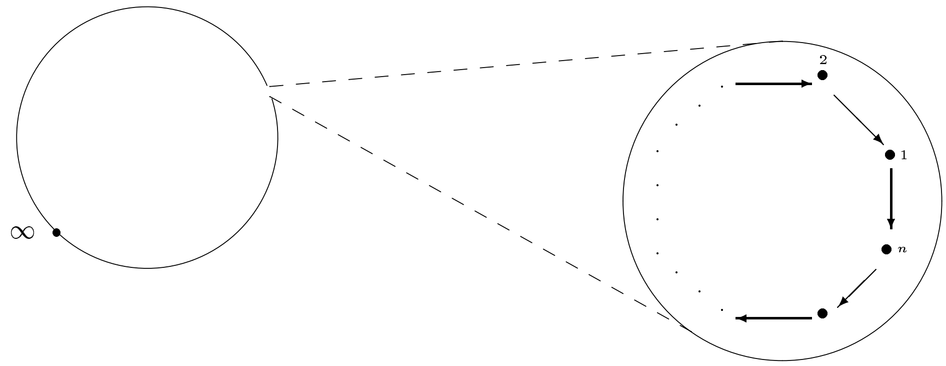
8.1 Let $A=k[x,y]$ be the $\mathbb{Z}$-graded commutative polynomial ring with $\deg x=2$ and $\deg y=3$. Examine $\mathsf{Tails}A$, and draw a picture of it.

8.2 Do the general case of the previous example including the case where $\deg x=1$ and $\deg y<0$.

8.3 Let $A=k[x,y,z]/(xy-z^2)$ be the $\mathbb{Z}$-graded commutative ring with $\deg z=2a$, $\deg x=b$ and $\deg y=2a-b$. Examine $\mathsf{Tails}A$, and draw a picture of it.

5.9 Some curves

If A is commutative, but not generated in degree one, then $\mathsf{Tails}A$ can be a non-commutative space. The next example exhibits a family of non-commutative curves C_n, $n \geq 1$. We have $C_1 \cong \mathbb{P}^1$, but C_n is non-commutative for $n \geq 2$. There are maps $f : C_n \to \mathbb{P}^1$. This is an isomorphism on $\mathbb{P}^1 \backslash \{\infty\}$, but $f^{-1}(\infty)$ consists of n closed points which are linked in a cycle. The curves C_n are projectivizations of the affine curves which were discussed in Proposition 3.5.23 and Example 3.14.4. Our picture of C_n for $n \geq 2$ is



This is a picture of the projective line with a point replaced by n points linked as in the magnified portion of the picture.

Example 9.1 Fix a positive integer n. Let $C = \mathsf{Tails}A$, where $A = k[x, y]$ is the commutative polynomial ring with $\deg x = 1$ and $\deg y = n$ where n is a positive integer. If $n = 1$, then $C \cong \mathbb{P}^1$. Suppose that $n > 1$.

Consider first the loci where x and y are zero. We denote these closed subspaces by $\mathcal{Z}(x)$ and $\mathcal{Z}(y)$ respectively.

The zero locus of y is $\mathsf{Tails}A/(y) \cong \mathsf{Tails}k[x]$. Since $\deg x = 1$, $\mathsf{Tails}k[x] \cong \mathsf{Mod}k$, so is a single point, $\operatorname{Proj} k[x]$. We call this the point at infinity, and label it ∞. The zero locus of x is $\mathsf{Tails}A/(x) = \mathsf{Tails}k[y]$. Since $\deg y = n$, this is isomorphic to the space in Example 1.2, namely $\operatorname{Spec} k^{\times n}$.

There are some obvious simple C-modules. The graded A-modules $A/(x^n - \lambda y)$, $0 \neq \lambda \in k$, and $A/(x)(i)$, $0 \leq i \leq n-1$, are all isomorphic to polynomial rings in one variable, so any proper quotient of them is finite dimensional. It follows that the images of these modules are simple C-modules. These deserve to be called points of C. The n points arising from $A/(x)(i)$, $0 \leq i \leq n-1$, are the points where x is zero, and $A/(y)$ is the point at infinity.

We now consider the open complements to the closed subspaces $\mathcal{Z}(x)$ and $\mathcal{Z}(y)$.

The complement to $\mathcal{Z}(x)$ is $\mathsf{Tails}k[x, x^{-1}, y]$. Since this ring has a homogeneous unit of positive degree, its only finite dimensional graded module is the zero module. Thus $\mathsf{Tails}k[x, x^{-1}, y] = \mathsf{GrMod}k[x, x^{-1}, y]$. Write

$D = k[x, x^{-1}, y]$. Since D has a unit of degree one, it follows that D is strongly graded, and that $\mathsf{GrMod}D \cong \mathsf{Mod}D_0$. But $D_0 = k[yx^{-n}]$, so $C \backslash \mathcal{Z}(x) \cong \mathbb{A}^1$.

We will show that $C\backslash\{\infty\}$ is isomorphic to the curve $\mathsf{Mod}Q$ where Q is the quiver

$$\begin{tikzcd} \underset{n}{\bullet} \arrow[r] & \underset{n-1}{\bullet} \arrow[r] & \bullet & \cdots & \bullet \arrow[r] & \underset{2}{\bullet} \arrow[r] & \underset{1}{\bullet} \arrow[llllll, bend right] \end{tikzcd}$$

A picture of this curve appears at (14-5) in chapter 3.

