Approximation Algorithms for Single and Multi-Commodity Connected Facility Location

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Alexander von Humboldt Stiftung / Foundation

Outline

1. Better approximation algorithm
for CONNECTED FACILITY LOCATION

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2. First $O(1)$-apx for Multi-Commodity Connected Facility Location

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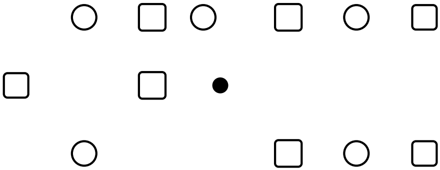
2. First $O(1)$-apx for Multi-Commodity Connected Facility Location
3. Improved hardness results for several problems

Part 1: Connected Facility Location

Given:

- graph $G = (V, E)$, with metric distances $c : E \to \mathbb{Q}^+$



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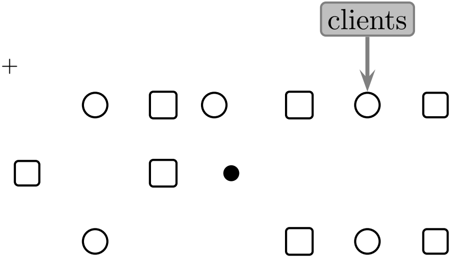

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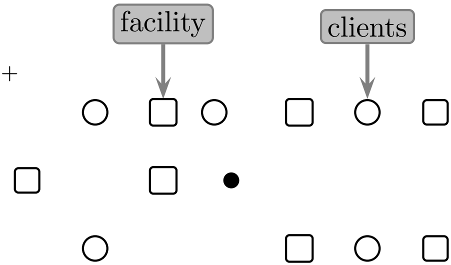

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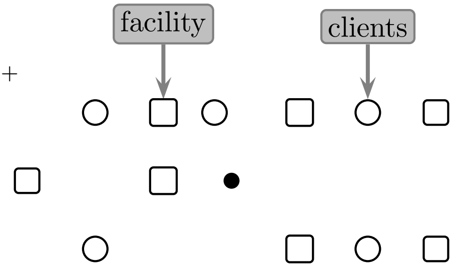

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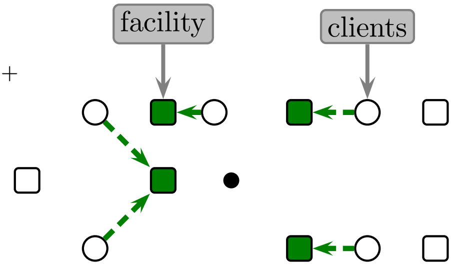

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- open facilities $F' \subseteq F$

minimizing

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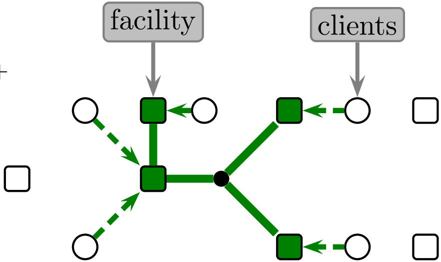

Goal:

- open facilities $F' \subseteq F$
- find Steiner tree T spanning opened facilities

minimizing

$$\underbrace{\sum_{f \in F'} o(f)}_{\text{opening cost}} + \underbrace{\sum_{j \in C} c(j, F')}_{\text{connection cost}} \quad + \quad \underbrace{M \sum_{e \in T} c(e)}_{\text{Steiner cost}}$$

Previous Results for Connected Facility Location

- **APX**-hard (reduction from Steiner Tree)
- 10.66-apx based on LP-rounding [Gupta et al '01].
- 8.55-apx primal-dual algorithm [Swamy, Kumar '02].
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Use existing algorithms for subproblems:

- Facility Location: 1.5-apx [Byrka '07]
- Steiner Tree: 1.39-apx [Byrka,Grandoni,R.,Sanità 10]

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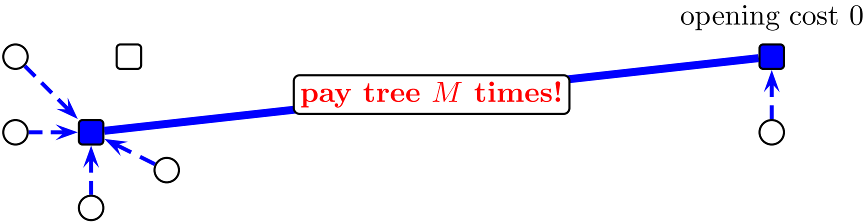

