Recognizing modules over a Dedekind domain

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Abstract

Let A be a locally noetherian category and M a 1-critical noetherian weak generator of A. Suppose that $D = \operatorname{End}_{\mathsf{A}} M$ is a Dedekind domain, and that $\operatorname{Hom}_{\mathsf{A}}(M, -)$ and $- \otimes_D M$ set up a bijection between the isomorphism classes of simple objects in A and simple D-modules. Then M is a progenerator of A and $\operatorname{Hom}_{\mathsf{A}}(M, -)$ implements an equivalence between A and the category of D-modules.

1 Introduction

It is always useful to recognize when a given abelian category is equivalent to the category of modules over a ring. Mitchell's theorem says that a Grothendieck category [6, p.63] is equivalent to the category of modules over a ring if and only if it contains a progenerator. However, recognizing whether one has a progenerator is not always easy. There is a need for results giving criteria which imply the existence of a progenerator. Proposition 1.1, due to O. Gabber, is an example of such a result.

First we need a definition. Let A be an abelian category. Objects in A will be called A-modules. An A-module M is a weak generator of A if every A-module is a subquotient of a (possibly infinite) direct sum of copies of M.

Proposition 1.1 *[3, 2.14] Let k be a field, and let A be a k-linear abelian category such that every A-module has finite length, and $\dim_k \operatorname{Hom}_k(M, N) < \infty$ for all A-modules M and N. If A has a weak generator, then it has a progenerator, and A is equivalent to the category of finite dimensional right modules over a finite dimensional k-algebra.*

In the same spirit, a result of Nastasescu [5, 3.3] says that a Grothendieck category with an artinian generator is equivalent to a category of modules over a right artinian ring. This was extended by Albu and Wisbauer [1] to the case of a Grothendieck category having a generator whose endomorphism ring is right perfect. Artinian rings have Krull dimension zero and perfect rings are close to artinian rings in some sense because a noetherian perfect ring is artinian.

In this paper we prove the following result which gives a criterion for a Grothendieck category to be equivalent to the category of modules over a commutative Dedekind domain.

Theorem 1.2 *Let* A *be a locally noetherian category. Suppose that*

1. M *is a 1-critical, noetherian, weak generator of* A*;*
2. $\operatorname{End}_{\mathsf{A}} M$ *is a Dedekind domain* D*;*
3. $\operatorname{Hom}_{\mathsf{A}}(M,-)$ *and* $-\otimes_D M$ *set up a bijection between the isomorphism classes of simple* A*-modules and simple* D*-modules.*

Then M *is a progenerator for* A*, and* $\operatorname{Hom}_{\mathsf{A}}(M,-) : \mathsf{A} \to \mathsf{Mod}D$ *is an equivalence of categories.*

A locally noetherian category A is a Grothendieck category having a set of noetherian generators. Krull dimension, as defined by Gabriel, makes sense in any locally noetherian category. It is denoted by $\operatorname{Kdim} M$. We say that M is d-critical if $\operatorname{Kdim} M = d$ and every proper quotient of M has strictly smaller Krull dimension. A d-critical module has no nonzero submodule of Krull dimension less than d. A noetherian module is 1-critical if it is not of finite length, but every proper quotient of it has finite length.

If M is a noetherian A-module, then M is a weak generator if the set of submodules of all finite direct sums of M generates A. If M is an A-module, then M is a weak generator of the smallest full subcategory of A containing M that is closed under direct sums and subquotients.

In [8] we use 1.3 to show that certain subspaces of some non-commutative spaces are affine lines. In that paper we need to know that the full subcategory weakly generated by a module M over a ring is equivalent to $\mathsf{Mod}k[X]$. The following theorem, which is a consequence of Theorem 1.2, gives us the result we need in [8].

Theorem 1.3 *Let* A *be a locally noetherian category. Suppose that*

1. M *is a 1-critical, noetherian, weak generator of* A *and every nonzero submodule of* M *is isomorphic to* M*;*
2. $D := \operatorname{End}_{\mathsf{A}} M$ *is a commutative PID (principal ideal domain);*
3. *if* S *is a simple* A*-module, then* $\operatorname{Hom}_{\mathsf{A}}(M,S)$ *is a simple* D*-module.*

Then M *is a progenerator, whence* $\operatorname{Hom}_{\mathsf{A}}(M,-) : \mathsf{A} \to \mathsf{Mod}D$ *is an equivalence.*

The proof of 1.2 (given in Section 3) depends crucially on the structure of noetherian modules over Dedekind domains. We don't know how to extend it to other commutative rings. However, Section 4 gives a partial extension of 1.2 and 1.3 when $\operatorname{End}_{\mathsf{A}} M$ is a commutative ring of higher Krull dimension.

2 Preliminaries

Let R be a ring, and write $\mathsf{Mod}R$ for the category of right R-modules. Let A be a Grothendieck category.

Definition 2.1 *Let M an A-module. An R-*module in A *is a pair (M,φ) consisting of an R-module M and a ring homomorphism $\varphi : R \to \mathrm{Hom}_{\mathsf{A}}(M,M)$. We will usually just write M for the module.*

If R is commutative and A is R-linear, then every A-module has an R-module structure in a canonical way.