By definition, $C\backslash\{\infty\} = \mathsf{Tails}E$, where $E = k[x, y, y^{-1}]$. A straightforward computation shows that $E_0 = k[t]$, where $t = x^n y^{-1}$, and that

$$E_m = \begin{cases} x^m E_0 & \text{if } m \geq 0, \\ x^m t E_0 & \text{if } m < 0. \end{cases}$$

Since y is a unit of positive degree in E, $\mathsf{Tails}E = \mathsf{GrMod}E$. Notice that E is not strongly graded because $E_{-1}E_1 \neq E_0$. It is not difficult to show that

$$P = E(n-1) \oplus \cdots \oplus E(1) \oplus E$$

is a progenerator in $\mathsf{GrMod}E$, so $\mathsf{GrMod}E \cong \mathsf{Mod}\,\mathrm{End}_{\mathrm{gr}}\,P$. If we view the elements of P as row vectors, then $\mathrm{End}_{\mathrm{gr}}\,P$ can be viewed as matrices acting by right multiplication. The ij^{th} entry of $\mathrm{End}_{\mathrm{gr}}\,P$ is

$$(\mathrm{End}_{\mathrm{gr}}\,P)_{ij} = \mathrm{Hom}_{\mathrm{gr}}(E(n-i), E(n-j)) = E_{i-j} = \begin{cases} x^{i-j}E_0 & \text{if } i \geq j, \\ x^{i-j}tE_0 & \text{if } i < j. \end{cases}$$

This is a subalgebra of $M_n(k[x, y, y^{-1}])$. Explicitly,

$$\mathrm{End}_{\mathrm{gr}}\,P \cong \begin{pmatrix} E_0 & E_{-1} & \dots & \dots & \dots & E_{1-n} \\ E_1 & E_0 & \dots & \dots & \dots & E_{2-n} \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ E_{n-2} & & \dots & \dots & & E_{-1} \\ E_{n-1} & & \dots & \dots & E_1 & E_0 \end{pmatrix}$$

To show that $C\backslash\{\infty\}$ is isomorphic to the affine curve above, it suffices to show that $\mathrm{End}_{\mathrm{gr}}\,P$ is isomorphic to the ring S described in Proposition 5.23. We do this by verifying that $v^{-1}(\mathrm{End}_{\mathrm{gr}}\,P)v = S$ where

$$v = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & x & 0 & & 0 \\ 0 & 0 & x^2 & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & x^{n-1} \end{pmatrix}$$

The i^{th} diagonal entry of v is $v_i = x^{i-1}$, so the entry in the ij^{th} position of $v^{-1}(\operatorname{End}_{\mathrm{gr}} P)v$ is

$$x^{1-i}(\operatorname{End}_{\mathrm{gr}} P)_{ij}x^{j-1} = \begin{cases} E_0 & \text{if } i \geq j, \\ tE_0 & \text{if } i < j. \end{cases}$$

Since $E_0 = k[t]$, it follows that $v^{-1}(\operatorname{End}_{\mathrm{gr}} P)v$ has entries $k[t]$ on and below the diagonal, and entries $tk[t]$ above the diagonal, so it is isomorphic to S, as claimed.

The n^{th}-Veronese subalgebra $A^{(n)} = k[x^n, y]$ is generated in degree one, so by Serre's Theorem, $\mathsf{Tails}A^{(n)} \cong \mathbb{P}^1$. There is a map of spaces $f : C \to \mathbb{P}^1$ induced by sending a graded A-module M to the graded $A^{(n)}$-module $M^{(n)}$. $\lozenge$

5.10 The affine line with a double point

Let $A = k[x, y]$ be the commutative polynomial ring with $\deg x = 1$ and $\deg y = -1$. The points in $Z = \mathsf{Tails}A$ are $\mathcal{O}_\lambda = A/(xy - \lambda)$ for $0 \neq \lambda \in k$ and $\mathcal{O}_p = A/(x)$ and $\mathcal{O}_q = A/(y)$.

Define $\mathcal{O}_Z$ to be the image of A in $\mathsf{Tails}A$.

5.11 Homological algebra in Tails$A$

In this section A denotes a noetherian connected graded k-algebra.

Since $\mathsf{GrMod}A$ is a Grothendieck category it has injective envelopes. An injective resolution

$$0 \to M \to E^0 \to E^1 \to \cdots$$

in $\mathsf{GrMod}A$ is $\mathsf{minimal}$ if E^j is the injective envelope of the image of E^{j-1} for all $j \geq 0$.

For each $q \geq 0$ we write $\operatorname{Ext}^q_{\mathsf{Gr}}(N, -)$ for the right derived functors of $\operatorname{Hom}_{\mathsf{Gr}}(N, -)$. We may compute these by taking injective resolutions of the argument in the usual way. $\operatorname{Ext}^q_{\mathsf{Gr}}(N, M)$ can also be computed by taking a projective resolution of N. We will use the following notation:

$$\begin{aligned} \operatorname{Ext}^q_A(N, M) &= \text{the usual Ext groups in } \mathsf{Mod}A, \\ \operatorname{Ext}^q_A(N, M)_d &= \text{the derived functors of } \operatorname{Hom}_A(N, -)_d, \\ \underline{\operatorname{Ext}}^q_A(N, M) &= \bigoplus_{n \in \mathbb{Z}} \operatorname{Ext}^q_A(N, M)_n. \end{aligned}$$

The field k is a graded algebra concentrated in degree zero. The $\mathsf{graded\ dual}$ of a graded vector space is

$$V^* := \underline{\operatorname{Hom}}_k(V, k).$$

Thus $(V^*)_d = \operatorname{Hom}_k(V_{-d}, k)$.

