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SECOND-ORDER OPTIMALITY CONDITIONS IN NONLINEAR PROGRAMMING OBTAINED BY WAY OF EPI-DERIVATIVES*†

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Dedicated to the memory of Robin W. Chaney

Second-order optimality conditions for finite-dimensional smooth and nonsmooth nonlinear programming are obtained by a new method that emphasizes a close connection with geometrical approximation of the essential objective function. The approximation is secured by the use of certain epi-derivatives defined by epiconvergence. The optimality conditions are expressed in a form that covers general interval constraints and their possible representation through penalties or an augmented Lagrangian. An abstract constraint involving restriction to a convex polyhedron is incorporated.

1. Introduction. Numerous authors have worked on the question of necessary and sufficient optimality conditions of first and second order in nonlinear programming. In order to accommodate modern problem formulations that may involve exact penalty terms or augmented Lagrangians, they have appealed in this effort to generalized concepts of differential approximation as well as the more traditional concepts of tangent cones. Especially to be cited in this vein are the papers of Ioffe [18]–[20], Ben-Tal and Zowe [3]–[6], Chaney [7]–[13], and Burke [6].

The reader may find it surprising, therefore, that in this subject there is still something left to be said, particularly in finite dimensions. In fact the contribution in this paper differs in several significant respects from what has been seen before. It is based on a problem formulation given recently in Rockafellar [27] that covers virtually all the optimization models commonly encountered in practice and yet is more specific than most of those dealt with by the authors just cited and therefore affords the possibility of sharper conclusions. It is based on a concept of derivative from paper [27] that has a stronger basis in geometric approximation and correspondingly a greater potential of stability and robustness. It extends the classical approach to optimality into a "neo-classical" approach where all one has to do is replace graphs of functions by epigraphs, and graphical convergence by epigraphical convergence.

In an abstract sense any finite-dimensional problem of optimization can be expressed in terms of minimizing a certain extended-real-valued function f over $\mathbf{R}^n$. The broad case that captures our attention here is the one where

$$f(x) = g(F(x)) \quad \text{for } F: \mathbf{R}^n \to \mathbf{R}^d, \quad g: \mathbf{R}^d \to \overline{\mathbf{R}}, \tag{1.1}$$

where F is a mapping of class $\mathscr{C}^2$ and g is a proper convex function that is *piecewise linear-quadratic* by the following definition: the effective domain of g (the set where g does not merely have the value ∞) is expressible as the union of finitely many polyhedral sets, relative to each of which g is given by a formula that is quadratic (or affine). Examples showing how typical problems in nonlinear programming can be cast in this mold have been furnished in [27], and we shall turn to them in due course later in this paper. The reader should bear in mind that the minimization of f over $\mathbf{R}^n$ when f is of form (1.1) carries with it the implicit constraint that $F(x) \in D$, where D is the effective domain of g and therefore, under the stated assumptions, is a convex polyhedron (possibly all of $\mathbf{R}^d$).

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The important fact to record about such functions f for the purpose of this introduction is that they are "twice epi-differentiable" as long as a basic constraint qualification is satisfied. This property, established in [27], expresses geometric features not heretofore recognized or put to use in connection with optimality conditions.

In §2 we briefly review the concept of epi-differentiability and develop in terms of it a simple, general theorem about necessary and sufficient conditions which does not depend on the particular structure in (1.1). Such structure is taken up however in §3, where we apply the epi-derivative formulas of [27], and in §4, where we specialize to a number of examples. The sufficient conditions obtained by this route are fully satisfactory, but the necessary conditions appear at first to fall short of the ones developed by other authors, because of the particular assumption of a constraint qualification. There is some philosophy to be offered on this, but in any case the discrepancy is only temporary and illusory, since by applying the results to an auxiliary problem one can quickly obtain all that might be desired. This is the topic of §5.

2. Abstract conditions of optimality. In this section f denotes an arbitrary lower semicontinuous function from $\mathbf{R}^n$ to the extended real number system $\overline{\mathbf{R}}$, not necessarily of the composite form (1.1), and x denotes a point where f is finite. We work with the first and second-order difference quotients

$$\varphi_{x,t}(\xi) = [f(x + t\xi) - f(x)]/t \quad \text{and} \tag{2.1}$$

$$\psi_{x,v,t}(\xi) = [f(x + t\xi) - f(x) - t\xi \cdot v]/\tfrac{1}{2}t^2, \tag{2.2}$$

the latter involving the choice of an additional vector v about which more will be said presently.

A family of sets S_t in $\mathbf{R}^n$ for $t > 0$ is said to *converge* to a set S as $t \downarrow 0$ if S is closed and one has

$$\lim_{t \downarrow 0} \operatorname{dist}(x, S_t) = \operatorname{dist}(x, S) \quad \text{for all } x \in \mathbf{R}^n.$$

This notion of set convergence can be characterized in many other equivalent ways, but we do not need to go into the details here; see [27] and [28] for additional description and references. A family of functions φ_t on $\mathbf{R}^n$ for $t > 0$ is said to *epiconverge* to a function φ if the epigraph sets epi φ_t converge as $t \downarrow 0$ to epi φ. Again this is a notion that has various expressions, but we merely refer to [27] and [28].

DEFINITION 2.1. The function f is said to be *epi-differentiable* at x if the functions $\varphi_{x,t}$ in (2.1) epiconverge as $t \downarrow 0$ and the limit function, denoted by f_x', has $f_x'(0) > -\infty$. It is said to be *twice epi-differentiable at x relative to the vector v* if it is epi-differentiable in the sense just described and the functions $\psi_{x,v,t}$ in (2.2) epiconverge as $t \downarrow 0$, and in addition the limit function, denoted by $f''_{x,v}$, has $f''_{x,v}(0) > -\infty$. In this event v must be an *epi-gradient* of f at x in the sense that $f'_x(\xi) \geq \xi \cdot v$ for all $\xi \in \mathbf{R}^n$ [27, Proposition 2.8]. We simply say therefore that f is *twice epi-differentiable at* x (without reference to a particular v) if f is epi-differentiable at x, has at least one epi-gradient there, and with respect to every epi-gradient v is twice epi-differentiable at x relative to v.

If f happens to be of class $\mathscr{C}^2$, one has

$$f'_x(\xi) = \xi \cdot \nabla f(x) \quad \text{for all } \xi,$$

and the unique epi-gradient at x is $v = \nabla f(x)$, and

$$f''_{x,v}(\xi) = \xi \cdot \nabla^2 f(x)\xi \quad \text{for all } \xi \text{ when } v = \nabla f(x).$$

In particular, f is twice epi-differentiable at x in this case.

A number of elementary properties of epi-derivatives have been furnished in [27, §2] along with further justification of the concept. Our attention here is focused on proving the following theorem that uses epi-derivatives in a prototype statement of optimality conditions. We use the terminology that f has a local minimum at x in the *strong sense* if

$$\text{(2.3)} \qquad \exists \alpha > 0 \quad \text{with } f(x') \geq f(x) + \alpha|x' - x|^2 \quad \text{for all } x' \text{ near } x.$$

THEOREM 2.2. *Let $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ be a lower semicontinuous function, and let x be a point where f is finite and twice epi-differentiable.*

(a) (*Necessary condition*). *If f has a local minimum at x, then 0 is a epi-gradient of f at x and*

$$\text{(2.4)} \qquad f''_{x,0}(\xi) \geq 0 \quad \text{for all } \xi.$$

(b) (*Sufficient condition*). *If 0 is a epi-gradient of f at x and*

$$f''_{x,0}(\xi) > 0 \quad \text{for all } \xi \neq 0,$$

then f has a local minimum at x in the strong sense.

PROOF. If f has a local minimum at x relative to the ball of radius δ around x, then the functions $\varphi_{x,t}$ and $\psi_{x,0,t}$ in (2.1) and (2.2) are nonnegative on the ball of radius $1/\delta$ around 0. Their epi limits f'_x and $f''_{x,0}$, if these indeed exist, must therefore be nonnegative on all of $\mathbf{R}^n$. The epi limit f'_x exists by the assumption that f is epi-differentiable; the condition $f'_x(\xi) \geq 0$ for all ξ means that 0 is a epi-gradient of f at x. Then $f''_{x,0}$ exists by the assumption of twice epi-differentiability. This establishes (a).