Let M be an R-module in A. For each A-module N, composition of morphisms $\mathrm{Hom}_{\mathsf{A}}(M,N) \times \mathrm{Hom}_{\mathsf{A}}(M,M) \to \mathrm{Hom}_{\mathsf{A}}(M,N)$ gives $\mathrm{Hom}_{\mathsf{A}}(M,N)$ the structure of a right R-module. The functor

$$g_* = \mathrm{Hom}_{\mathsf{A}}(M,-) : \mathsf{A} \to \mathsf{Mod}R$$

is left exact and commutes with products, so has a left adjoint

$$g^* = - \otimes_R M : \mathsf{Mod}R \to \mathsf{A}$$

(cf. [6, p.108]). In particular, $- \otimes_R M$ commutes with colimits. If M is finitely presented, then $\mathrm{Hom}_{\mathsf{A}}(M,-)$ also commutes with direct limits (this can be taken as either a definition or result depending on one's point of view). The basic properties of this tensor product are given in [2], although in [2] morphisms in A are viewed as acting on the *right* of an object, so there one works with the category of left R-modules.

We denote the natural transformations associated to the adjoint pair (g^*, g_*) by

$$\tau : \mathrm{id}_{\mathsf{Mod}R} \to g_* g^*$$

and

$$\eta : g^* g_* \to \mathrm{id}_{\mathsf{A}}\,.$$

The adjunction isomorphism

$$\mathrm{Hom}_R(R, \mathrm{Hom}_{\mathsf{A}}(M,M)) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{A}}(R \otimes_R M, M)$$

sends $\varphi : R \to \mathrm{Hom}_{\mathsf{A}}(M,M)$ to a distinguished map $\nu : R \otimes_R M \to M$. Using the Yoneda lemma, it is not difficult to show that this map is an isomorphism. If $\delta : R \to R$ is left multiplication by x, then the induced map $\nu\delta\nu^{-1} : M \to M$ coincides with the action of $\varphi(x)$ as an endomorphism of M. Hence if one uses ν to identify $R \otimes_R M$ with M, then the action of x on $R \otimes_R M$ agrees with the action of $\varphi(x)$ on M. We write xM for the image of M under $\varphi(x)$. If I is a right ideal of R, then there is an exact sequence

$$I \otimes_R M \to R \otimes_R M \to R/I \otimes_R M \to 0.$$

By the preceding remarks, the image of the composition

$$I \otimes_R M \longrightarrow R \otimes_R M \xrightarrow{\nu} M.$$

is $\sum_{x \in I} xM$. We denote this by IM. It follows that $R/I \otimes_R M \cong M/IM$.

In the rest of this section we prove some lemmas which will be used in the next sections.

Let D be a Dedekind domain. Every noetherian D-module is isomorphic to a direct sum of various ideals of D and quotients of the form $D/\mathfrak{m}^n$ for various maximal ideals $\mathfrak{m}$ and positive integers n. We call this the structure theorem for D-modules.

Lemma 2.2 *Let D be a Dedekind domain. Suppose that M is a 1-critical* A-*module, and that* $\operatorname{End}_{\mathsf{A}} M = D$. *Then*

1. $g^* = - \otimes_D M$ *is exact;*
2. g^* *is faithful;*
3. *if I and J are ideals of D such that $I \subset J$ but $I \neq J$, then IM is properly contained in JM. In particular, if $d \in D \backslash I$, then $dM \not\subset IM$.*

Proof. (1) First, we show that if I is a non-zero ideal of D, then $I \otimes_D M$ is 1-critical. There is an ideal J such that $I \oplus J \cong D \oplus D$. Thus $I \otimes_D M \oplus J \otimes_D M$ is isomorphic to $M \oplus M$. Since M is 1-critical and since $I \otimes_D M$ and $J \otimes_D M$ are non-zero, $I \otimes_D M$ is 1-critical.

The map $I \otimes_D M \to D \otimes_D M \cong M$ is monic because it is a non-zero map between 1-critical modules. Therefore $I \otimes_D M \cong IM$ and the sequence

$$0 \to I \otimes_D M \to D \otimes_D M \to D/IM \to 0$$

is exact.

The functor $\operatorname{Tor}_1^D(-, M)$ commutes with direct limits by [2, C1.7], so it suffices to show this vanishes on noetherian D-modules. By the structure theorem for noetherian D-modules and the long exact sequence for $\operatorname{Tor}^D(-, M)$, it is enough to prove that $\operatorname{Tor}_1^D(-, M)$ vanishes on cyclic D-modules and on non-zero ideals. Since ideals of D are projective, $\operatorname{Tor}_1^D(-, M)$ vanishes on ideals. By the previous paragraph, and [2, C1.7], $\operatorname{Tor}_1^D(D/I, M) = 0$ for all non-zero ideals I.