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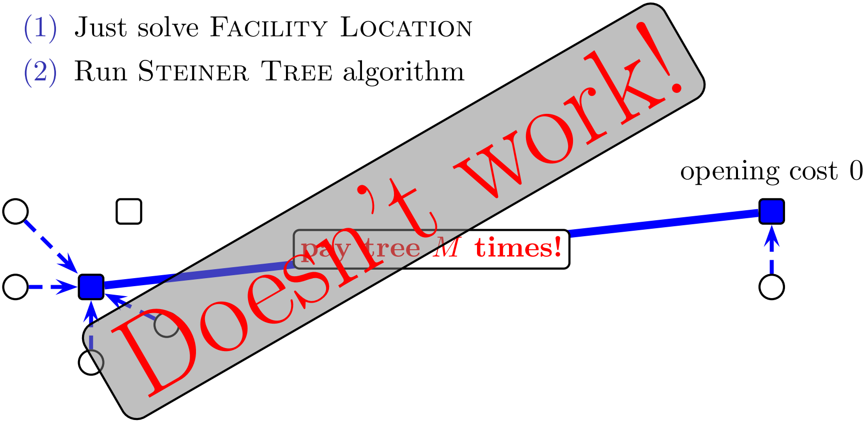
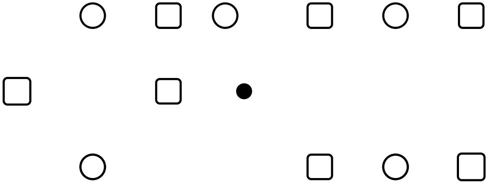


The CFL algorithm



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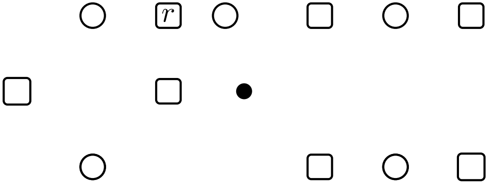

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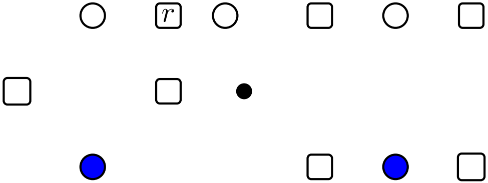

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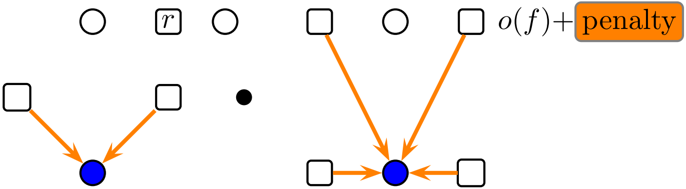

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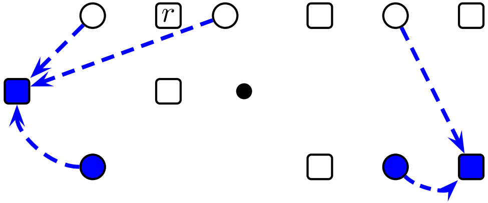

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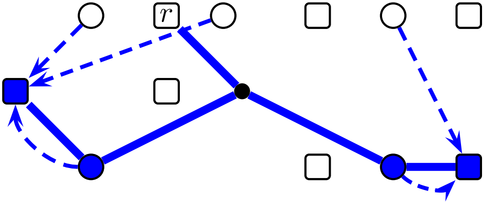

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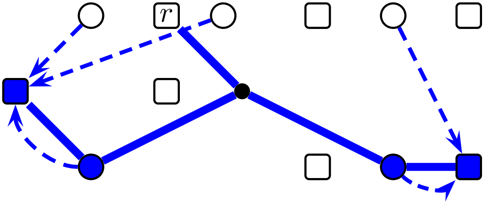

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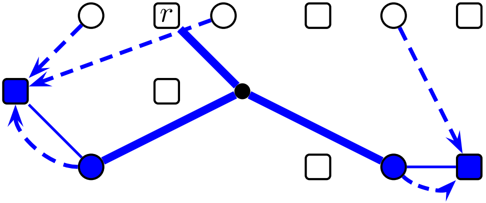

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$$OPT = \underbrace{O^*}_{\text{opening cost}} + \underbrace{S^*}_{\text{Steiner tree cost}} + \underbrace{C^*}_{\text{connection cost}}$$

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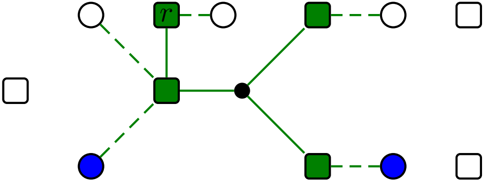