Proposition 11.1 *The injective envelope of the trivial module is isomorphic to* $A^* = \underline{\mathrm{Hom}}_k(A,k)$ *with right action of* $x \in A$ *given by* $(\lambda.x)(a) = \lambda(xa)$ *for* $\lambda \in A^*$. *The copy of* k *inside* A^* *is* $k\epsilon$, *where* $\epsilon : A \to k$ *is the projection with kernel* $A_{\geq 1}$.

Proof. It is easy to see that A^* is an essential extension of $k\epsilon$. To see that A^* is injective, suppose that $f : N \to M$ is an injective map of graded A-modules and that $\alpha : N \to A^*$ is a graded A-module map. Let $\alpha^* : A \to N^*$ and $f^* : M^* \to N^*$ be the maps dual to α and f. These are homomorphisms of graded left A-modules. Since f^* is surjective and A is projective, there is a homomorphism $\theta : A \to M^*$ of left A-modules such that $f^* \circ \theta = \alpha^*$. Now define $\beta : M \to A^*$ by

$$\beta(m)(a) = \theta(a)(m),$$

Then β is a right A-module map satisfying $\beta \circ f = \alpha$. It follows that A^* is injective. $\square$

Lemma 11.2 *An essential extension of a torsion (respectively, torsion-free) module is torsion (respectively, torsion-free). In particular,* $\mathsf{Fdim}A$ *is closed under injective envelopes.*

Proof. Let $M \subset E$ be an essential extension. If τE is non-zero it has non-zero intersection with M, so τM is non-zero. Thus, if M torsion-free so is E. Conversely, suppose that M is torsion. Let $e \in E$. Then $eA \cap M$ is torsion, hence finite dimensional since A is noetherian. Thus $eA_{\geq n} \cap M = 0$ for $n \gg 0$, whence $\dim_k eA < \infty$, since A is locally finite. Thus E is a sum of finite dimensional modules, hence torsion. $\square$

Lemma 11.3
1. *Each injective in* $\mathsf{GrMod}A$ *decomposes as a direct sum of a torsion injective and a torsion-free injective.*
2. *If* A *is connected, then every torsion injective is a direct sum of shifts of* $A^* = \underline{\mathrm{Hom}}_k(A,k)$.

Proof. (1) Let E an injective. Being injective it contains a copy of the injective envelope of τE, say I. Since I is injective, $E = I \oplus Q$ for some other submodule Q; being a summand of an injective, Q is also injective, and torsion-free since $\tau E \subset I$. Finally, by Lemma 11.2, I is torsion.

(2) Let I be a torsion injective in $\mathsf{GrMod}A$. If $0 \neq M \in \mathsf{Fdim}A$, then $\underline{\mathrm{Hom}}_A(k,M) \neq 0$. We may consider $S = \underline{\mathrm{Hom}}_A(k,I)$ as a submodule of I; it is a (possibly infinite) direct sum of shifts of ${}_Ak$. If M is a non-zero submodule of I then, since M is torsion, $\mathrm{Hom}_A(k,M) \neq 0$, whence $M \cap S \neq 0$, so S is essential in I; thus $I = E(S)$. Since A is right noetherian, a direct sum of injective modules is injective, whence $E(S)$ is a (possibly infinite) direct sum of shifts of $E({}_Ak) \cong A^*$. $\square$

Corollary 11.4 *Let A be a connected graded k-algebra. If $i \geq 1$, and $\mathcal{M} \in \mathsf{Tails}A$, then $R^i\omega\mathcal{M}$ is a torsion module.*

Proof. It follows from Theorem 2.14.15, that $(R^i\omega) \circ \pi \cong R^{i+1}\tau$ for all $i \geq 0$. But, $R^{i+1}\tau$ takes values in $\mathsf{Fdim}A$ by definition. $\square$

Proposition 11.5 *For each $M \in \mathsf{GrMod}A$, there is an exact sequence*

$$0 \to \tau M \to M \to \omega\pi M \to \varinjlim \underline{\mathrm{Ext}}^1_A(A/A_{\geq n}, M) \to 0.$$

Proof. Over directed sets $\varinjlim$ is an exact functor, so taking direct limits of the exact sequences

$$0 \to \underline{\mathrm{Hom}}_A(A/A_{\geq n}, M) \to \underline{\mathrm{Hom}}_A(A, M) \to \underline{\mathrm{Hom}}_A(A_{\geq n}, M) \to \underline{\mathrm{Ext}}^1_A(A/A_{\geq n}, M) \to 0$$

yields the result, because $\varinjlim \underline{\mathrm{Hom}}_A(A/A_{\geq n}, M) = \tau M$. $\square$

If A, B, C are graded algebras, and ${}_AM_B$ and ${}_AN_C$ are graded bimodules, then $\underline{\mathrm{Ext}}^q_A(N, M)$ is a graded C-B-bimodule.

The Ext-groups inherit good properties from their second argument.