(2) Since g^* is exact, it suffices to show that $g^* V \neq 0$ for any simple D-module $V = D/I$. Let $\alpha : I \to D$ be the inclusion. By adjointness, the following diagram commutes

$$\begin{array}{ccc} \operatorname{Hom}_{\mathsf{A}}(g^* D, M) & \xrightarrow{\operatorname{Hom}(g^*(\alpha), M)} & \operatorname{Hom}_{\mathsf{A}}(g^* I, M) \\ \cong \downarrow & & \downarrow \cong \\ \operatorname{Hom}_D(D, g_* M) & \xrightarrow{\operatorname{Hom}(\alpha, g_* M)} & \operatorname{Hom}_D(I, g_* M). \end{array}$$

The bottom arrow is the natural inclusion $\operatorname{Hom}(\alpha, D) : D \to I^{-1}$. This is not an epimorphism, so neither is the top arrow. Therefore $\operatorname{Ext}^1_{\mathsf{A}}(g^*(D/I), M) \neq 0$. It follows that $g^*(D/I) \neq 0$.

(3) Follows from (1) and (2). $\square$

Lemma 2.3 *Suppose that the hypotheses of 2.2 hold.*

1. *If g^* sends simple modules to simple modules, then g^* preserves the length of finite length modules.*

2. *If g_* sends simple modules to simple modules, then g_* does not increase the length of finite length modules.*

Proof. Part (1) follows by induction and 2.2, and part (2) follows by induction on the length. $\square$

Lemma 2.4 *If a simple A-module S is a subquotient of a direct sum $\oplus_{i=1}^n M_i$, then it is a subquotient of some M_i.*

Proof. Let S be a quotient of a submodule $N \subset \oplus_{i=1}^n M_i$. Consider the chain of submodules

$$0 = N_0 \subset N_1 \subset \cdots \subset N_n = N$$

where $N_j = N \cap (\oplus_{i=1}^j M_i)$. Hence N_j/N_{j-1} is a submodule of M_j. Since $\mathrm{Hom}_{\mathsf{A}}(N, S) \neq 0$, $\mathrm{Hom}_{\mathsf{A}}(N_j/N_{j-1}, S) \neq 0$ for some j and S is a subquotient of M_j. $\square$

3 The proofs of 1.1 and 1.2

In this section we assume

Hypothesis 3.1

1. *A is a locally noetherian category;*

2. *M is a 1-critical noetherian A-module, and $\mathrm{End}_{\mathsf{A}} M$ is a Dedekind domain D;*

3. *$\mathrm{Hom}_{\mathsf{A}}(M, -)$ and $- \otimes_D M$ set up a bijection between the isomorphism classes of simple A-modules and simple D-modules.*

We use the notation $g_* = \mathrm{Hom}_{\mathsf{A}}(M, -)$ and $g^* = - \otimes_D M$.

Proposition 3.2 *Suppose that hypothesis 3.1 holds. Then the natural transformation $\tau : \mathrm{id}_{\mathsf{Mod}D} \to g_*g^*$ is a natural isomorphism.*

Proof. Since g^* is a right adjoint it commutes with direct limits. Since M is a noetherian module in a locally noetherian category it is finitely presented, so $\mathrm{Hom}_{\mathsf{A}}(M, -)$ commutes with direct limits. Therefore g_*g^* commutes with direct limits. Since A and $\mathsf{Mod}D$ are locally noetherian, and since $\mathrm{id}_{\mathsf{Mod}D}$ and g_*g^* commute with direct limits, it suffices to show that $\tau_V : V \to g_*g^*V$ is an isomorphism whenever V is noetherian. By the structure theorem for noetherian

D-modules it suffices to prove this when V is an ideal of D and when V is a quotient of D.

Consider the case when $V = I$ is an ideal. There is a commutative diagram

$$\begin{tikzcd}
I \arrow[r] \arrow[d, "\tau_I"'] & D \arrow[r] \arrow[d, "\tau_D"] & D/I \arrow[d, "\tau_{D/I}"] \\
\mathrm{Hom}_{\mathsf{A}}(M, I \otimes M) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathsf{A}}(M, D \otimes M) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathsf{A}}(M, D/I \otimes M) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathsf{A}}(M, IM) \arrow[r] & \mathrm{Hom}_{\mathsf{A}}(M, M) = D \arrow[r] & \mathrm{Hom}_{\mathsf{A}}(M, M/IM)
\end{tikzcd}$$

in $\mathsf{Mod}D$. Hence if $d \in I$, then $\tau_I(d) : M \to IM$ is simply the map $d : M \to M$ considered as a map to IM. In particular, if d is non-zero, so is $\tau_I(d)$. Thus τ_I is injective. To see that τ_I epic, let θ be an element of $\mathrm{Hom}_{\mathsf{A}}(M, I \otimes_D M)$. Then θ may be considered as a map from M to M whose image is contained in IM. It follows from 2.2(3) that θ is given by an element of I; that is, $\theta = \tau_I(d)$ for some $d \in I$. Hence τ_I is an isomorphism.