Under the assumptions in (b) one can let $\beta = \min_{|\xi|=1} f''_{x,0}(\xi)$ and have $\beta > 0$, because $f''_{x,0}$ is lower semicontinuous (its epigraph being a closed set by virtue of the definition of epiconvergence). Then actually

$$f''_{x,0}(\xi) \geq \beta|\xi|^2 \quad \text{for all } \xi,$$

because $f''_{x,0}$ is a function that is positively homogeneous of degree 2 [27, Proposition 2.7]. This says in the terminology of [27] that the matrix $H = \beta I$ is a "epi-Hessian" of f at x relative to $v = 0$. Then by [27, Proposition 2.8] one has

$$f(x') \geq f(x) + \tfrac{1}{2}(x' - x) \cdot H(x' - x) + o(|x' - x|^2),$$

and the assertion about f having a strong local minimum at x in the sense of (2.3) is justified. ■

The prime feature of Theorem 2.2 is that, despite its broad generality, it precisely mirrors the classical theorem for an unconstrained minimum, to which it reduces when f happens to be twice differentiable at x. The hypothesis of the theorem needs to be judged in this light, since it is not the weakest that might serve for the same conclusions. Neither the necessary condition nor the sufficient condition fully requires the second-order epi-differentiability of f at x. Both could actually be expressed in terms of certain "lower" epi-derivatives that always exist, and in this way one would obtain a seemingly more subtle result. This is true, however, for the classical theorem as well: "lower" second derivatives are all one really needs to deal with in order to characterize optimality.

Why then the classical assumption of twice differentiability? The reason obviously is that twice differentiability is a convenient and readily verifiable property with many consequences besides those of the theorem itself. The derivatives it involves can specifically be calculated for a large class of functions, and the optimality conditions in terms of them can therefore be incorporated into the analysis of computational methods and approximation schemes for problems with particular structure. Our argument for Theorem 2.2 is that much the same holds for twice epi-differentiability. This claim is based, of course, on the calculations of epi-derivatives that were carried out in [27] and which will be utilized in the next part of this paper.

Theorem 2.2 should also be compared with the kind of result that can be stated for the generalized second derivatives of Ben-Tal and Zowe [3]–[5]. Those derivatives, defined only for functions with finite values (and therefore not directly usable in a neoclassical approach to optimality where constraints are represented by an infinite penalty, as here), have been shown to exist in a number of important cases in which they can be used to characterize optimality. On the abstract level corresponding to Theorem 2.2 they cover necessity but not sufficiency, at least without drastically restricting attention to functions f that are differentiable and whose gradient mapping is Lipschitz continuous [5].

The general optimality conditions comparable to Theorem 2.2 that have been obtained by Chaney [7]–[13] concern a Lipschitz continuous function f and a different concept of second derivative, defined using directional limits of subgradients. Again the case of constraints represented by ∞ is not *directly* included. In contrast to the results of Ben-Tal and Zowe, necessity rather than sufficiency requires an extra assumption (semismoothness of f in the sense of Mifflin [22]), but this is not so stringent, and the two conditions are therefore more satisfyingly close together. Chaney's second derivatives turn out to coincide with ours for the main class of functions for which they have been calculated by him, which is also the main class treated by Ben-Tal and Zowe [3], as we have shown in [27] on the basis of Chaney's formulas in [10]. Their relationship with second-order epi-derivatives in other cases (under the assumption that f is locally Lipschitzian) is not settled.

3. Chain rule formula and the main theorem. For Theorem 2.2 to lead to optimality conditions that are illuminating in specific applications, one must be able to identify cases of interest where f is indeed twice epi-differentiable at x, and to calculate then the corresponding derivatives. For this purpose we shall rely on a chain rule proved in [22] for functions f of the form (1.1). Our first task is to state the relevant parts of this chain rule in a convenient manner for reference. We must begin with the underlying properties of the functions g appearing in (1.1).

THEOREM 3.1 [27, Theorem 3.1]. *Let $g: \mathbf{R}^d \to \overline{\mathbf{R}}$ be a proper convex function which is piecewise linear quadratic, and let D denote its effective domain. Then at any point $u \in D$, g is twice epi-differentiable, the derivatives being given by simple limits along rays:*

$$(3.1) \qquad g'_u(\omega) = \lim_{t \downarrow 0} [g(u + t\omega) - g(u)]/t,$$

$$(3.2) \qquad g''_{u,y}(\omega) = \lim_{t \downarrow 0} [g(u + t\omega) - g(u) - t\omega \cdot y]/\tfrac{1}{2}t^2.$$

The epi-gradients y in the second formula are the same as the subgradients of g, the elements of the subdifferential $\partial g(u)$ of convex analysis, which is given by

$$(3.3) \qquad \partial g(u) = \{ y \mid y \cdot \omega \leq g'_u(\omega), \forall \omega \}.$$

In fact one has

$$(3.4) \qquad g''_{u,y}(\omega) = \begin{cases} \gamma_u(\omega) & \text{if } y \cdot \omega = g'_u(\omega), \\ \infty & \text{if } y \cdot \omega < g'_u(\omega), \end{cases}$$

where for ω with $g'_u(\omega) < \infty$ one defines

$$(3.5) \qquad \gamma_u(\omega) = \lim_{t \downarrow 0} [g(u + t\omega) - g'(u) - t g'_u(\omega)]/\tfrac{1}{2}t^2 < \infty$$

$$[= 0 \text{ if } g \text{ is actually piecewise linear}].$$

DEFINITION 3.2. Let $F: \mathbf{R}^n \to \mathbf{R}^d$ be a $\mathscr{C}^2$ mapping and consider the constraint condition

$$(3.6) \qquad F(x) \in D,$$

where $D \subset \mathbf{R}^d$ is a convex polyhedron. At a point x for which this constraint is satisfied we shall say that the *basic constraint qualification* holds if the only vector $y \in \mathbf{R}^d$ having

$$(3.7) \qquad y \in N_D(u) \quad \text{and} \quad \nabla(yF)(x) = 0$$

is $y = 0$.

Here $N_D(u)$ denotes the normal cone to D in the sense of convex analysis, and yF is the $\mathscr{C}^2$ function from $\mathbf{R}^n$ to $\mathbf{R}$ defined by $(yF)(x) = y \cdot F(x)$. One can also write

$$(3.8) \qquad \nabla(yF)(x) = y\nabla F(x)$$

where $\nabla F(x)$ denotes the $d \times n$ dimensional Jacobian matrix of the mapping F at x.

The basic constraint qualification is a natural extension of the Mangasarian-Fromovitz constraint qualification to systems of the general form (3.3), as explained in [27, §4].

THEOREM 3.3 [27, Theorem 4.5]. *Let $f(x) = g(F(x))$, where $F: \mathbf{R}^n \to \mathbf{R}^d$ is a $\mathscr{C}^2$ mapping and $g: \mathbf{R}^d \to \overline{\mathbf{R}}$ is a proper convex function which is piecewise linear-quadratic with effective domain D. Let C denote the effective domain of f, i.e., $C = \{x \mid F(x) \in D\}$. Then at any $x \in C$ where the basic constraint qualification is satisfied, f is twice epi-differentiable. One has*

$$(3.9) \qquad f'_x(\xi) = g'_{F(x)}(\nabla F(x)\xi).$$

The corresponding set of epi-gradients v of f at x coincides with the set $\partial f(x)$ of generalized subgradients in the sense of Clarke and is given by

$$(3.10) \qquad \partial f(x) = \partial g(F(x))\nabla F(x) = \{ y\nabla F(x) \mid y \in \partial g(F(x)) \}.$$

For each $v \in \partial f(x)$ one has

$$(3.11) \qquad f''_{x,v}(\xi) = \begin{cases} \gamma_{F(x)}(\nabla F(x)\xi) + \max\limits_{y \in Y_v(x)} \xi \cdot \nabla^2(yF)(x)\xi & \text{if } \xi \in \Xi_v(x), \\ \infty & \text{if } \xi \notin \Xi_v(x), \end{cases}$$

where

$$(3.12) \qquad Y_v(x) = \{ y \in \partial g(F(x)) \mid y\nabla F(x) = v \}$$

is a nonempty, bounded, polyhedral convex set,

$$(3.13) \qquad \Xi_v(x) = \{ \xi \in \mathbf{R}^n \mid g'_{F(x)}(\nabla F(x)\xi) \leq v \cdot \xi \}$$

is a polyhedral convex cone, and $\gamma_{F(x)}(\nabla F(x)\xi)$ is the expression defined from g by (3.5).