Proposition 11.6 *Let A be right noetherian, and $\mathbb{N}$-graded. If $N \in \mathsf{grmod}A$ and $M \in \mathsf{GrMod}A$, then*

1. *if M is left (or right) bounded, so is $\underline{\mathrm{Ext}}^q_A(N, M)$;*
2. *if M is locally finite, so is $\underline{\mathrm{Ext}}^q_A(N, M)$;*
3. *if M is a graded A-B bimodule, where B is a right noetherian graded algebra, and $M \in \mathsf{grmod}B$, then $\underline{\mathrm{Ext}}^q_A(N, M) \in \mathsf{grmod}B$ too.*

Proof. Take a projective resolution for N, each term of which is a finite direct sum of shifts of A. Apply $\underline{\mathrm{Hom}}_A(-, M)$ to get a complex in which each term is a finite direct sum of shifts of M. Each $\underline{\mathrm{Ext}}^q_A(N, M)$ is a subquotient of these terms, so inherits the relevant property from M. $\square$

In section 5.13 we define cohomology groups for projective spaces. To establish the basic properties of the cohomology groups requires an understanding of injectives in $\mathsf{Tails}A$. The following is a special case of Theorem 2.14.14.

Proposition 11.7 1. *$\mathsf{Tails}A$ has enough injectives.*

2. *If $\mathcal{Q} \in \mathsf{Tails}A$ is injective, then $\omega\mathcal{Q}$ is a torsion-free injective.*
3. *If $Q \in \mathsf{GrMod}A$ is torsion-free injective, then πQ is injective and $Q \cong \omega\pi Q$.*

If $\mathcal{F} \in \mathsf{Tails}A$, then $\mathrm{Hom}_{\mathsf{Tails}}(\mathcal{F}, -)$ is left exact, so we may define its right derived functors, and compute them via injective resolutions. That is, if $\mathcal{G} \to \mathcal{E}^\bullet$ is an injective resolution in $\mathsf{Tails}A$, then

$$\mathrm{Ext}^q(\mathcal{F}, \mathcal{G}) := h^q(\mathrm{Hom}_{\mathsf{Tails}}(\mathcal{F}, \mathcal{E}^\bullet)),$$

the q^{th} homology group of the complex. We also define

$$\underline{\mathrm{Ext}}^q(\mathcal{F}, \mathcal{G}) := \bigoplus_{d \in \mathbb{Z}} \mathrm{Ext}^q(\mathcal{F}, \mathcal{G}[d]).$$

These Ext groups are k-vector spaces. We will show that they can be computed in terms of Ext groups in $\mathsf{GrMod}A$ by using ω.

Proposition 11.8 *Let $N \in \mathsf{grmod}A$ and $M \in \mathsf{GrMod}A$. Let $E^\bullet M$ be a minimal injective resolution of M, and write $E^\bullet M = I^\bullet M \oplus Q^\bullet M$, where $I^\bullet M$ is the torsion part of $E^\bullet M$ (it is a subcomplex) and $Q^\bullet M$ is a torsion-free complement. Write $\mathcal{N} = \pi N$ and $\mathcal{M} = \pi M$. Then*

1. $\mathrm{Ext}^q(\mathcal{N}, \mathcal{M}) = h^q(\mathrm{Hom}_{\mathsf{Gr}}(N, Q^\bullet M))$
2. $\underline{\mathrm{Ext}}^q(\mathcal{N}, \mathcal{M}) \cong \varinjlim \underline{\mathrm{Ext}}^q_A(N_{\geq n}, M)$

Proof. (1) Although $Q^\bullet M$ is not usually a subcomplex of $E^\bullet M$, we may identify it with the complex $E^\bullet M / I^\bullet M$. The exactness of π implies that $\mathcal{M} \to \pi E^\bullet \simeq \pi Q^\bullet$ is an injective resolution of $\mathcal{M}$ in $\mathsf{Tails}A$. But

$$\mathrm{Hom}(\mathcal{N}, \pi Q^\bullet) \cong \mathrm{Hom}_{\mathsf{Gr}}(N, \omega\pi Q^\bullet \cong Q^\bullet),$$

so the result follows.

(2) First observe that $\varinjlim \underline{\mathrm{Hom}}_A(N_{\geq n}, I^\bullet) = 0$: if $f : N_{\geq n} \to I^\bullet$, then $N_{\geq n}/\ker f$ is finite dimensional because $I^\bullet$ is torsion and N is noetherian, whence $N_{\geq r} \subseteq \ker f$ for $r \gg 0$, which implies that in the direct limit f becomes zero. Therefore

$$\begin{aligned} \varinjlim \underline{\mathrm{Ext}}^q_A(N_{\geq n}, M) &= \varinjlim h^q(\underline{\mathrm{Hom}}_A(N_{\geq n}, I^\bullet \oplus Q^\bullet)) \\ &= h^q(\varinjlim \underline{\mathrm{Hom}}_A(N_{\geq n}, Q^\bullet)) \\ &\cong h^q(\underline{\mathrm{Hom}}(\mathcal{N}, \pi Q^\bullet)) \\ &= \underline{\mathrm{Ext}}^q(\mathcal{N}, \mathcal{M}), \end{aligned}$$

as required. □

5.12 The condition χ

Throughout this section A is a noetherian connected graded k-algebra.

In order to prove a non-commutative version of Serre's Finiteness Theorem in the next section we need some technical results.