Now suppose $V = D/I$ for some non-zero ideal I. The commutative diagram shows that $\tau_{D/I}([d + I])$ is the composition

$$M \xrightarrow{d} M \longrightarrow M/IM.$$

Therefore, to show that $\tau_{D/I}$ is an isomorphism, we must show that the map

$$D = \mathrm{Hom}_{\mathsf{A}}(M, M) \to \mathrm{Hom}_{\mathsf{A}}(M, M/IM)$$

is surjective and that its kernel is no larger than I. 2.2(3) shows that the kernel can be no larger than I. Hence $\tau_{D/I}$ provides an injective map $D/I \to \mathrm{Hom}_{\mathsf{A}}(M, M/IM) = g_*g^*(D/I)$. However, the length of $g_*g^*(D/I)$ is at most the length of D/I (see 2.3), so the map $D/I \to g_*g^*(D/I)$ is an isomorphism. Thus $\tau_{D/I}$ is an isomorphism. $\square$

Corollary 3.3 *Suppose that hypothesis 3.1 holds. Then g^* is full and faithful.*

Proof. By the dual version of [4, Theorem 1, p.88], this is equivalent to the fact that $g_*g^* \cong \mathrm{id}_{\mathsf{Mod}D}$. $\square$

Proposition 3.4 *Suppose that hypothesis 3.1 holds. Then every submodule of a finite direct sum of copies of JM, for various ideals $J \subset D$, is isomorphic to a direct sum of copies of IM for various ideals I of D.*

Proof. First we show that every simple A-module is of the form g^*V for some simple D-module V. If S is a simple A-module, it follows from adjointness and 2.3 that $\eta_S : g^*g_*S \to S$ is a non-zero map of simple modules. Therefore $S \cong g^*V$ where $V = g_*S$.

Let E be a submodule of $N = N_1 \oplus \ldots \oplus N_r$ where each N_i is isomorphic to $J_i M$ for some ideal J_i of D. Thus $N \cong g^* N'$ where $N' = J_1 \oplus \ldots \oplus J_r$. If some $N_i \cap E$ is zero, then E embeds in N/N_i. Hence we can assume that $E \cap N_i \neq 0$ for all i. Therefore N/E has finite length. Let $E = E_0 \subset E_1 \subset \ldots \subset E_s = N$ be a chain of A-submodules such that each E_j/E_{j-1} is simple. By a downwards induction argument it suffices to prove the result, namely that E is isomorphic to a direct sum of various IM, when N/E is simple.

Hence we assume that $S = N/E$ is simple. Let $\theta : N \to S$ be an A-module map with kernel E. By the first paragraph, $S \cong g^* V$ for some simple D-module V. Since g^* is fully faithful, there is an isomorphism

$$g^* : \operatorname{Hom}_D(N', V) \to \operatorname{Hom}_{\mathsf{A}}(N, S).$$

In particular, there is a map $\alpha : N' \to V$ such that $g^*(\alpha) = \theta$. Since g^* is exact, E, which is equal to $\ker \theta$, is isomorphic to $g^*(\ker \alpha)$. But $\ker \alpha$ is a submodule of N', so is isomorphic to a direct sum of various ideals of D (by the structure theorem of D-modules). Hence E is isomorphic to a direct sum of various IM. $\square$

Proof of 1.2. Since g^* commutes with direct limits, it suffices to show that g^* implements an equivalence of categories between the full subcategories of noetherian modules. Since g^* is full and faithful (see 3.3), it suffices to show that every noetherian A-module L is isomorphic to $g^* L'$ for some D-module L'.

Since L is noetherian, it is a subquotient of a finite direct sum of copies of M. Thus, by 3.4, L is the cokernel of a map $\theta : P \to Q$, where P and Q are finite direct sums of IM for various ideals I of D. In particular, there are finitely generated D-modules, say P' and Q', such that $P = g^* P'$ and $Q = g^* Q'$. Since g^* is full, $\theta = g^*(\alpha)$ for some $\alpha \in \operatorname{Hom}_D(P', Q')$ and, since g^* is exact, $L = \operatorname{coker} \theta = g^*(\operatorname{coker} \alpha)$. $\square$

Theorem 1.3 is a special case of the next result.

Corollary 3.5 *Let* A *be a locally noetherian category. Suppose that*

1. *M is a 1-critical, noetherian, weak generator of* A.
2. *$D := \operatorname{End}_{\mathsf{A}} M$ is a Dedekind domain.*
3. *Every nonzero submodule of M is isomorphic to IM for some ideal $I \subset D$.*
4. *If S is a simple A-module, then $\operatorname{Hom}_{\mathsf{A}}(M, S)$ is a simple D-module.*

Then $\operatorname{Hom}_{\mathsf{A}}(M, -) : \mathsf{A} \to \mathsf{Mod} D$ *is an equivalence.*

Proof. The result will follow from 1.2 once we show that g^* and g_* set up a bijection between isomorphism classes of simple modules.

Hypothesis (4) ensures that g_* sends simple modules to simple modules.

Now we show that g^* sends simple modules to simple modules. Let $U = D/J$ be a simple D-module for some maximal ideal $J \subset D$. Then $g^* U \cong M/JM$.