It may be noted that in (3.13) one actually has

$$(3.14) \qquad \Xi_v(x) = \{ \xi \in \mathbf{R}^n \mid g'_{F(x)}(\nabla F(x)\xi) = v \cdot \xi \}$$

because of the assumption that $v \in \partial f(x)$. The latter implies by (3.9) and (3.10) that

$$(3.15) \qquad g'_{F(x)}(\nabla F(x)\xi) \geq v \cdot \xi \quad \text{for all } \xi \in \mathbf{R}^n.$$

Our main theorem on optimality conditions can now be displayed. It applies to any problem that can be represented in the general form

$$(P_0) \qquad \text{minimize } f(x) = g(F(x)) \quad \text{subject to } F(x) \in D,$$

where F, g and D are as in Theorem 3.3. This is a very rich class of problems; examples will be recalled in §4.

THEOREM 3.4. *Let $f: \mathbf{R}^n \to \overline{\mathbf{R}}$ be a function having the form specified in Theorem 3.3, and let x be a point of the effective domain C of f where the basic constraint qualification is satisfied. Let*

$$(3.16) \qquad Y_0(x) = \{ y \in \partial g(F(x)) \mid y\nabla F(x) = 0 \}$$

(this will be the set of first-order multiplier vectors), let

$$\Xi_0(x) = \{\xi \in \mathbf{R}^n \mid g'_{F(x)}(\nabla F(x)\xi) \le 0\} \tag{3.17}$$

(this will be the cone giving the first-order critical directions), and denote by $\gamma_{F(x)}(\nabla F(x)\xi)$ *the expression defined from g by (3.5).*

(a) (*Necessary condition*). *If f has a local minimum at x, then* $Y_0(x) \ne \emptyset$ *and*

$$\forall \xi \in \Xi_0(x), \quad \exists y \in Y_0(x) \quad \textit{with } \gamma_{F(x)}(\nabla F(x)\xi) + \xi \cdot \nabla^2(yF)(x)\xi \ge 0. \tag{3.18}$$

(b) (*Sufficient condition*). *If* $Y_0(x) \ne \emptyset$ *and*

$$\forall \xi \in \Xi_0(x) \setminus \{0\}, \quad \exists y \in Y_0(x) \quad \textit{with} \quad \gamma_{F(x)}(\nabla F(x)\xi) + \xi \cdot \nabla^2(yF)(x)\xi > 0, \tag{3.19}$$

then f has a local minimum at x in the strong sense (as defined by (2.3)).

PROOF. According to Theorem 3.3 one has $Y_0(x) \ne \emptyset$ if and only if 0 is a epi-gradient vector for f at x. Formulas (3.11) and (3.14) yield

$$f''_{x,0}(\xi) = \begin{cases} \gamma_{F(x)}(\nabla F(x)\xi) + \max\limits_{y \in Y_0(x)} \xi \cdot \nabla^2(yF)(x)\xi & \text{if } \xi \in \Xi_0(x), \\ \infty & \text{if } \xi \notin \Xi_0(x). \end{cases} \tag{3.20}$$

It is immediate then that (3.18) and (3.19) are realizations in the present context of the second derivative conditions in Theorem 2.2 ∎

The fact that Theorem 3.4 assumes a constraint qualification for the sufficient condition as well as for the necessary condition puts this result somewhat out of the usual pattern for the literature on optimality, as we have already mentioned in connection with Theorem 2.2 and will discuss further in the sequel. We remind the reader that this feature is largely one of the mathematical presentation rather than of the refinements that ultimately drop out. The hypothesis of Theorem 3.4, like that of the classical theorem for unconstrained minimization, supports not only the optimality conditions in question but their robust interpretation in terms of a *geometrical approximation* to f at x, as expressed through Theorem 3.3 by the epiconvergence that underlies the concept of epi-differentiation.

4. Specialization to models in nonlinear programming. Two specific types of problem formulation in nonlinear programming will serve to illustrate the content of our general result in Theorem 3.4. The first is

$$(P_1) \qquad \begin{aligned} &\text{minimize } f_0(x) \quad \text{subject to } f_i(x) \in I_i \quad \text{for} \quad i = 1, \dots, m \\ &\text{and } x \in X, \quad \text{where } f_0(x) = \max_{j=1,\dots,s} f_{0j}(x). \end{aligned}$$

In this problem X denotes a nonempty convex polyhedron in $\mathbf{R}^n$, I_i is for $i = 1, \dots, m$ a nonempty closed interval in $\mathbf{R}$, and the functions f_i and f_{0j} are all of class $\mathcal{C}^2$.

The interval I_i can be of any kind; $I_i = [c_i, \infty)$ corresponds to an inequality $f_i(x) \ge c_i$, $I_i = (-\infty, c_i]$ corresponds to $f_i(x) \le c_i$, and $I_i = [c_i, c_i]$ corresponds to an

equality constraint $f_i(x) = c_i$. The polyhedron X may be $\mathbf{R}^n_+$ or all of $\mathbf{R}^n$, for instance; it may be used also as a representation of an arbitrary system of finitely many linear constraints, which can give extra power later when verifying that a constraint qualification is satisfied. Obviously covered too are cases where $s = 1$, so that f_0 itself is a $\mathcal{C}^2$ function rather than just a max function, or when $m = 0$, i.e. no side conditions of the form $f_i(x) \in I_i$ are present.

The second of our formulations involves the same data elements and assumptions but corresponds rather to something like a penalty representation or Lagrangian representation of (P_1):

$$(P_2) \qquad \begin{aligned} &\text{minimize } f_0(x) + \sum_{i=1}^m \rho_i\big(d_{I_i^*}(f_i(x))\big) \quad \text{over all } x \in X, \quad \text{with} \\ &f_0(x) = \max_{j=1,\dots,s} f_{0j}(x), \quad \text{where} \end{aligned}$$

$$\begin{aligned} &I_i^* = \text{some nonempty closed interval (possibly } I_i^* \ne I_i), \\ &d_{I_i^*}(u_i) = \text{distance of } u_i \text{ from } I_i^*. \end{aligned} \tag{4.1}$$

The functions $\rho_i\colon \mathbf{R}_+ \to \mathbf{R}$ are assumed to be (affine or) quadratic and increasing (therefore convex).

An example of this structure is

$$\rho_i(\theta_i) = q_i\theta_i \quad \text{with } q_i > 0; \quad I_i^* = I_i. \tag{4.2}$$

One then has an "exact" penalty representation of (P_1) of l_1 type, with the q_i's as the penalty parameters. Alternatively one could take

$$\rho_i(\theta_i) = \tfrac{1}{2}r_i\theta_i^2 \quad \text{with } r_i > 0; \quad I_i^* = I_i. \tag{4.3}$$

This would correspond to a traditional smooth penalty representation of l_2 type. The standard (quadratic-based) augmented Lagrangian representation of (P_1), as generalized to the interval constraints $f_i(x) \in I_i$, takes the form of (P_2) with

$$\rho_i(\theta_i) = \tfrac{1}{2}r_i\theta_i^2 - \tfrac{1}{2}(\lambda_i/r_i)^2 \quad \text{with } r_i > 0; \quad I_i^* = I_i - (\lambda_i/r_i). \tag{4.4}$$

In this case λ_i corresponds to the usual Lagrange multiplier. The reader can verify for instance that when $I_i = [0, 0]$ one has under (4.4) that

$$\rho_i\big(d_{I_i^*}(f_i(x))\big) = \lambda_i f_i(x) + \tfrac{1}{2}r_i f_i(x)^2; \tag{4.5}$$

this is the expression introduced by Hestenes [16] and Powell [23] for an equality constraint $f_i(x) = 0$. When $I_i = (-\infty, c_i]$, on the other hand, one has under (4.4) that

$$\rho_i\big(d_{I_i^*}(f_i(x))\big) = \begin{cases} \lambda_i f_i(x) + \tfrac{1}{2}r_i f_i(x)^2 & \text{if } f_i(x) \ge -\lambda_i/r_i, \\ -\lambda_i^2/2r_i & \text{if } f_i(x) \le -\lambda_i/r_i; \end{cases} \tag{4.6}$$

this is the augmented Lagrangian term introduced by Rockafellar [25] for an inequality constraint $f_i(x) \le 0$.

In order to represent (P_1) as a problem of minimizing $f(x) = g(F(x))$ over all $x \in \mathbf{R}^n$ we need only define

$$F(x) = (x, f_{01}(x), \dots, f_{0s}(x), f_1(x), \dots, f_m(x)), \tag{4.7}$$

$$D = X \times \mathbf{R}^s \times I_1 \times \cdots \times I_m, \tag{4.8}$$

$$g(u) = g(w, u_{01}, \dots, u_{0s}, u_1, \dots, u_m) = \begin{cases} \max\{u_{01}, \dots, u_{0s}\} & \text{if } w \in X \text{ and } u_i \in I_i, \\ \infty & \text{otherwise.} \end{cases} \tag{4.9}$$

Obviously F is a $\mathscr{C}^2$ mapping and g is a piecewise linear convex function with effective domain D. One has $f(x) = f_0(x)$ if x satisfies the constraints in (P_1), but $f(x) = \infty$ otherwise.

