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ON A SPECIAL CLASS OF CONVEX FUNCTIONS

R.Tyrrell Rockafellar[1]

1 Department of Mathematics, University of Washington, Seattle, Washington 98195, USA

1. A property of convex functions

I answer in the affirmative to a conjecture concerning convex functions. In Ref. 1 it is claimed that, if $f : \mathbb{R}^n \to \mathbb{R}$, with $n \geq 2$, is convex, then the limit

$$\lim_{t \downarrow 0} \nabla f(x(t)) \qquad (1)$$

where $x(t) := (t, 0, \ldots, 0) \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $t>0$, may not exist. Here a function will be shown which does not admit (1). First of all I recall the following theorem due to Danskin (see Ref. 2).

THEOREM. If $f(z) = \max_{w \in W} g_w(z)$, where W is compact and g_w is a differentiable functions with $g_w(z)$ and $\nabla g_w(z)$ depending continously on (w,z), then f has one-sided directional derivatives given by

$$f'(z;h) = \max_{w \in W_z} \langle \nabla g_w(z), h \rangle ,$$

where $W_z := \operatorname{argmax}_{w \in W} g_w(z)$.

The following corollary is a straightforward consequence of Danskin's Theorem (see Ref. 2).

COROLLARY. Under the assumptions of Danskin's Theorem, if in addition $g_w(z)$ is convex in z, and $\bar{z}$ is a point such that $W_{\bar{z}} = \{\bar{w}\}$, then f is differentiable at $\bar{z}$ with

$$\nabla f(\bar{z}) = \nabla_{g_{\bar{w}}}(\bar{z}).$$

PROOF. The above theorem gives $f'(\bar{z};h) = \langle \nabla g_{\bar{w}}(\bar{z}), h \rangle$. Moreover a convex function, whose directional derivative function at $\bar{z}$ is linear, must actually be differentiable at $\bar{z}$.

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2. **An Example.**

Let us consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, given by

$$f(x,y) := \max_{(u,v)\in W} (ux+vy - u^2/2) \quad , \quad (x,y)\in \mathbb{R}^2,$$

where

$$W:=\{(u,v)\in R^2 : 0<u\leq 1;\ v=\sin(1/2)\}\cup\{(u,v)\in R^2 : u=0;\ |v|\leq 1\}.$$

The set W is obviously compact. The function f is convex with respect to (x,y), since it is the pointwise maximum of affine functions. Let us set:

$$z=(x,y) \ , \ w=(u,v) \ , \ g_w(z)=g_{(u,v)}(x,y)=ux+vy - u^2/2,$$

so that $\nabla g_{(u,v)}(x,y) = (u,v)$. Then we can apply the Corollary. If $z=(x,0)$ and $0<|x|<1$, we deduce that

$$W_z = \operatorname*{argmax}_{(u,v)\in W} g_w(x,y) = \operatorname*{argmax}_{(u,v)\in W} (ux - u^2/2)=\{(x,\ \sin(1/2))\},$$

so that card $W_z=1$. Hence we obtain

$$\nabla f(x,0) = (x,\ \sin(1/2)) \quad , \quad 0<|x|<1 \ ,$$

$$\partial f(0,0) = \{0\}\times[-1,1] \quad ,$$

which shows that $\nabla f(x,0)$ exists for all x such that $0<x<1$, and that $\lim_{x\downarrow 0} \nabla f(x,0)$ does not exist.

REMARK. A function similar to the above f(x,y) has been subsequently proposed also by E. De Giorgi.

REFERENCES

1 F. GIANNESSI, A problem on convex functions. Journal of Optimization Theory and Applications. Vol. 59, No. 3, 1988, p. 525.

2 J.M. DANSKIN, The theory of max-min, with applications. SIAM Journal on Applied Mathematics, Vol. 14, No. 4, 1966, pp. 641-644.