Lemma 12.1 *Write* $[l,r] = \{T \in \mathsf{GrMod}A \mid T_{<l} = T_{>r} = 0\}$*.*
If $\underline{\mathrm{Ext}}^j_A(A/A_{\geq 1}, M) \in [l', r']$ *for all* $j \leq i$*, and* $T \in [l,r]$*, then*

$$\underline{\mathrm{Ext}}^j_A(T, M) \in [l' - r, r' - l]$$

for all $j < i$*.*

Proof. By induction on $r - l$, we reduce to $r - l = 1$, in which case T is a direct sum of shifts of $A/A_{\geq 1}$; the lemma is easy for such T. □

Proposition 12.2 *Let* $M \in \mathsf{grmod}A$ *and fix* $i \geq 0$*. The following are equivalent:*

1. *for all* $j \leq i$*,* $\underline{\mathrm{Ext}}^j_A(A/A_{\geq 1}, M)$ *is finite dimensional;*
2. *for all* $j \leq i$*,* $\underline{\mathrm{Ext}}^j_A(A/A_{\geq n}, M)$ *is finite dimensional for all* n*;*
3. *for all* $j \leq i$ *and all* $N \in \mathsf{grmod}A$*,* $\underline{\mathrm{Ext}}^j_A(N/N_{\geq n}, M)$ *has a right bound independent of* n*;*
4. *for all* $j \leq i$ *and all* $N \in \mathsf{grmod}A$*,* $\varinjlim \underline{\mathrm{Ext}}^j_A(N/N_{\geq n}, M)$ *is right bounded.*

Proof. First, by Proposition 11.6, if $T \in \mathsf{grmod}A$, $\underline{\mathrm{Ext}}^q_A(T, M)$ is a subquotient of a finite direct sum of shifts of M, so is left bounded and locally finite.

We will prove the result by induction on i. For $i = 0$, (1)–(4) all hold because $\dim_k T < \infty$ implies that $\underline{\mathrm{Hom}}_A(T, M) \subseteq \underline{\mathrm{Hom}}_A(T, \tau M)$ which is finite dimensional since $\dim_k(\tau M) < \infty$; notice that (4) holds because $\varinjlim \underline{\mathrm{Hom}}_A(A/A_{\geq n}, M) = \tau M$. So suppose the Proposition is true for $i - 1$; i.e., the four conditions are equivalent.

(1) ⇔ (2) If (1) holds, the previous lemma implies that $\underline{\mathrm{Ext}}^j_A(A/A_{\geq n}, M)$ is bounded, and hence finite dimensional by the first paragraph; thus (2) holds. The converse is a tautology.

(1) ⇒ (3) The exact sequence $0 \to T \to N/N_{\geq n+1} \to N/N_{\geq n} \to 0$ yields an exact sequence.

$$\underline{\mathrm{Ext}}^{j-1}_A(T, M) \to \underline{\mathrm{Ext}}^j_A(N/N_{\geq n}, M) \to \underline{\mathrm{Ext}}^j_A(N/N_{\geq n+1}, M) \to \underline{\mathrm{Ext}}^j_A(T, M).$$

But $T \in [n, n]$, so by Lemma 12.1, the first and last terms are bounded, and their right bounds approaches $-\infty$ as $n \to \infty$. Hence, given $d \in \mathbb{Z}$, there is a natural isomorphism

$$\underline{\mathrm{Ext}}^j_A(N/N_{\geq n}, M)_{\geq d} \xrightarrow{\sim} \underline{\mathrm{Ext}}^j_A(N/N_{\geq n+1}, M)_{\geq d} \qquad (12\text{-}1)$$

for $n \gg 0$. By Lemma 12.1, these are right bounded, so have a right bound which is independent of n.

(3) ⇒ (4) This is immediate.

(4) ⇒ (1) Consider the exact sequence

$$\underline{\mathrm{Ext}}^{i-1}_A(A_{\geq 1}/A_{\geq n}, M) \to \underline{\mathrm{Ext}}^i_A(A/A_{\geq 1}, M) \to \underline{\mathrm{Ext}}^i_A(A/A_{\geq n}, M).$$

By hypothesis the direct limit of the last term is right bounded. Since (4) holds for i, and hence for $i-1$, the direct limit of the first term is right bounded. Hence so is the direct limit of the middle term. But that is simply $\underline{\mathrm{Ext}}^i_A(A/A_{\geq 1}, M)$, which we already know is left bounded and locally finite, whence it is finite dimensional. Thus (1) is true. □

Definition 12.3 Let $M \in \mathsf{grmod}A$. We say that

- $\chi_i(M)$ holds if the equivalent conditions of Proposition 12.2 hold;
- $\chi(M)$ *holds* if $\chi_i(M)$ holds for all i;
- A satisfies χ if $\chi(M)$ holds for all $M \in \mathsf{grmod}A$.

Proposition 12.4 *If $M \in \mathsf{grmod}A$, the following are equivalent:*

1. $\chi_1(M)$ *holds;*
2. $\operatorname{coker}(M \to \omega\pi M)$ *is right bounded;*
3. $(\omega\pi M)_{\geq d}$ *is finitely generated for all $d \in \mathbb{Z}$.*

Proof. We will use the exact sequence

$$0 \to \tau M \to M \to \omega\pi M \to \varinjlim \underline{\mathrm{Ext}}^1_A(A/A_{\geq n}, M) \to 0.$$

The equivalence of (1) and (2) is a restatement of the equivalence of (1) and (4) in Proposition 12.2, noting that the proof of (4) implies (1) only used the truth of (4) for $N = A$.

(1) $\Rightarrow$ (3). Fix $d \in \mathbb{Z}$, and consider

$$M_{\geq d} \to (\omega\pi M)_{\geq d} \to \varinjlim \underline{\mathrm{Ext}}^1_A(A/A_{\geq n}, M)_{\geq d} \to 0. \tag{12-2}$$

By hypothesis the first term is finitely generated. Since $\chi_1(M)$ holds, part (3) of Proposition 12.2 ensures that the last term of (12-2) is right bounded and hence finite dimensional. It follows that $(\omega\pi M)_{\geq d}$ is finitely generated too.