The representation of (P_2) as a problem of minimizing $f(x) = g(F(x))$ over all $x \in \mathbf{R}^n$ uses the same $\mathscr{C}^2$ mapping F as in (4.7) but takes

$$D = X \times \mathbf{R}^s \times \mathbf{R}^m, \tag{4.10}$$

$$g(u) = g(w, u_{01}, \dots, u_{0s}, u_1, \dots, u_m) = \begin{cases} \max\{u_{01}, \dots, u_{0s}\} + \sum_{i=1}^{m} \rho_i(d_{I_i}(u_i)) & \text{if } w \in X, \\ \infty & \text{otherwise.} \end{cases} \tag{4.11}$$

Then g is a piecewise linear-quadratic convex function with effective domain D, provided of course that every ρ_i is (linear or) quadratic and nondecreasing on $\mathbf{R}_+$, as already specified.

The meaning of the basic constraint qualification of Definition 3.2 in the case of these representations of problems (P_1) and (P_2) must be determined next. It will be of help to us in this task to use the notation $y_i \in N_i(x)$, where

$$N_i(x) := N_{I_i}(f_i(x)) = \text{normal cone to } I_i \text{ at } f_i(x). \tag{4.12}$$

Inasmuch as I_i is merely a closed interval in $\mathbf{R}$, this notation is just a convenient way of expressing the conditions on the signs of the multipliers that have long been familiar in nonlinear programming. Thus if we write $I_i = [c_i^-, c_i^+]$ (where c_i^- or c_i^+ might be finite), we have

$$N_i(x) = \begin{cases} [0,0] & \text{if } c_i^- < f_i(x) < c_i^+, \\ (-\infty, 0] & \text{if } c_i^- = f_i(x) < c_i^+, \\ [0, \infty) & \text{if } c_i^- < f_i(x) = c_i^+, \\ (-\infty, \infty) & \text{if } c_i^- = f_i(x) = c_i^+. \end{cases} \tag{4.13}$$

We will also use the notation

$$N_X(x) = \text{normal cone to the polyhedron } X \text{ at } x. \tag{4.14}$$

THEOREM 4.1. *In problem (P_1) as represented by (4.7), (4.8) and (4.9), a feasible solution x satisfies the basic constraint qualification if and only if the only multiplier vector $(y_1, \dots, y_m)$ satisfying*

$$y_i \in N_i(x) \quad \text{and} \quad -\sum_{i=1}^{M} y_i \nabla f_i(x) \in N_X(x) \tag{4.15}$$

is the vector $(0, \dots, 0)$.

In problem (P_2) as represented by (4.7), (4.10) and (4.11), every feasible solution x automatically satisfies the basic constraint qualification.

PROOF. For (P_1) one calculates from (4.8) that

$$N_D(F(x)) = N_X(x) \times \{(0, \dots, 0)\} \times N_1(x) \times \cdots \times N_m(x).$$

A vector $y = (z, y_{01}, \dots, y_{0s}, y_1, \dots, y_m)$ belongs to this cone if and only if $(y_{01}, \dots, y_{0s}) = (0, \dots, 0)$, $z \in N_X(x)$ and $y_i \in N_i(x)$, and for such a vector one has

$$\nabla(yF)(x) = z + \sum_{i=1}^{m} y_i \nabla f_i(x).$$

This equals 0 if and only if $-\sum_{i=1}^{m} y_i \nabla f_i(x) = z$. The basic constraint qualification requires that such be the case only when $(y_1, \dots, y_m) = 0$ and $z = 0$. This is equivalent to the condition stated in the theorem.

For (P_2) one has from (4.10) simply that

$$N_D(F(x)) = N_X(x) \times \{(0, \dots, 0)\} \times \{(0, \dots, 0)\}.$$

In this case a vector $y = (z, y_{01}, \dots, y_{0s}, y_1, \dots, y_m)$ belongs to $N_D(F(x))$ if and only if $(y_{01}, \dots, y_{0s}) = (0, \dots, 0)$ and $(y_1, \dots, y_m) = (0, \dots, 0)$, in which event $\nabla(yF)(x) = z$. Such a vector y satisfies $\nabla(yF)(x) = 0$ only when $y = 0$. Thus the basic constraint qualification is in this case satisfied automatically. ■

For the theorem that comes next we require the tangent cones that are polar to the normal cones in (4.12) and (4.14):

$$T_i(x) := T_{I_i}(f_i(x)) = \text{tangent cone to } I_i \text{ at } f_i(x), \tag{4.16}$$

$$T_X(x) = \text{tangent cone to } X \text{ at } x. \tag{4.17}$$

In the case of the intervals I_i, the tangent cones merely express sign conditions: parallel to (4.13) we have for $I_i = [c_i^-, c_i^+]$ (where c_i^- or c_i^+ might be infinite) that

$$T_i(x) = \begin{cases} (-\infty, \infty) & \text{if } c_i^- < f_i(x) < c_i^+, \\ [0, \infty) & \text{if } c_i^- = f_i(x) < c_i^+, \\ (-\infty, 0] & \text{if } c_i^- < f_i(x) = c_i^+, \\ [0, 0] & \text{if } c_i^- = f_i(x) = c_i^+. \end{cases} \tag{4.18}$$

It will also help to have the notation

(4.19) $$J(x) = \{ j \in [1, s] \mid f_{0j}(x) = f_0(x) \}.$$

(4.20) $$S(x) = \left\{ y_0 = (y_{01}, \dots, y_{0s}) \,\middle|\, \sum_{j=1}^{s} y_{0j} = 1 \text{ with } y_{0j} \geq 0 \text{ for } j \in J(x),\ y_{0j} = 0 \text{ for } j \notin J(x) \right\}.$$

THEOREM 4.2. *Suppose in problem* (P_1) *that* x *is a feasible solution for which the constraint qualification in Theorem* 4.1 *is satisfied: there does not exist* $(y_1, \dots, y_m) \neq (0, \dots, 0)$ *such that* (4.15) *holds. Consider the cone*

(4.21) $$\Xi(x) = \{ \xi \in T_X(x) \mid \nabla f_i(x) \cdot \xi \in T_i(x) \text{ for } i = 1, \dots, m \text{ and } \nabla f_{0j}(x) \cdot \xi \leq 0 \text{ for all } j \in J(x) \},$$

and the multiplier set

(4.22) $$Y(x) = \left\{ (y_{01}, \dots, y_{0s}, y_1, \dots, y_m) \mid (y_{01}, \dots, y_{0s}) \in S(x),\ y_i \in N_i(x) \text{ for } i = 1, \dots, m, \text{ and } -\sum_{j=1}^{s} y_{0j} \nabla f_{0j}(x) - \sum_{j=1}^{m} y_i \nabla f_i(x) \in N_X(x) \right\}.$$

(a) (*Necessary condition*). *If* x *is a locally optimal solution to* (P_1), *then* $Y(x) \neq \varnothing$ *and*

(4.23) $$\forall \xi \in \Xi(x), \exists (y_{01}, \dots, y_{0s}, y_1, \dots, y_m) \in Y(x) \text{ with } \xi \cdot \left[\sum_{j=1}^{s} y_{0j} \nabla^2 f_{0j}(x) + \sum_{i=1}^{m} y_i \nabla^2 f_i(x) \right] \xi \geq 0.$$

(b) (*Sufficient condition*). *If* $Y(x) \neq \varnothing$ *and*

(4.24) $$\forall \xi \in \Xi(x) \setminus \{0\}, \exists (y_{01}, \dots, y_{0s}, y_1, \dots, y_m) \in Y(x) \text{ with } \xi \cdot \left[\sum_{j=1}^{s} y_{0j} \nabla^2 f_{0j}(x) + \sum_{i=1}^{m} y_i \nabla^2 f_i(x) \right] \xi > 0,$$

then x *is a locally optimal solution to* (P_1) *in the strong sense, namely: for some* $\alpha > 0$ *one has*

$$f_0(x') > f_0(x) + \alpha |x' - x|^2 \quad \textit{for all feasible } x' \textit{ near } x.$$

PROOF. We have here a case of Theorem 3.4 where g is piecewise linear, so that $\gamma_{F(x)}(\nabla F(x)\xi) = 0$. All we need to do is calculate $Y_0(x)$ and $\Xi_0(x)$ in (3.16) and (3.17) using the structure of F and g in (4.7), (4.8) and (4.9), and to verify that the necessary and sufficient conditions in Theorem 3.4 thereby reduce to the ones now claimed.