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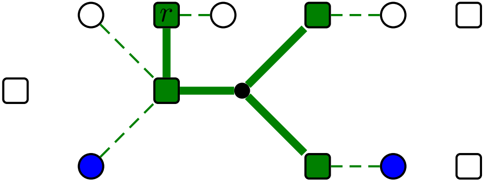

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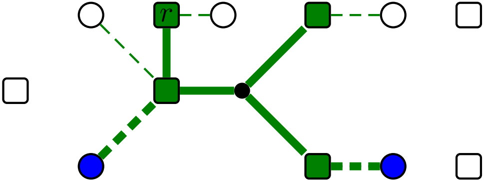

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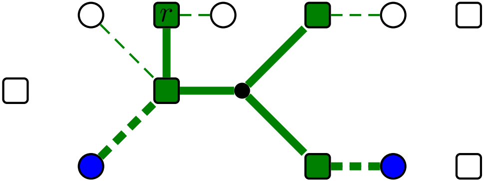

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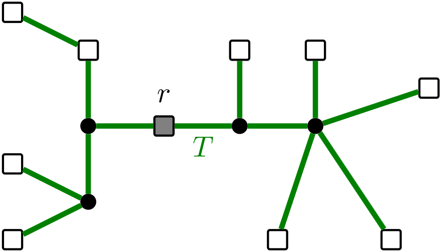
Core Detouring Theorem [EGSR '08]

Given a spanning tree T with root $r \in T$ and distinguished terminals $D \subseteq V$. Sample any terminal in D with prob. $p \in]0,1]$

$$E\Big[\sum_{v \in D} \mathrm{dist}(v, \text{sampled node} \cup \{r\})\Big] \leq \frac{0.81}{p} \cdot c(T)$$



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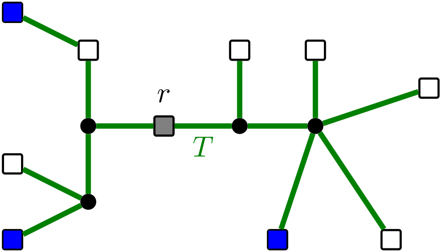

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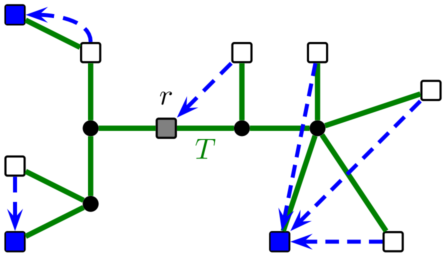

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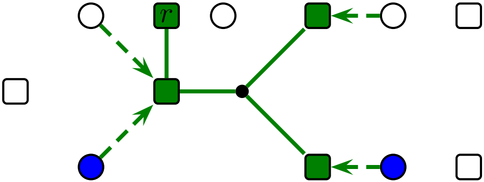
Theorem

$E[apx \text{ Facility Location } cost] \leq 1.5 \cdot (O^* + 2C^* + 0.81S^*)$



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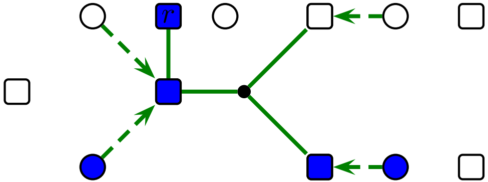

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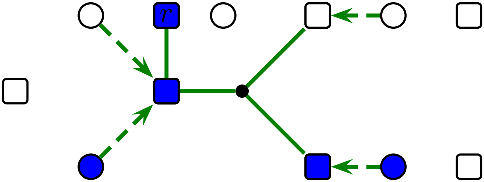

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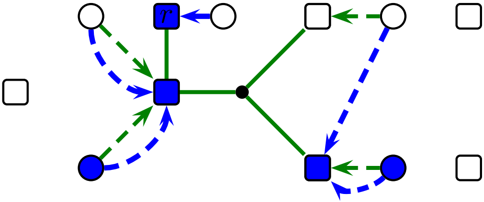
- $E[\text{connection cost}] \leq C^* +$



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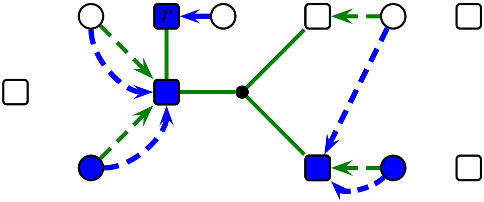
- $E[\text{connection cost}] \leq C^* + \frac{0.81}{1/M} \cdot \underbrace{\frac{S^*}{M}}_{\text{tree}}$

Use **Core Detouring Theorem** with $T = S^*$ and $p := \frac{1}{M}$.