(3) $\Rightarrow$ (1). The hypothesis ensures that the last term of (12-2) is finitely generated, but it is also torsion, hence finite dimensional. Therefore part (4) of Proposition 12.2 holds for $i = 1$ (with $N = A$) and, as noted, this ensures that part (1) of Proposition 12.2 holds too; i.e., $\chi_1(M)$ holds. □

Rephrasing part (2) of Proposition 12.4, if A satisfies χ_1, then $\omega\pi M$ is finitely generated up to torsion whenever $M \in \mathsf{grmod}A$ (Example 3.8 showed that $\omega\pi M$ is not generally finitely generated). Part (3) of Proposition 12.4 says that if $\omega\pi M$ is considered as a rather nice module with respect to torsion, then M is not too far from being nice—at least $M_{\geq d} \cong (\omega\pi M)_{\geq d}$ for $d \gg 0$.

The condition χ is a non-commutative phenomenon. The next two results show that quotients of polynomial rings satisfy it, and the example which follows these positive results exhibits a non-commutative algebra which does not satisfy χ_1.

Definition 12.5 A locally finite connected k-algebra, A say, is Artin-Schelter regular of dimension $n+1$ if

- $\operatorname{gldim} A = n+1 < \infty$,
- $\operatorname{GKdim} A < \infty$, and
- A is Gorenstein, meaning that $\operatorname{Ext}^i_A({}_Ak, A) = \begin{cases} 0 & \text{if } i \neq n+1 \\ k & \text{if } i = n+1 \end{cases}$. ◊

Polynomial rings, and more generally iterated Ore extensions

$$k[X_0][X_1;\sigma_1,\delta_1]\cdots[X_n;\sigma_n,\delta_n]$$

where each σ_i is an automorphism and $\deg(X_i)=1$ for all i, are Artin-Schelter regular; so too are the Sklyanin algebras.

Theorem 12.6 *Noetherian Artin-Schelter regular algebras satisfy χ.*

Proof. Let A be such an algebra, and $M \in \mathsf{grmod}A$. We proceed by induction on $\operatorname{pdim}(M)$. If $\operatorname{pdim}(M)=0$, then A is a finite direct sum of shifts of A; but $\underline{\operatorname{Ext}}^j_A(A/A_{\geq 1}, A)$ is finite dimensional by the Gorenstein hypothesis, so $\chi_1(M)$ holds. If $\operatorname{pdim}(M) > 0$, write $0 \to K \to P \to M \to 0$ with P projective, and $\operatorname{pdim}(K) = \operatorname{pdim}(M)-1$. By the induction hypothesis, the last term of the exact sequence

$$\underline{\operatorname{Ext}}^j_A(k,P) \to \underline{\operatorname{Ext}}^j_A(k,M) \to \underline{\operatorname{Ext}}^{j+1}_A(k,K)$$

is finite dimensional, as is the first term, whence so is the middle term. □

Proposition 12.7 *If A is noetherian and satisfies χ_i, so does A/I for all ideals I.*

Proof. Write $B=A/I$ and let $M \in \mathsf{grmod}(B)$. We will proceed by induction on i; since B satisfies χ_0, we will assume the result is true for $i-1$. Thus B satisfies χ_{i-1}, and we must show B satisfies χ_i.

Consider the spectral sequence

$$E_2^{pq} = \underline{\operatorname{Ext}}^P_B(\operatorname{Tor}^A_q(B, A/A_{\geq n}), M) \Rightarrow \underline{\operatorname{Ext}}^{p+q}_A(A/A_{\geq n}, M).$$

Since A is noetherian, each term in the minimal projective resolution of B_A is a finite direct sum of shifts of A, whence each $\operatorname{Tor}^A_q(B, A/A_{\geq n})$ is finite dimensional. In particular, it is right bounded. Since A is projective,

$$\operatorname{Tor}^A_q(B, A/A_{\geq n}) \cong \operatorname{Tor}^A_{q-1}(B, A_{\geq n})$$

for $q \geq 2$, and

$$\operatorname{Tor}^A_1(B, A/A_{\geq n}) \subseteq B \otimes_A A_{\geq n}.$$

By taking a minimal resolution of $A_{\geq n}$, it is easy to see that

$$\mathrm{Tor}^A_{q-1}(B, A_{\geq n}) \in [n, \infty)$$

for all $q \geq 1$, whence

$$\mathrm{Tor}^A_q(B, A/A_{\geq n}) \in [n, \infty)$$

for all $q \geq 1$. Since B satisfies χ_{i-1}, Lemma 12.1 with $T = \mathrm{Tor}^A_q(B, A/A_{\geq n})$ implies that, for all $p \leq i-1$, the right bound of E_2^{pq} tends to $-\infty$ as $n \to \infty$. Thus, given $d \in \mathbb{Z}$, $p \leq i-1$, and $q \geq 1$,

$$(E_2^{pq})_{\geq d} = 0$$

for $n \gg 0$. Hence, for all $p \leq i$,

$$(E_2^{p0})_{\geq d} \cong \underline{\mathrm{Ext}}^p_A(A/A_{\geq n}, M)_{\geq d}$$

for all $n \gg 0$. That is, for all $p \leq i$, and all $n \gg 0$,

$$\underline{\mathrm{Ext}}^p_B(B/B_{\geq n}, M)_{\geq d} \cong \underline{\mathrm{Ext}}^p_A(A/A_{\geq n}, M)_{\geq d}.$$

But A satisfies χ_i, so the condition in part (3) of Proposition 12.2 implies that B satisfies χ_i too. $\square$

Example 12.8 [236] Fix $0 \neq q \in k$, and suppose that q is not a root of unity. Let $B = k[x, y]$, with defining relation $xy - qyx = y^2$. (It is easy to show that $B \cong k[u, v]$ with relation $vu = quv$.) Define $A = k + xB$.