To assist ourselves in the calculation we shall write

(4.25) $$u_0 = (u_{01}, \dots, u_{0s}), \quad y_0 = (y_{01}, \dots, y_{0s}),$$

(4.26) $$\sigma(u_0) = \max_{j=1,\dots,s} u_{0j}, \quad J_0(u_0) = \operatorname*{argmax}_{j=1,\dots,s} u_{0j}.$$

In (4.9) we have

$$g(u) = g(w, u_0, u_1, \dots, u_m) = \delta_X(w) + \sigma(u_0) + \sum_{i=1}^{m} \delta_{I_i}(u_i),$$

where the δ's symbolize indicator functions. Let

$$T_{I_i}(u_i) = \text{tangent cone to } I_i \text{ at } u_i, \quad N_{I_i}(u_i) = \text{normal cone to } I_i \text{ at } u_i.$$

It is evident from the limit formula (3.1) for $g_u'(\omega)$ that

$$g_u'(\omega) = g_u'(\psi, \omega_0, \omega_1, \dots, \omega_m) = \delta_{T_X(w)}(\psi) + \sigma_{u_0}'(\omega_0) + \sum_{i=1}^{m} \delta_{T_{I_i}(u_i)}(\omega_i),$$

where $\omega_0 = (\omega_{01}, \dots, \omega_{0s})$. At the same time,

$$\partial g(u) = \partial g(w, u_0, u_1, \dots, u_m) = \left(\partial \delta_X(w), \partial \sigma(u_0), \partial \delta_{I_1}(u_1), \dots, \partial \delta_{I_m}(u_m) \right) = \left(N_X(w), \partial \sigma(u_0), N_{I_1}(u_1), \dots, N_{I_m}(u_m) \right)$$

by formulas of convex analysis. Here

(4.27) $$\sigma_{u_0}'(\omega_0) = \max_{j \in J_0(u_0)} \omega_{0j}, \quad \text{and}$$

(4.28) $$\partial \sigma(u_0) = \left\{ y_0 \,\middle|\, \sum_{j=1}^{s} y_{0j} = 1,\ y_{0j} \geq 0 \text{ for } j \in J_0(u_0),\ y_{0j} = 0 \text{ for } j \notin J_0(u_0) \right\}.$$

Observing that

(4.29) $$\nabla F(x)\xi = (\xi, \nabla f_{01}(x) \cdot \xi, \dots, \nabla f_{0s}(x) \cdot \xi, \nabla f_1(x) \cdot \xi, \dots, \nabla f_m(x) \cdot \xi)$$

[illegible] (4.[illegible]) we [illegible] that

$$g_{I,\ldots}(\nabla f(x)\xi) = \begin{cases} \max \nabla f(x)\,\xi & \text{if } \xi \in N_I(x) \ldots \\ \ldots & \text{otherwise} \end{cases}$$

The set $\Xi(x)$ in (4.21) does therefore specialize the set $\Xi(x)$ in Theorem 3.4. On the other hand, the elements of the set $Y(x)$ in (4.22) correspond one to one with the vectors $y = (z, y_{\ldots}, \ldots, y_m)$ in

$$\partial g(F(x)) = (N_X(x), N(x), N_I(x) \ldots)$$

such that

(4.30) $$0 = z + \sum y_{i}\nabla f_{i}(x) + \sum y\nabla f(x) \ldots \nabla f(x)$$

For such vectors one has

(4.31) $$\nabla^2 (yF)(x) = \sum y_{i}\nabla^2 f_{i}(x) + \cdots + \sum y\nabla^2 f(x)$$

Conditions (4.23) and (4.24) in terms of $Y(x)$ do turn out then to express the same thing as conditions (3.18) and (3.19) of Theorem 3.4 in terms of $Y(x)$. ■

The reader will notice that the constraint qualification in Theorem 4.2 is assumed for the sufficient as well as the necessary condition. This is not the limitation it may seem. The matter will be addressed in §5 along with refinements in the constraint qualification which can be derived simply by applying Theorem 4.2 to an auxiliary problem.

The final multiplier condition in (4.22) takes the more familiar form

$$\sum y_{i}\nabla f_{i}(x) + \sum y\nabla f(x) = 0,$$

of course, if $x \in \operatorname{int} X$, as for instance when $X = \mathbb{R}^n$, i.e. no abstract linear constraints are incorporated into the problem formulation; similarly for the constraint qualification (4.15).

As our next project, we shall apply our basic Theorem 3.4 to problem (P). A preliminary step is the clarification of properties of the distance functions d_{I^*} that appear in (P). Generally speaking the interval I^* is closed but not necessarily bounded; it may reduce on the other hand to a single point. It is expedient to adopt the following notation:

$u > I^*$ means: u lies to the right of I^*,

$u_i < I^*$ means: u_i lies to the left of I^*.

In parallel with (4.12) and (4.16) we let

(4.32) $$N^*(x) = N_{I^*}(f(x)) \quad \text{normal cone to } I^* \text{ at } f(x)$$

(4.33) $$T^*(x) = T_{I^*}(f(x)) \quad \text{tangent cone to } I^* \text{ at } f(x);$$

these sets specify sign patterns relative to x just as in (4.13) and (4.18), but with I replaced by the possibly different interval I^*. We also write

(4.34) $$d_{T^*(x)}(\omega) = [\text{distance of } \omega \text{ from } T^*(x)]$$

PROPOSITION 4.3. *One has*

(4.35) $$(d_{I^*})'_{\ldots}(\omega) = \begin{cases} \omega & \text{if } f(x) > I^* \\ d_{T^*(x)}(\omega) & \text{if } f(x) \in I^*, \\ -\omega & \text{if } f(x) < I^*, \end{cases}$$

and correspondingly

(4.36) $$\partial d_{I^*}(f(x)) = \begin{cases} [1,1] & \text{if } f(x) > I^* \\ N^*(x) \cap [-1,1] & \text{if } f(x) \in I^* \\ [-1,-1] & \text{if } f(x) < I^* \end{cases}$$

PROOF. The assertions of (4.32) are obvious in the cases where $f(x) \notin I^*$, since in a neighborhood of such values $f(x)$ the function d_{I^*} is affine with slope 1 or -1. In the case where $f(x) \in I^*$, one has $d_I(f(x) + t\omega) = 0$ for small $t > 0$ if $\omega \in T^*(x)$; $d_I(f(x) + t\omega) = t|\omega|$ if $\omega \notin T^*(x)$. This implies that

$$\lim_{t \downarrow 0} \left[d_{I^*}(f(x) + t\omega) - d_{I^*}(u) \right] / t = \begin{cases} 0 & \text{if } \omega \in T^*(x) \\ |\omega| & \text{if } \omega \notin T^*(x) \end{cases}$$

Since $T^*(x)$ can only be one of the four intervals $(-\infty, 0]$, $[0, \infty)$, $[0, 0]$ or $(-\infty, \infty)$, the middle formula in (4.35) is correct. If we now invoke from convex analysis the fact that $(d_{I^*})'_{u}$ is the support function of $\partial d_{I^*}(u)$, we get (4.36) by taking $u = f(x)$. ■

THEOREM 4.4. *In problem (P) where the functions ρ are [illegible] and increasing on $\mathbb{R}$, let*

(4.37) $$r = [\text{the constant value of } \rho']$$

(4.38) $$p(x) = \rho(d_{I^*}(f(x))) = 0,$$

(4.39) $$d(\omega) = (d_{I^*})'_{\ldots}(\omega) \quad \text{as in (4.35)}$$

Let $J(x)$ and $N(x)$ be as in (4.19) and (4.20); let

(4.40) $$\Xi(x) = \{ \ldots \mid f_0(x) = \max \nabla f(x)\,\xi + \sum p(\ldots) \ldots \}$$

and define

$$(4.41)\qquad Y(x) = \Big\{(y_{01}, \dots, y_{0s}, y_1, \dots, y_m) \mid (y_{01}, \dots, y_{0s}) \in S(x),$$

$$y_i \in p_i(x)\, \partial d_{I_i^*}(f_i(x)) \text{ for } i = 1, \dots, m,$$

$$\text{and } -\sum_{j=1}^{s} y_{0j} \nabla f_{0j}(x) - \sum_{i=1}^{m} y_i \nabla f_i(x) \in N_X(x)\Big\},$$

where $\partial d_{I_i^*}(f_i(x))$ is the interval in (4.36).