We must show that JM is a maximal submodule of M. If not, then there is a submodule M' strictly between JM and M. By hypothesis (3), there is an ideal $I \subset D$ such that $M' \cong IM$. Hence the evaluation map

$$\operatorname{Hom}_{\mathsf{A}}(M, M') \otimes_D M \to M'$$

is epic. Since $I' := \operatorname{Hom}_{\mathsf{A}}(M, M')$ is an ideal of D, $M' = I'M$. Therefore by 2.2 I' is not contained in J and

$$M = (I' + J)M = I'M + JM = M' + JM = M',$$

which yields a contradiction. □

4 A partial generalization

In this section we prove an ungraded version of a part of the main result of [7]. This result is an analogue of 1.3 when $\operatorname{End}_{\mathsf{A}} M$ is a commutative ring of Krull dimension > 1.

The basic tool we use is localization, which we review next. We refer to [2] for the basic definitions and properties of localization and base change for Grothendieck categories.

Let R be a commutative noetherian ring, and let A denote an R-linear Grothendieck category. Let T be another commutative ring and $R \to T$ a ring homomorphism. The T-modules in A form a Grothendieck category, which is denoted by A_T. By [2, B2.2], A_T is T-linear and the restriction functor $\mathsf{A}_T \to \mathsf{A}$ is an exact functor commuting with limits and colimits. The restriction functor has a left adjoint, namely the tensor functor $T \otimes_R - : \mathsf{A} \to \mathsf{A}_T$.

Let S be a multiplicative set in R, and let T be the localization RS^{-1}. The tensor product $T \otimes_R -$ is a direct limit with respect to a filtering system [2, B6.1]. Because T_R is flat, the functor $T \otimes_R - : \mathsf{A} \to \mathsf{A}_T$ is exact [2, B6.2]. An A-module M can be given a T-module structure if and only if $x : M \to M$ is an isomorphism for all $x \in S$, and there is at most one T-module structure on M. Note that $\operatorname{Hom}_{\mathsf{A}_T}(M, N) = \operatorname{Hom}_{\mathsf{A}}(M, N)$ for all $M, N \in \mathsf{A}_T$.

Let B be another R-linear category and $F : \mathsf{A} \to \mathsf{B}$ an R-linear functor. Then for every T-module M in A, $F(M) \in \mathsf{B}$ has a unique T-module structure because the maps $F(x) : F(M) \to F(M)$ are isomorphisms for all $x \in S$. Therefore F induces a functor $\mathsf{A}_T \to \mathsf{B}_T$, which we denote by F_T. Note that $F_T(M) = F(M)$.

If F commutes with direct limits, then F commutes with the localization $T \otimes_R -$ because the localization can be interpreted as a direct limit [2, B6.1]. Therefore we have

$$F_T(T \otimes_R M) = F(T \otimes_R M) \cong T \otimes_R F(M)$$

for all $M \in \mathsf{A}$. Consequently, F_T commutes with direct limits.

Let F and F' be R-linear functors from A to B. If $\tau : F \to F'$ is a natural transformation, then τ induces a natural transformation $\tau_T : F_T \to F'_T$. If F and F' commute with direct limits, then $\tau_{T\otimes_R M} = T \otimes_R \tau_M$.

For any prime ideal $\mathfrak{p} \subset R$, we write $\mathsf{A}_\mathfrak{p}$, $F_\mathfrak{p}$, and $\tau_\mathfrak{p}$, for $\mathsf{A}_{R_\mathfrak{p}}$, $F_{R_\mathfrak{p}}$, and $\tau_{R_\mathfrak{p}}$ respectively.

Let Φ be a set of prime ideals of R. We say that A is Φ-faithful if for every non-zero A-module M, the localization $M_\mathfrak{p} := R_\mathfrak{p} \otimes_R M$ is nonzero for some $\mathfrak{p} \in \Phi$.

Lemma 4.1 *Let A and B be R-linear locally noetherian categories. Let F and F' be R-linear functors from A to B that commute with direct limits. Let $\tau : F \to F'$ be a natural transformation. Suppose that B is Φ-faithful for some $\Phi \subset \operatorname{Spec} R$. If $\tau_\mathfrak{p} : F_\mathfrak{p} \to F'_\mathfrak{p}$ is an isomorphism for all $\mathfrak{p} \in \Phi$, then τ is an isomorphism.*

Proof. Let M be an A-module. Consider the exact sequence

$$0 \to K \to F(M) \xrightarrow{\tau_M} F'(M) \to C \to 0.$$

Tensoring with $R_\mathfrak{p}$ yields an exact sequence

$$0 \to K_\mathfrak{p} \to F_\mathfrak{p}(M_\mathfrak{p}) \to F'_\mathfrak{p}(M_\mathfrak{p}) \to C_\mathfrak{p} \to 0.$$

Since $\tau_\mathfrak{p}$ is an isomorphism for all $\mathfrak{p} \in \Phi$, $K_\mathfrak{p} = C_\mathfrak{p} = 0$. Since B is Φ-faithful, $K = C = 0$. $\square$