It is standard that B is (right and left) noetherian, and not too difficult to deduce from this that A is also noetherian. As a right A-module, B is finitely generated, namely $B = A + yA$. In contrast, as a left A-module, B is not finitely generated: indeed, as a left A-module,

$$B/A \cong k(-1) \oplus k(-2) \oplus \cdots$$

is an infinite direct sum of shifts of the trivial A-module ${}_Ak = A/A_{\geq 1}$. To see this, simply observe that B/A has a basis given by the images of $\{y^i | i \geq 1\}$, and that $A_{\geq 1}y = xBy \subseteq A$. Since A is a domain, $\tau A = 0$, whence $A \subseteq \omega\pi A$. Since $\mathrm{Fract}(A) = \mathrm{Fract}(B)$, B is an essential extension of A; since ${}_A(B/A)$ is torsion, it follows from the definition of ω that $A \subset B \subset \omega\pi A$. Thus $\mathrm{coker}(A \to \omega\pi A)$ is not right bounded, so $\chi_1(A)$ does not hold. Alternatively, one can see from the description of B/A that $\underline{\mathrm{Ext}}^1_A(A/A_{\geq 1}, A)$ is not finite dimensional. $\Diamond$

5.13 Cohomology for projective spaces

Throughout this section A denotes a noetherian connected graded k-algebra, and X denotes the enriched projective space

$$(X, \mathcal{O}_X) = \mathrm{Proj}\, A = (\mathsf{Tails} A, \pi A)$$

with homogeneous coordinate ring A.

Definition 13.1 The cohomology groups of an X-module $\mathcal{F}$ are

$$H^q(X,\mathcal{F}) := \mathrm{Ext}^q(\mathcal{O}_X,\mathcal{F})$$

and the cohomology modules

$$\underline{\mathrm{H}}^q(X,\mathcal{F}) := \underline{\mathrm{Ext}}^q(\mathcal{O}_X,\mathcal{F}),$$

which are graded by

$$\underline{\mathrm{H}}^q(X,\mathcal{F})_d := \underline{\mathrm{Ext}}^q(\mathcal{O}_X,\mathcal{F}(d)).$$

$\Diamond$

If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact, there is a long exact cohomology sequence

$$0 \to H^0(X,\mathcal{F}') \to H^0(X,\mathcal{F}) \to H^0(X,\mathcal{F}'') \to H^1(X,\mathcal{F}') \to H^1(X,\mathcal{F}) \to \cdots .$$

We have already observed that $\underline{\mathrm{Hom}}(\mathcal{O}_X,-) \simeq \omega$, so the $\underline{\mathrm{H}}^q(X,-)$ are the right derived functors of ω.

If X is a scheme then the Čech cohomology groups $H^q(X,-)$ agree with the derived functors of the global section functor $\Gamma(X,-)$ on $\mathcal{O}_X$. But $\Gamma(X,-) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X,-)$, so $H^q(X,-)$ are the derived functors of $\mathrm{Hom}(\mathcal{O}_X,-)$. Hence by Serre's equivalence of categories (Theorem **??**), this definition of cohomology agrees with the classical one for projective schemes.

The following result is mostly a specialization of earlier results.

Proposition 13.2 *Let $M \in \mathsf{GrMod}A$ and write $\mathcal{M} = \pi M$. Then*

1. $\underline{\mathrm{H}}^0(X,\mathcal{M}) \cong \omega\pi M$;
2. $\underline{\mathrm{H}}^q(X,\mathcal{M}) \cong \varinjlim \underline{\mathrm{Ext}}^q_A(A_{\geq n},M)$;
3. $\underline{\mathrm{H}}^q(X,\mathcal{M}) \cong \varinjlim \underline{\mathrm{Ext}}^{q+1}_A(A/A_{\geq n},M)$ *for* $q \geq 1$;
4. $\underline{\mathrm{H}}^q(X,\mathcal{M}) \cong h^{q+1}(I^\bullet M)$ *for* $q \geq 1$, *where* $I^\bullet M$ *is the torsion part of the minimal injective resolution of* M.

Proof. (1) and (2) follow from Proposition 11.8 and Proposition 3.4.

(3) For $q \geq 1$, the long exact sequence for $\underline{\mathrm{Ext}}_A(-,M)$ gives

$$\underline{\mathrm{Ext}}^q_A(A_{\geq n},M) \cong \underline{\mathrm{Ext}}^{q+1}_A(A/A_{\geq n},M)$$

since A is projective, so (3) follows from (2).