(a) (*Necessary condition*). *If x is a locally optimal solution to* (P_2), *then $Y(x) \neq \emptyset$ and*

$$\forall \xi \in \Xi(x), \exists (y_{01}, \dots, y_{0s}, y_1, \dots, y_m) \in Y(x) \text{ with}$$

$$(4.42)\qquad \sum_{i=1}^{m} r_i d_i^2(\nabla f_i(x) \cdot \xi) + \xi \cdot \left[\sum_{j=1}^{s} y_{0j} \nabla^2 f_{0j}(x) + \sum_{i=1}^{m} y_i \nabla^2 f_i(x)\right] \xi \geq 0.$$

(b) (*Sufficient condition*). *If $Y(x) \neq \emptyset$ and*

$$\forall \xi \in \Xi(x) \setminus \{0\}, \quad \exists (y_{01}, \dots, y_{0s}, y_1, \dots, y_m) \in Y(x) \quad \text{with}$$

$$(4.43)\qquad \sum_{i=1}^{m} r_i d_i^2(\nabla f_i(x) \cdot \xi) + \xi \cdot \left[\sum_{j=1}^{s} y_{0j} \nabla^2 f_{0j}(x) + \sum_{i=1}^{m} y_i \nabla^2 f_i(x)\right] \xi > 0,$$

then x is a locally optimal solution to (P_2) in the strong sense.

PROOF. Our intention is to derive this straightforwardly from Theorem 3.4 by calculating what the conditions in Theorem 3.4 mean when F, D and g are specified by (4.7), (4.10) and (4.11). Making use once more of the notation (4.25), (4.26), we write

$$(4.44)\qquad g(u) = g(w, u_0, u_1, \dots, u_m) = \delta_X(w) + \sigma(u_0) + \sum_{i=1}^{m} \rho_i(d_{I_i^*}(u_i)).$$

The limit formula (3.1) for $g_u'(\omega)$ then gives us

$$(4.45)\qquad g_u'(\omega) = g_u'(\psi, \omega_0, \omega_1, \dots, \omega_m)$$

$$= \delta_{T_X(w)}(\psi) + \sigma_{u_0}'(\omega_0) + \sum_{i=1}^{m} \rho_i'(d_{I_i^*}(u_i))(d_{I_i^*})_{u_i}'(\omega_i),$$

where $\omega_0 = (\omega_{01}, \dots, \omega_{0s})$. Because g_u' is the support function of $\partial g(u)$, we must have

$$\partial g(u) = \big(N_X(w), \partial\sigma(u_0), \rho_1'(d_{I_1^*}(u_1))\, \partial d_{I_1^*}(u_1), \dots, \rho_m'(d_{I_m^*}(u_m))\, \partial d_{I_m^*}(u_m)\big).$$

Recalling (4.27) and (4.29), we get (under the notation (4.35), (4.36))

$$(4.46)\qquad g_{F(x)}'(\nabla F(x)\xi)$$

$$= \begin{cases} \max\limits_{j \in J(x)} \nabla f_{0j}(x) \cdot \xi + \sum\limits_{i=1}^{m} p_i(x) d_i(\nabla f_i(x) \cdot \xi) & \text{if } \xi \in T_X(x), \\ \infty & \text{if } \xi \notin T_X(x). \end{cases}$$

Furthermore

$$(4.47)\qquad \partial g(F(x))$$

$$\big(N_X(x), \partial\sigma(F_0(x)), p_1(x)\, \partial d_{I_1^*}(f_1(x)), \dots, p_m(x)\, \partial d_{I_m^*}(f_m(x))\big)$$

where

$$(4.48)\qquad \partial\sigma(F_0(x)) = \partial\sigma(f_{01}(x), \dots, f_{0s}(x)) = S(x)$$

(see (4.28)).

In view of (4.46) the set $\Xi_0(x)$ in Theorem 3.4 coincides with the set $\Xi(x)$ defined by (4.40). The elements of the set $Y(x)$ in (4.41) correspond one-to-one under (4.47), (4.48), with the vectors $(z, y_{01}, \dots, y_{0s}, y_1, \dots, y_m) \in \partial g(F(x))$ such that (4.30) holds. Moreover such vectors give (4.31). The only thing left to verify then, in order to see that conditions (3.18) and (3.19) of Theorem 3.4 in terms of $Y_0(x)$ reduce to the conditions (4.42) and (4.43) claimed here, is the equation

$$(4.49)\qquad \gamma_{F(x)}(\nabla F(x)\xi) = \sum_{i=1}^{m} r_i d_i^2(\nabla f_i(x) \cdot \xi).$$

Calculating from the definition (3.5) and our current formulas (4.44) and (4.45), we find

(4.50)

$$\gamma_u(\omega) = \lim_{t \downarrow 0} \left[\sigma(u_0 + t\omega_0) - \sigma(u_0) - t\sigma_{u_0}'(\omega_0)\right] / \tfrac{1}{2}t^2$$

$$+ \sum_{i=1}^{m} \lim_{t \downarrow 0} \left[\rho_i(d_{I_i^*}(u_i + t\omega_i)) - \rho_i(d_{I_i^*}(u_i)) - t\rho_i'(d_{I_i^*}(u_i))(d_{I_i^*})_{u_i}'(\omega_i)\right] / \tfrac{1}{2}t^2.$$

The first limit in this expression vanishes, because σ is a piecewise linear function. Taking advantage of the quadratic nature of the function ρ_i in terms of the expression

$$\rho_i(\theta_i) - \rho_i(d_{I_i^*}(u_i)) = \rho_i'(d_{I_i^*}(u_i))\left[\theta_i - d_{I_i^*}(u_i)\right] + \tfrac{1}{2}r_i\left[\theta_i - d_{I_i^*}(u_i)\right]^2,$$

where r_i is the constant in (4.34), and setting

$$\Delta_i(t) = \left[\theta_i - d_{I_i^*}(u_i)\right]/t \quad \text{for } \theta_i = d_{I_i^*}(u_i + t\omega_i),$$

we can convert (4.50) into

$$\gamma_u(\omega) = \sum_{i=1}^{m} \lim_{t \downarrow 0} \Big(r_i \Delta_i^2(t) + 2\rho_i'\big(d_{I_i}\cdot(u_i)\big)\big[\Delta_i(t) - (d_{I_i}\cdot)'_{u_i}(\omega_i)\big]\Big).$$

But $\Delta_i(t) = (d_{I_i}\cdot)'_{u_i}(\omega_i)$ for $t > 0$ sufficiently small, inasmuch as $d_{I_i}\cdot$ is a piecewise linear function. It follows that

$$\gamma_u(\omega) = \sum_{i=1}^{m} r_i \big[(d_{I_i}\cdot)'_{u_i}(\omega_i)\big]^2.$$

In the notation (4.39) we therefore have (4.49) as promised. ∎

Theorem 4.4 can instantly be specialized to the cases of (P_2) cited in (4.2), (4.3) and (4.4), which correspond respectively to an l_1 penalty representation, an l_2 penalty representation and an augmented Lagrangian representation of problem (P_1). It is to be noted that *no constraint qualification at all* is involved in this, despite the presence of the abstract (linear) constraint condition $x \in X$.

Previous work on problems of type (P_2) has not incorporated such an abstract constraint condition. Aside from this feature there is nothing essentially new in Theorem 4.4 except for the general formulation encompassing all three of the cases just mentioned and doing so in terms of arbitrary interval constraints $f_i(x) \in I_i$. With $X = \mathbb{R}^n$ one could derive much the same result from the theory of Ben-Tal and Zowe [3]–[5], for instance, or alternatively by the approach of Chaney [7]–[13]. In doing so, however, one would miss an important property made clear by the methodology used here, namely the connection between these necessary and sufficient conditions and the geometrically-based first and second-order epi-derivatives of the objective function in (P_2) —which have been shown always to exist.

To help with comparisons and facilitate an application of Theorem 4.4 that we shall make in §5, we state as a corollary the case that corresponds to a typical penalty representation of (P_1) as analyzed around a feasible solution to (P_1).