Lemma 4.2 *If (F, G) is an adjoint pair of functors between R-linear categories A and B, then $(F_\mathfrak{p}, G_\mathfrak{p})$ is an adjoint pair of functors between $\mathsf{A}_\mathfrak{p}$ and $\mathsf{B}_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof. For every $M \in \mathsf{A}_\mathfrak{p}$ and $N \in \mathsf{B}_\mathfrak{p}$, we have

$$\operatorname{Hom}_{\mathsf{B}_\mathfrak{p}}(F_\mathfrak{p} M, N) = \operatorname{Hom}_{\mathsf{B}}(FM, N) \cong \operatorname{Hom}_{\mathsf{A}}(M, GN) = \operatorname{Hom}_{\mathsf{A}_\mathfrak{p}}(M, G_\mathfrak{p} N).$$

$\square$

Lemma 4.3 *Let (F, G) be a pair of adjoint functors between two locally noetherian categories A and B. If F preserves noetherian modules, then G commutes with direct limits.*

Proof. To prove $G(\operatorname{colim} N_i) \cong \operatorname{colim} G(N_i)$ for any direct limit $\operatorname{colim} N_i$ in B, it suffices to show that

$$\operatorname{Hom}_{\mathsf{A}}(M, G(\operatorname{colim} N_i)) \cong \operatorname{Hom}_{\mathsf{A}}(M, \operatorname{colim} G(N_i))$$

for all noetherian modules $M \in \mathsf{A}$. But this follows from the adjointness of (F, G) and the fact that $\operatorname{Hom}_{\mathsf{A}}(M, -)$ and $\operatorname{Hom}_{\mathsf{B}}(FM, -)$ commute with direct limits (because M and FM are noetherian). $\square$

Proposition 4.4 *Let* A *and* B *be* R*-linear locally noetherian categories. Let* $F : \mathsf{A} \to \mathsf{B}$ *be an* R*-linear functor that commutes with direct limits and preserves noetherian modules. Let* $\Phi \subset \operatorname{Spec} R$*, and suppose that* A *and* B *are* Φ*-faithful. If the induced functor* $F_{\mathfrak{p}} : \mathsf{A}_{\mathfrak{p}} \to \mathsf{B}_{\mathfrak{p}}$ *is an equivalence for every* $\mathfrak{p} \in \Phi$*, then* F *is an equivalence.*

Proof. First we show that F is exact. Let

$$C : 0 \to L \to M \to N \to 0$$

be a short exact sequence in A. Since F commutes with direct limits, $F_{\mathfrak{p}}(R_{\mathfrak{p}} \otimes_R C) \cong R_{\mathfrak{p}} \otimes_R F(C)$. Now localization $R_{\mathfrak{p}} \otimes_R -$ is exact [2, B6.2] and $F_{\mathfrak{p}}$ is an equivalence, so $R_{\mathfrak{p}} \otimes_R F(C)$ is exact for all $\mathfrak{p} \in \Phi$. Since B is Φ-faithful, $F(C)$ is exact.

Since F commutes with direct limits it has a right adjoint, say $G : \mathsf{B} \to \mathsf{A}$ [9, 2.1]. Since F preserves noetherian modules, G commutes with direct limits by 4.3.

There are natural transformations $\tau : \mathrm{id}_{\mathsf{A}} \to GF$ and $\eta : FG \to \mathrm{id}_{\mathsf{B}}$. By 4.2, $(F_{\mathfrak{p}}, G_{\mathfrak{p}})$ is an adjoint pair of functors between $\mathsf{A}_{\mathfrak{p}}$ and $\mathsf{B}_{\mathfrak{p}}$, and $\tau_{\mathfrak{p}}$ and $\eta_{\mathfrak{p}}$ are the induced natural transformations. By the hypothesis, $F_{\mathfrak{p}}$ is an equivalence, so $\tau_{\mathfrak{p}}$ and $\eta_{\mathfrak{p}}$ are natural isomorphisms. Since A and B are Φ-faithful, 4.1 shows that τ and η are natural isomorphisms. Therefore F is an equivalence. $\square$

We now introduce a category $\mathcal{V}_R$ of Krull dimension 1 that is a ungraded version of the category $\mathbb{V}^1_n$ studied in [7].

Let R be a commutative noetherian domain of global dimension $d \geq 1$. Define Tor to be the full Grothendieck subcategory of $\mathsf{Mod}R$ generated by noetherian modules whose associated primes have height at least 2. Let

$$\mathcal{V}_R := \mathsf{Mod}R / \operatorname{Tor}$$

denote the quotient category. If Φ is the set of all height one primes in R, then $\mathcal{V}_R$ is Φ-faithful. Using the ideas in [7, 3.4], one can prove that the Krull and global dimensions of $\mathcal{V}_R$ are one.