(4) Consider the exact sequence of complexes

$$0 \to I^\bullet M \to E^\bullet M \to Q^\bullet M \to 0.$$

Since $Q^\bullet M$ is torsion free, and $A/A_{\geq n}$ is torsion, there is an isomorphism of complexes

$$\underline{\mathrm{Hom}}_A(A/A_{\geq n}, I^\bullet M) \cong \underline{\mathrm{Hom}}_A(A/A_{\geq n}, E^\bullet M).$$

Taking direct limits and homology yields

$$h^{q+1}(\varinjlim \underline{\mathrm{Hom}}_A(A/A_{\geq n}, I^\bullet M)) \cong \underline{\mathrm{H}}^q(X, \mathcal{M}).$$

But $I^\bullet M$ is torsion, so the sum of its finite dimensional submodules, whence

$$\varinjlim \underline{\mathrm{Hom}}_A(A/A_{\geq n}, I^\bullet M) \cong I^\bullet M. \qquad \square$$

Each $\underline{\mathrm{H}}^q(X, \mathcal{M})$ has a natural right A-module structure arising from the left action of A on $A_{\geq n}$ in Proposition 13.2(2). Its degree d component is equal to $H^q(X, \mathcal{M}(d))$.

One of the first things to do after defining sheaf cohomology is to compute the cohomology groups $H^q(\mathbb{P}^n, \mathcal{O}(d))$ of the line bundles on $\mathbb{P}^n$. The next example extends that computation to a larger class of spaces.

Example 13.3 Let A be Artin-Schelter regular of dimension $n+1 \geq 2$. We compute $H^q(X, \mathcal{O}_X(d))$ for $d \in \mathbb{Z}$.

First we show that $\underline{\mathrm{H}}^0(X, \mathcal{O}_X) = \omega\pi A$. The Gorenstein property ensures that $\underline{\mathrm{Hom}}_A(A/A_{\geq 1}, M) = 0$, whence $\tau A = 0$. Also, $\underline{\mathrm{Ext}}^1_A(A/A_{\geq 1}, M) = 0$ by the Gorenstein property, whence $\underline{\mathrm{Ext}}^1_A(A/A_{\geq n}, M) = 0$ for all n (by induction). Hence by Proposition 11.5, $A \cong \omega\pi A$. That is,

$$\underline{\mathrm{H}}^0(X, \mathcal{O}_X) = A \qquad \text{and} \qquad H^0(X, \mathcal{O}_X(d)) = A_d.$$

Now suppose that $q \geq 1$. Since $\underline{\mathrm{Ext}}^{n+1}_A(k, A) \cong k(l)$, the trivial right A-module shifted by some integer l, it follows that for any finite dimensional A-module T, $\underline{\mathrm{Ext}}^{n+1}_A(T, A) \cong T^*(l)$; one argues by induction on the length of T, the case of a shift of k being obviously true. Hence

$$\begin{aligned}
\underline{\mathrm{H}}^q(X, \mathcal{O}_X) &= \varinjlim \underline{\mathrm{Ext}}^{q+1}_A(A/A_{\geq n}, A) \\
&= \varinjlim \begin{cases} 0 & q \neq n, \\ (A/A_{\geq n})^*(l) & q = n. \end{cases} \\
&= \begin{cases} 0 & q \neq n \\ A^*(l) & q = n. \end{cases}
\end{aligned}$$

Thus $H^n(X, \mathcal{O}_X(d)) = (A^*)_{l+d} = (A_{-l-d})^*$.

When A is a polynomial ring on $n+1$ generators the Koszul complex gives a linear resolution of the trivial module ${}_Ak$, so $l = n+1$, whence we recover the usual result for $H^q(\mathbb{P}^n, \mathcal{O}(d))$. $\Diamond$

Theorem 13.4 *(Serre's Finiteness Theorem). Let A be a noetherian connected graded k-algebra satisfying χ. Let $X = \mathrm{Proj}\, A$. If $\mathcal{F} \in \mathsf{mod}X$, then*

1. $\dim_k H^q(X, \mathcal{F}) < \infty$ *for all q, and*

2. *if* $q \geq 1$, *then* $H^q(X, \mathcal{F}(n)) = 0$ *for* $n \gg 0$.

Conversely, if A *satisfies* χ_1, *and (2) holds for all* $\mathcal{F} \in \mathsf{mod}X$, *then* A *satisfies* χ.

Proof. Write $\mathcal{F} = \pi M$ where $M \in \mathsf{grmod}A$.

Suppose that $q = 0$. Since $\chi_1(M)$ holds, $(\omega\pi M)_{\geq 0}$ is finitely generated, hence locally finite. In particular, $(\omega\pi M)_0 = H^0(X, \mathcal{F})$ is finite dimensional.

Suppose that $q \geq 1$. Since A satisfies χ_{q+1},

$$\varinjlim \underline{\mathrm{Ext}}_A^{q+1}(A/A_{\geq n}, M)$$

is right bounded; but this equals $\underline{\mathrm{H}}^q(X, \mathcal{F})$, so (2) follows because $\underline{\mathrm{H}}^q(X, \mathcal{F})_n = H^q(X, \mathcal{F}(n))$. The proof of Proposition 12.2 showed that, given $d \in \mathbb{Z}$,

$$\varinjlim \underline{\mathrm{Ext}}_A^{q+1}(A/A_{\geq n}, M)_{\geq d} \cong \underline{\mathrm{Ext}}_A^{q+1}(A/A_{\geq r}, M)_{\geq d}$$

for $r \gg 0$; in particular, this is locally finite, which proves (1) for $q \geq 1$.

Conversely, (2) implies that $\underline{\mathrm{H}}^{i-1}(X, \mathcal{F})$ is right bounded for $i \geq 2$, but this is isomorphic to $\varinjlim \underline{\mathrm{Ext}}_A^i(A/A_{\geq n}, M)$; thus, since $\chi_1(M)$ holds, condition (4) in Proposition 12.2 is satisfied for all i. Thus A satisfies χ. $\square$

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