COROLLARY 4.5. *Consider the case of (P_2) where*

$$(4.51)\qquad \rho_i(\theta_i) = q_i\theta_i + \tfrac{1}{2}r_i\theta_i^2, \quad I_i^* = I_i$$

(with $q_i \geq 0$, $r_i \geq 0$, $q_i + r_i > 0$), so that (P_2) is simple penalty representation of (P_1) (either l_1 or l_2 or a mixture). Suppose x is a feasible solution to (P_1) and introduce the notation $J(x)$ and $S(x)$ as in (4.19) and (4.20), along with

$$(4.52)\qquad \Xi(x) = \Big\{ \xi \in T_X(x) \,\Big|\, \max_{j \in J(x)} \nabla f_{0j}(x)\cdot\xi + \sum_{i=1}^{q} q_i d_{T_i(x)}(\nabla f_i(x)\cdot\xi) \leq 0 \Big\},$$

$$(4.53)\qquad Y(x) = \Big\{ (y_{01},\dots,y_{0s},y_1,\dots,y_m) \,\Big|\, (y_{01},\dots,y_{0s}) \in S(x),$$
$$y_i \in N_i(x) \text{ with } |y_i| \leq q_i \text{ for } i = 1,\dots,m,$$
$$-\sum_{j=1}^{s} y_{0j}\nabla f_{0j}(x) - \sum_{i=1}^{m} y_i\nabla f_i(x) \in N_X(x) \Big\}.$$

(a) (*Necessary condition*). *If x is a locally optimal solution to (P_2), then $Y(x) \neq \varnothing$ and*

$$(4.54)\qquad \forall \xi \in \Xi(x),\ \exists (y_{01},\dots,y_{0s},y_1,\dots,y_m) \in Y(x) \quad \textit{with}$$
$$\sum_{i=1}^{m} r_i d_{T_i(x)}^2(\nabla f_i(x)\cdot\xi) + \xi\cdot\Big[\sum_{j=1}^{s} y_{0j}\nabla^2 f_{0j}(x) + \sum_{i=1}^{m} y_i\nabla^2 f_i(x)\Big]\xi \geq 0.$$

(b) (*Sufficient condition*). *If $Y(x) \neq \varnothing$ and*

$$(4.55)\qquad \forall \xi \in \Xi(x)\setminus\{0\},\ \exists (y_{01},\dots,y_{0s},y_1,\dots,y_m) \in Y(x) \quad \textit{with}$$
$$\sum_{i=1}^{m} r_i d_{T_i(x)}^2(\nabla f_i(x)\cdot\xi) + \xi\cdot\Big[\sum_{j=1}^{s} y_{0j}\nabla^2 f_{0j}(x) + \sum_{i=1}^{m} y_i\nabla^2 f_i(x)\Big]\xi > 0,$$

then x is a locally optimal solution to (P_2) in the strong sense.

Here, of course $d_{T_i(x)}(\omega_i)$ denotes the distance of ω_i from $T_i(x)$. Thus in accordance with (4.18) one has that if I_i is written as $[c_i^-, c_i^+]$ (with c_i^- or c_i^+ possibly infinite) then

$$(4.56)\qquad d_{T_i(x)}(\nabla f_i(x)\cdot\xi) = \begin{cases} 0 & \text{if } c_i^- < f_i(x) < c_i^+, \\ \max\{0, -\nabla f_i(x)\cdot\xi\} & \text{if } c_i^- = f_i(x) < c_i^+, \\ \max\{0, \nabla f_i(x)\cdot\xi\} & \text{if } c_i^- < f_i(x) = c_i^+, \\ |\nabla f_i(x)\cdot\xi| & \text{if } c_i^- = f_i(x) = c_i^+. \end{cases}$$

5. Refinements of the constraint qualification. In this section we address the questions raised by the presence in Theorem 4.2, for both the necessary and sufficient conditions for problem (P_1), of a constraint qualification generalizing the standard one of Mangasarian and Fromovitz. We demonstrate that by applying Theorem 4.2 to auxiliary problems related to (P_1), instead of to (P_1) itself, it is possible quickly to obtain results that fit the more customary pattern for this subject.

Before doing this, however, we wish to emphasize that Theorem 4.2, in departing from the customary pattern, should not merely be seen as a sort of stepping stone toward a subsequent goal, which will be reached here. If in the theory of optimality one adopts the philosophy that necessary and sufficient conditions should be tied as closely as possible to something like uniform local approximations of the essential objective (which incorporates the constraints), and much justification can be given for this, then Theorem 4.2 serves *better* than the traditional type of theorem by virtue of focusing on the circumstances where such approximations, as expressed by epi-differentiation do exist.

THEOREM 5.1. *The condition in (b) of Theorem 4.2 is sufficient for the strong local optimality of x in (P_1) even if the constraint qualification in that theorem is not satisfied at x.*

PROOF. Suppose condition (4.24) is satisfied. Our argument will be that in this case the corresponding sufficient condition (4.55) in Corollary 4.5 is satisfied for $r_i = 0$ and $q_i = q$ for some sufficiently high value of $q > 0$. Then x must be locally optimal for (P_2) in the strong sense, and the same must be true for (P_1) because the essential objective in (P_1) majorizes the one in (P_2) but agrees with it on the set of feasible solutions to (P_1).

First we establish that although the set $\Xi(x)$ in (4.52) is larger in general than the one in (4.21), the two sets agree when the q_i's are sufficiently high. Specifically: consider

$$(5.1) \qquad \varphi(\xi) = \max_{j \in J(x)} \nabla f_{0j}(x) \cdot \xi,$$

$$(5.2) \qquad \psi(\xi) = \sum_{i=1}^{m} d_{I_i(x)}(\nabla f_i(x) \cdot \xi),$$

$$(5.3) \qquad K = \{\xi \mid \psi(\xi) = 0\} = \{\xi \mid \nabla f_i(x) \cdot \xi \in T_i(x) \text{ for } i = 1, \ldots, m\}.$$

In (4.52) with $q_i = q > 0$ for all i one has

$$(5.4) \qquad \Xi(x) = \{\xi \in T_X(x) \mid \varphi(\xi) + q\psi(\xi) \leq 0\},$$

whereas in (4.21) one has

$$(5.5) \qquad \Xi(x) = \{\xi \in T_X(x) \mid \varphi(\xi) \leq 0 \text{ and } \xi \in K\}.$$

In the second case actually $\varphi(\xi) = 0$ for the vectors in question, because of our assumption that (4.24) holds; this follows from the nonemptiness of the multiplier set in (4.22). Thus our hypothesis implies that the minimum value of φ over $T_X(x) \cap K$ is 0, attained precisely at the elements in the set (5.5). Observe now that φ is globally Lipschitz continuous with a certain modulus λ (it suffices actually to take λ equal to the largest of the norms $|\nabla f_{0j}(x)|$ for $j \in J(x)$), and that

$$\psi(\xi) > \epsilon d_K(\xi) \quad \text{for some } \epsilon > 0.$$

We claim that when $q > \lambda/\epsilon$, the set in (5.4) is no larger than the one in (5.5), for if it were there would be a vector $\xi \in T_X(x) \setminus K$ with

$$0 \geq \varphi(\xi) + q\psi(\xi) > \varphi(\xi) + \lambda d_K(\xi).$$

Then for the (Euclidean) projection ξ' of ξ on $T_X(x) \cap K$ we would have $D_K(\xi) = |\xi' - \xi|$, $\varphi(\xi') > 0$, so that

$$\varphi(\xi') > \varphi(\xi) + \lambda|\xi' - \xi|$$

in contradiction to the specification of λ as a Lipschitz modulus for φ.

The next step in our argument is this: if q is sufficiently high (beyond the value already described), it is possible in our assumed condition (4.24) always (no matter which ξ is given) to choose the multipliers in such a way that

$$(5.6) \qquad |y_i| \leq q \text{ for } i = 1, \ldots, m.$$

Compactness here does the trick. We need only consider the vectors ξ in the closed, bounded set

$$(5.7) \qquad \{\xi \in \Xi(x) \mid |\xi| = 1\},$$

in view of positive homogeneity. For each ξ in this set there is a corresponding vector in the multiplier set (4.22) such that (4.24) holds—and this multiplier vector then works also for some neighborhood of ξ. Since the set (5.7) can be covered by finitely many such neighborhoods, only finitely many of the multiplier vectors in (4.22) are needed in obtaining (4.24). We need only choose q high enough that (5.6) holds for these finitely many vectors, which of course is always possible.

All that remains is to observe that with the value of q as high as we have described, it is possible to conclude that not only (4.24) but the seemingly stronger condition (4.55) in the sufficiency part of Corollary 4.5 is satisfied (with $q_i = q$, $r_i = 0$). This is all we had to verify in carrying out the proof. ■

THEOREM 5.2. *Suppose in Theorem 4.2 that the multiplier set $Y(x)$ is replaced by $\tilde{Y}(x)$, which is the same except that the set*

$$\tilde{S}(x) = \{(y_{01}, \ldots, y_{0s}) \mid y_{0j} \geq 0 \text{ for } j \in J(x),\ y_{0j} = 0 \text{ for } j \notin J(x)\}$$

is substituted for $S(x)$ (i.e. the requirement that $\sum_{j=1}^{s} y_{0j} = 1$ is dropped), and the zero multiplier vector is excluded as an element. Then with this modification the condition in (a) of Theorem 4.2 is necessary for the local optimality of x in (P_1) even if the constraint qualification in that theorem is not satisfied at x.