Theorem 4.5 *Let* R *be a commutative noetherian domain of global dimension* $d \geq 1$*. Let* A *be an* R*-linear locally noetherian category, and let* M *be an* A*-module. Suppose that*

1. M *is a* 1*-critical, noetherian, weak generator of* A*;*
2. $\operatorname{End}_{\mathsf{A}} M = R$*;*
3. *Every nonzero submodule of* M *is isomorphic to* IM *for some ideal* $I \subset R$*;*
4. *If* S *is a simple module in* A*, then* $\operatorname{End} S \cong Q(R/\mathfrak{p})$*, the field of fractions of* $R/\mathfrak{p}$ *for some height one prime* $\mathfrak{p}$*.*

Then A *is equivalent to* $\mathcal{V}_R$*.*

We need two lemmas before proving 4.5.

Lemma 4.6 *Suppose that the hypotheses of 4.5 hold. If N is a 1-critical* A-*module, then N contains a submodule which is isomorphic to M.*

Proof. We may assume N is noetherian. So N is a subquotient of $\oplus_{i=1}^{r} M$ for some integer r. Let P be a submodule of $\oplus_{i=1}^{r} M$ mapping epicly to N. We have a filtration of P,

$$0 = P_0 \subset P_1 \subset \cdots \subset P_r = P$$

where $P_j = P \cap \oplus_{i=1}^{j} M$. Let j_0 be the minimal j such that $P_j \to N$ is nonzero. Since P_{j_0}/P_{j_0-1} is isomorphic to a non-zero submodule of M, it is 1-critical. Hence the map $P_{j_0}/P_{j_0-1} \to N$ is injective, or equivalently, N contains a nonzero submodule which is a submodule of M. By hypothesis (3) of 4.5, every submodule of M is isomorphic to IM for some ideal $I \subset R$. Finally, IM contains a submodule $xM \cong M$ for every nonzero element $x \in I$. $\square$

Lemma 4.7 *Suppose that the hypotheses of 4.5 hold. Then the hypotheses of Theorem 1.3 hold for the $\mathsf{A}_\mathfrak{p}$-module $M_\mathfrak{p}$ for every height one prime $\mathfrak{p}$ in R. Therefore $\mathsf{A}_\mathfrak{p}$ is equivalent to $\mathsf{Mod}R_\mathfrak{p}$ via $\operatorname{Hom}_{\mathsf{A}_\mathfrak{p}}(M_\mathfrak{p}, -)$ and $- \otimes_{R_\mathfrak{p}} M_\mathfrak{p}$.*

Proof. Since R has finite global dimension, $R_\mathfrak{p}$ has global dimension and Krull dimension 1. Hence $R_\mathfrak{p}$ is a DVR (discrete valuation ring) and hence a PID. Since localization commutes with Hom [2, C3.1(iv)], we have $\operatorname{End}_{\mathsf{A}_\mathfrak{p}}(M_\mathfrak{p}) = R_\mathfrak{p}$. This verifies hypothesis (2) of 1.3. This also shows that $M_\mathfrak{p}$ is not simple.

Since localization is exact, it follows that $M_\mathfrak{p}$ is a weak generator of $\mathsf{A}_\mathfrak{p}$.

By [2, B6.2], $M_\mathfrak{p}$ is noetherian of Krull dimension ≤ 1. Since $M_\mathfrak{p}$ is not simple, it has a proper submodule $N_\mathfrak{p}$ for some nonzero 1-critical A-submodule $N \subset M$. By 4.6, N contains a submodule that is isomorphic to M. Hence $M_\mathfrak{p} \subset N_\mathfrak{p} \subset M_\mathfrak{p}$. Thus M is not artinian. Hence $\operatorname{Kdim} M = 1$. Every proper quotient of $M_\mathfrak{p}$ is a localization of a proper quotient of M, so is artinian. Thus $M_\mathfrak{p}$ is 1-critical.

By hypothesis (3) of 4.5, every submodule N of M is isomorphic to IM for some ideal $I \subset R$. Hence $N_\mathfrak{p} \cong (IM)_\mathfrak{p}$, which is an image of $I_\mathfrak{p} \otimes_{R_\mathfrak{p}} M$. Since $R_\mathfrak{p}$ is a DVR, $I_\mathfrak{p} \cong R_\mathfrak{p}$ and $I_\mathfrak{p} \otimes_{R_\mathfrak{p}} M \cong M$. Therefore there is an epimorphism from $M_\mathfrak{p}$ to $N_\mathfrak{p}$. This is an isomorphism because both $M_\mathfrak{p}$ and $N_\mathfrak{p}$ are 1-critical. Thus hypothesis (1) of 1.3 holds. Similarly one can show that $N_\mathfrak{p}$ is 1-critical for every noetherian 1-critical A-module N.

Let S be a simple $\mathsf{A}_\mathfrak{p}$-module. Then $S = R_\mathfrak{p} \otimes_R S'$ for any noetherian critical A-submodule $S' \subset S$. If S' is 1-critical, then by 4.6 we may take $S' \cong M$. This contradicts the fact that $M_\mathfrak{p}$ is 1-critical. Hence S' is a simple module. Then hypothesis (4) of 4.5 shows that $\operatorname{End}_{\mathsf{A}} S = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. We must show that $\operatorname{Hom}_{\mathsf{A}_\mathfrak{p}}(M_\mathfrak{p}, S) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$.