PROOF. In proving this we follow a well-traveled route. Fixing the x in question we examine an auxiliary problem in the variable vector $\tilde{x}$, which has $\tilde{x} = x$ as a locally optimal solution. We apply Theorem 4.2 to this problem and obtain the result.

There is no real loss of generality, but a considerable gain in notational simplicity, in supposing for the purpose at hand that

$$(5.8) \qquad I_i = \begin{cases} (-\infty, 0] & \text{for } i = 1, \ldots, r, \\ [0, 0] & \text{for } i = r+1, \ldots, m \end{cases}$$

(so that $f_i(\tilde{x}) \in I_i$ represents the constraint $f_i(\tilde{x}) \leq 0$ for $i = 1, \ldots, r$, but $f_i(\tilde{x}) = 0$ for $i = r+1, \ldots, m$). Define the value

$$(5.9) \qquad \alpha = f_0(x) = f_{0j}(x) \quad \text{for all } j \in J(x),$$

and the function

$$(5.10) \qquad \tilde{f}_0(\tilde{x}) = \max\{f_{01}(\tilde{x}) - \alpha, \ldots, f_{0s}(\tilde{x}) - \alpha, f_1(\tilde{x}), \ldots, f_r(\tilde{x})\}.$$

The auxiliary problem we consider is

$(\tilde{P}_1)$ minimize $\tilde{f}_0(\tilde{x})$ over all $\tilde{x} \in X$ satisfying $f_i(\tilde{x}) = 0$ for $i = r+1, \ldots, m$.

The given x is obviously locally optimal in $(\tilde{P}_1)$ just as it was in (P_1). We divide our analysis now into three cases.

Case 1. The constraint qualification of Theorem 4.2 is satisfied for $(\tilde{P}_1)$ at $\tilde{x} = x$; in other words there is no vector $(y_{r+1}, \ldots, y_m) \neq (0, \ldots, 0)$ such that

$$(5.11) \qquad \sum_{i=r+1}^{m} y_i \nabla f_i(x) \in N_X(x).$$

Theorem 4.2 is then applicable to $(\tilde{P}_1)$, and the multiplier vectors it yields can be recognized at once as elements of $\tilde{Y}_0(x)$, thereby furnishing the desired conclusion.

Case 2. The constraint qualification of Theorem 4.2 for $(\tilde{P}_1)$ at $\tilde{x} = x$ fails in the strong sense that a vector $(y_{r+1}, \ldots, y_m) \neq (0, \ldots, 0)$ exists for which not only (5.11) holds but also

$$-\sum_{i=r+1}^{m} \bar{y}_i \nabla f_i(x) \in N_X(x). \tag{5.12}$$

(Note that this is the only alternative if $X = \mathbf{R}^n$, since then $N_X(x) = \{0\}$, or if $x \in \operatorname{int} X$ or even merely if $x \in \operatorname{ri} X$, since then $N_X(x)$ is a subspace of $\mathbf{R}^n$.) In this situation our generalized necessary condition is trivial: the set $\bar{Y}(x)$ contains the vector $(0,\dots,0,\bar{y}_{r+1},\dots,\bar{y}_m)$ and also its negative; for each $\xi \in \Xi(x)$ the inequality in (4.23) must be satisfied for one or the other of these.

Case 3 (which comes into play only in dealing with a nontrivial polyhedron X). Neither of the circumstances designated as Case 1 or Case 2 is present. We shall demonstrate that this can be taken care of by a modification of $(\tilde{P}_1)$ that puts us back into Case 1, in effect. For this purpose we introduce a local representation of X around x by constraints

$$(5.13)\qquad \begin{aligned} l_k(\tilde{x}) = a_k \cdot \tilde{x} - a_k \cdot x &\le 0 \quad \text{for } k = 1,\dots,p,\\ &= 0 \quad \text{for } k = p+1,\dots,q. \end{aligned}$$

This can be done in such a manner that

$$(5.14)\qquad N_X(x) = \left\{ \sum_{k=1}^{q} \lambda_k a_k \,\middle|\, \lambda_k \in \mathbf{R}_+ \text{ for } k = 1,\dots,p; \ \lambda_k \in \mathbf{R} \text{ for } k = p+1,\dots,q \right\}$$

and

$$(5.15)\qquad \text{no vector } (\lambda_1,\dots,\lambda_q) \neq (0,\dots,0) \text{ with } \lambda_k \ge 0 \text{ for } k = 1,\dots,p, \text{ yields } \sum_{k=1}^{q} \lambda_k a_k = 0.$$

(This is seen by taking L to be the subspace $N_X(x) \cap [-N_X(x)]$ and K to be the *pointed* cone $N_X(x) \cap L^{\perp}$. Select $a_1,\dots,a_p$ to be nonzero vectors that generate K and then $a_{p+1},\dots,a_q$ to be a basis for L.) The vectors

$$\nabla f_{r+1}(x),\dots,\nabla f_m(x),\ a_{p+1},\dots,a_q \tag{5.16}$$

must be linearly independent, for otherwise we would still be in Case 2: if

$$\sum_{i=r+1}^{m} \bar{y}_i \nabla f_i(x) + \sum_{k=p+1}^{q} \lambda_k a_k = 0$$

with coefficients that are not all 0, then $(\bar{y}_{r+1},\dots,\bar{y}_m) \neq (0,\dots,0)$ because of (5.15), so that (5.11) and (5.12) would both be satisfied in view of (5.14). Consider now the function

$$\hat{f}_0(x) = \max\{\tilde{f}_0(x), l_1(x),\dots,l_p(x)\}$$

and the auxiliary problem

$$(\hat{P}_1)\qquad \begin{aligned} &\text{minimize} \quad \hat{f}_0(\tilde{x}) \text{ over all } \tilde{x} \text{ satisfying}\\ &f_i(\tilde{x}) = 0 \quad \text{for } i = r+1,\dots,m, \quad \text{and}\\ &l_k(\tilde{x}) = 0 \quad \text{for } k = p+1,\dots,q. \end{aligned}$$

As with $(\tilde{P}_1)$, the given x is a locally optimal solution to $(\hat{P}_1)$. Moreover $(\hat{P}_1)$ satisfies the constraint qualification of Theorem 4.2 at $\tilde{x} = x$ by virtue of the linear independence of the vectors in (5.16), the a_k's being the gradients of the functions l_k in (5.13). The necessary condition of Theorem 4.2 for $(\hat{P}_1)$ then gives us what we want, just as in Case 1, by way of (5.14) and (5.15). ∎

The necessary condition provided by Theorem 5.2 fits with a long tradition of research on this topic. The somewhat novel features are the incorporation of the abstract constraint $x \in X$ and the admission of f_0 as a "max function". The traditional setting, where $X = \mathbf{R}^n$ and f_0 is of class $\mathscr{C}^2$, has been treated by many authors. The first published result appears to have been that of Cox [15] in 1944, as referenced by Hestenes [17, Chapter 6, Theorem 10.4]. This is almost the same as what Theorem 5.2 yields in the traditional setting, but in certain situations where active inequality constraints might not be associated with nonzero multipliers the set of direction vectors ξ used by Cox might be smaller than the $\Xi(x)$ in (4.21). The full version without this discrepancy was stated by Ioffe [20] in 1979 and attributed to a 1974 paper of Levitin, Miljutin and Osmolovskii [21] that appears in a book in Russian. (These authors assume that the gradients of the equality constraints are linearly independent, not realizing, it seems, that if this is not true the desired necessary condition holds trivially; cf. Case 2 of the preceding proof.) The result can be seen also in a 1980 paper of Ben-Tal [1], based independently on his unpublished 1975 report [2].

Of course the story of second-order necessary conditions does not end there, because many generalizations have been made to infinite-dimensional spaces, infinitely many constraints, other degrees of nonsmoothness, etc. Our purpose in presenting Theorem 5.2 has not been to push in those directions, however, but to confirm that for the classes of problems most frequently seen in computation and mathematical modeling, the "neoclassical" approach of using differential approximations to the epigraph of the essential objective leads to results as sharp or sharper than any that are known.

It would be possible also to demonstrate that the constraint qualification in Theorem 4.2 (or for that matter the one in Theorem 3.4) can be weakened to a "calmness" condition such as used by Clarke [14] and Rockafellar [26]. The pattern of development would be to show that the calmness condition supports a local l_1 penalty representation of the constraints around the given point x, to which Theorem 4.4 can be applied. The details will not be given here, however.

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