First, since $M_\mathfrak{p}$ is a weak generator, S is a subquotient of $M_\mathfrak{p}$ by 2.4. We have already shown that every submodule of $M_\mathfrak{p}$ is isomorphic to $M_\mathfrak{p}$. This implies that S is a quotient of $M_\mathfrak{p}$. If $M_\mathfrak{p}$ contains a unique maximal submodule, then $\mathrm{Hom}_{\mathsf{A}_\mathfrak{p}}(M,S) = \mathrm{Hom}_{\mathsf{A}_\mathfrak{p}}(S,S) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ as required. If not, then there are at least two maximal submodules M_1 and M_2 such that $M_\mathfrak{p}/M_i \cong S$ for $i = 1,2$. By the last paragraph, there are isomorphisms $f_i : M_\mathfrak{p} \to M_i$. Each f_i can be viewed as an element in $R_\mathfrak{p}$ so we can write $M_i = f_i M_\mathfrak{p}$. By 2.2, $- \otimes_{R_\mathfrak{p}} M_\mathfrak{p}$ is exact and faithful, which implies that ideals (f_i) are maximal and distinct. But this contradicts the fact that R is a DVR so we conclude that $\mathrm{Hom}(M_\mathfrak{p}, S) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is a simple module in $\mathsf{Mod}R_\mathfrak{p}$, thus verifying hypothesis (3) of 1.3. □

Note that if $d > 1$, we don't know a priori whether M is R-flat, or equivalently whether $g^* = - \otimes_R M : \mathsf{Mod}R \to \mathsf{A}$ is exact. Hence even if g^* maps modules in Tor to zero, we don't know if it induces a functor from $\mathcal{V}_R$ to A (because g^* is not exact). So, in the proof of 4.5, we work with the functor $g_* = \mathrm{Hom}_\mathsf{A}(M, -)$ instead .

Proof. The proof is based on 4.4. We write Φ for the set of height one primes of R.

Let F be the composition of $\mathrm{Hom}_\mathsf{A}(M,-) : \mathsf{A} \to \mathsf{Mod}R$ followed by the quotient functor $\mathsf{Mod}R \to \mathcal{V}_R$. We will show that F is an equivalence. By 4.4, it suffices to show the following:

(i) F commutes with direct limits;
(ii) $F_\mathfrak{p}$ is an equivalence for every $\mathfrak{p} \in \Phi$;
(iii) A and $\mathcal{V}_R$ are Φ-faithful.
(iv) F preserves noetherian modules;

(i) Since M is noetherian, $\mathrm{Hom}_\mathsf{A}(M,-)$ commutes with direct limits. The subcategory Tor is closed under direct sums in $\mathsf{Mod}R$, so the quotient functor $\mathsf{Mod}R \to \mathcal{V}_R$ has a right adjoint, and hence it commutes with direct limits. Therefore F commutes with direct limits.

(ii) For every $\mathfrak{p} \in \Phi$, $(\mathrm{Tor})_\mathfrak{p} = 0$, so the quotient functor $(\mathsf{Mod}R)_\mathfrak{p} = \mathsf{Mod}R_\mathfrak{p} \to (\mathcal{V}_R)_\mathfrak{p}$ is an equivalence. Hence $F_\mathfrak{p} \cong \mathrm{Hom}_{\mathsf{A}_\mathfrak{p}}(M_\mathfrak{p}, -)$. By 4.7, $F_\mathfrak{p}$ is an equivalence.

(iii) It is clear that $\mathcal{V}_R$ is Φ-faithful, and hypothesis (4) ensures that A is Φ-faithful.

(iv) By exact sequence arguments, this reduces to showing that $\mathrm{Hom}_\mathsf{A}(M,N)$ is noetherian in $\mathcal{V}_R$ when N is a noetherian 1-critical and when N is simple. If N is noetherian 1-critical, by 4.6 we may assume $N = M$ and the assertion follows from hypothesis (2). Suppose $N = S$ is simple. By hypothesis (4), $\mathrm{End}_\mathsf{A} S \cong Q(R/\mathfrak{p})$ for some $\mathfrak{p} \in \Phi$. So $\mathrm{Hom}_\mathsf{A}(M,S) \cong Q(R/\mathfrak{p})^r$ for some r. We need to show that r is finite. Localizing at $\mathfrak{p}$, we have $\mathrm{Hom}_{\mathsf{A}_\mathfrak{p}}(M_\mathfrak{p}, S_\mathfrak{p}) \cong Q(R/\mathfrak{p})^r$. It follows from (ii) that

$$\mathrm{Hom}_{\mathsf{A}_\mathfrak{p}}(M_\mathfrak{p}, S_\mathfrak{p}) \cong \mathrm{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}, R_\mathfrak{p}/(\mathfrak{p})) \cong Q(R/\mathfrak{p}).$$

So $r = 1$. □

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