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Conclusion:

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$$\begin{aligned} E[APX] &\leq 1.39 \cdot (S^* + C^*) + 1.5 \cdot (O^* + 2C^* + 0.81S^*) \\ &\leq 1.5 \cdot O^* + 2.7 \cdot S^* + 4.39 \cdot C^* \end{aligned}$$

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Improvement to 3.19:

- Adapting the sampling probability
- Using a bi-factor Facility Location algorithm

Part 2: Multi-Commodity Connected Facility Location

Input:

- Undirected graph $G = (V, E)$, metric distances $c : E \to \mathbb{Q}^+$
- source-sink pairs $(s_1, r_1), \ldots, (s_k, r_k)$
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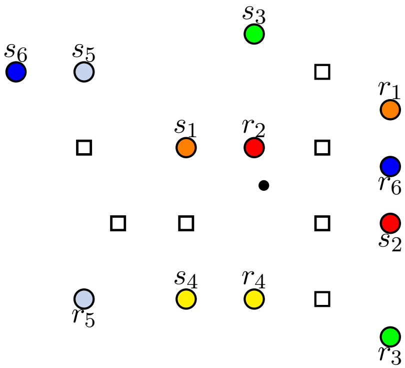

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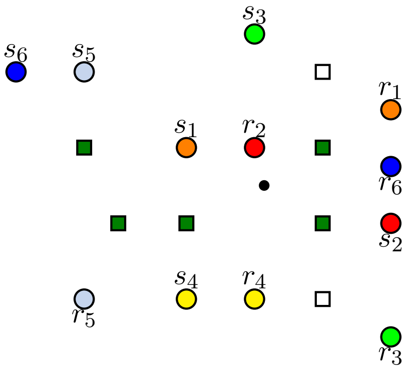

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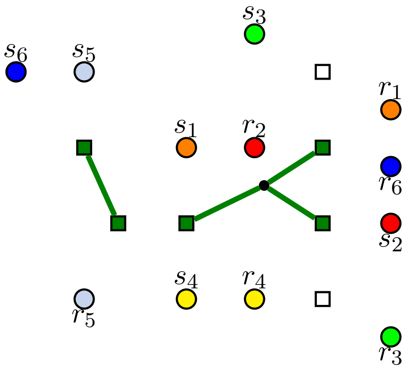

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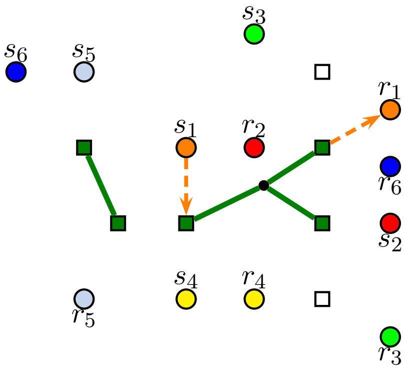

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$$\underbrace{\sum_{f \in F'} o(f)}_{\text{opening cost}} + \underbrace{M \cdot c(T)}_{\text{forest cost}} + \sum_{i=1}^{k} \underbrace{\mathrm{dist}_{F',T}(s_i, r_i)}_{\substack{\text{0 distance between}\\ \text{opened \& connected}\\ \text{facilities}}}$$

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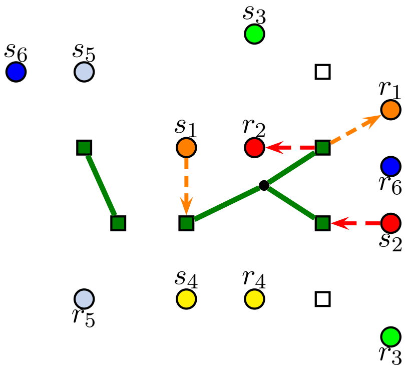

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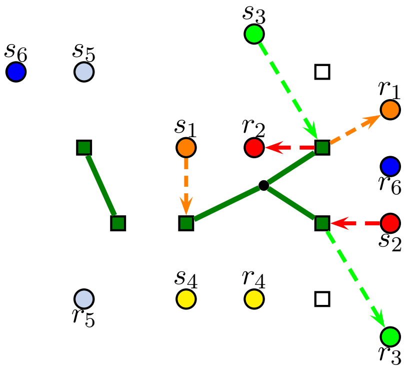
Part 2: Multi-Commodity Connected Facility Location

Input:

- Undirected graph $G = (V, E)$, metric distances $c : E \to \mathbb{Q}^+$
- source-sink pairs $(s_1, r_1), \ldots, (s_k, r_k)$
- a set of facilities $F \subseteq V$ with opening costs $o : F \to \mathbb{Q}^+$
- parameter $M \geq 1$



Goal: Find

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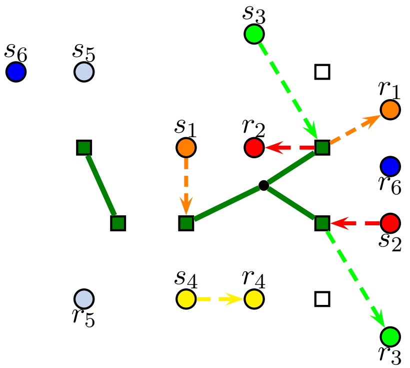

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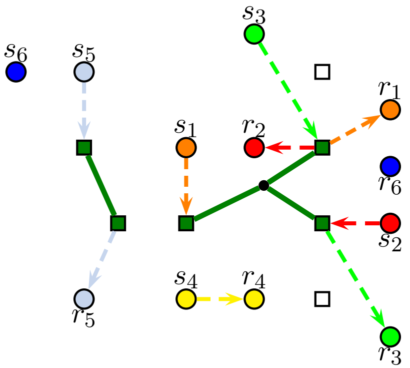

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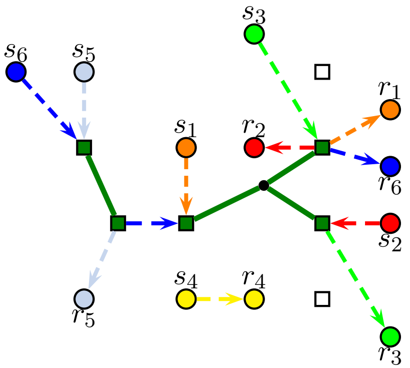

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Our result

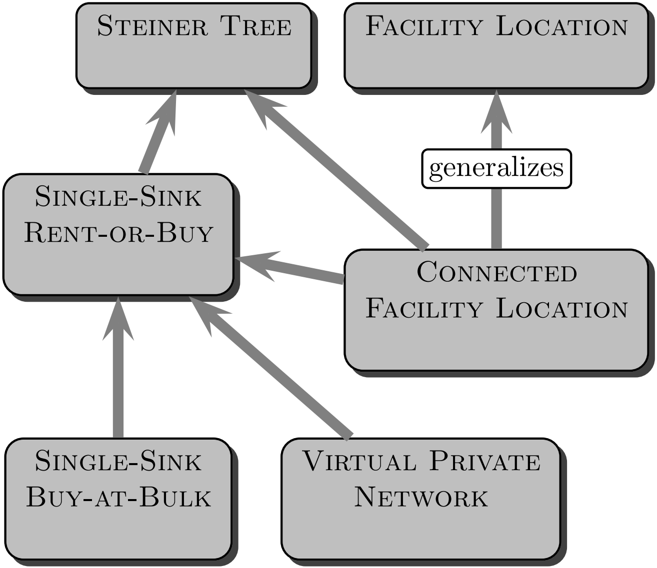
Our result

Simple 16.2-approximation algorithm for Multi-Commodity Connected Facility Location.

Ingredients:

- Random-sampling
- Use algorithms for
  - Price-Collecting Facility Location
  - Steiner Forest

Part 3: Improved hardness results



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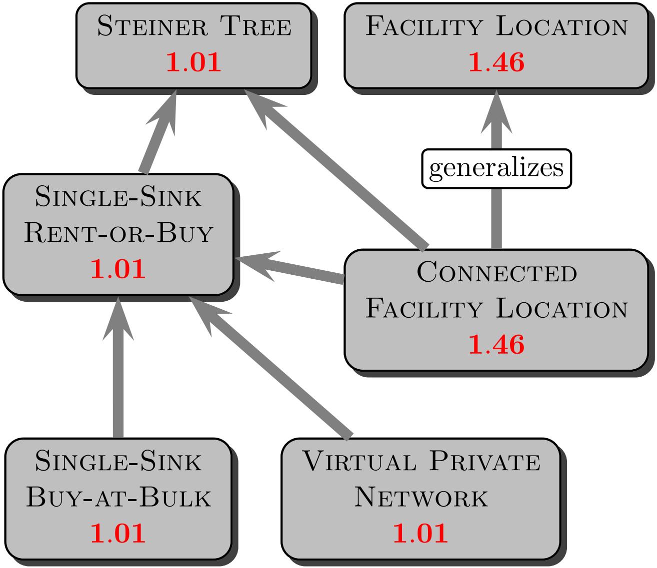

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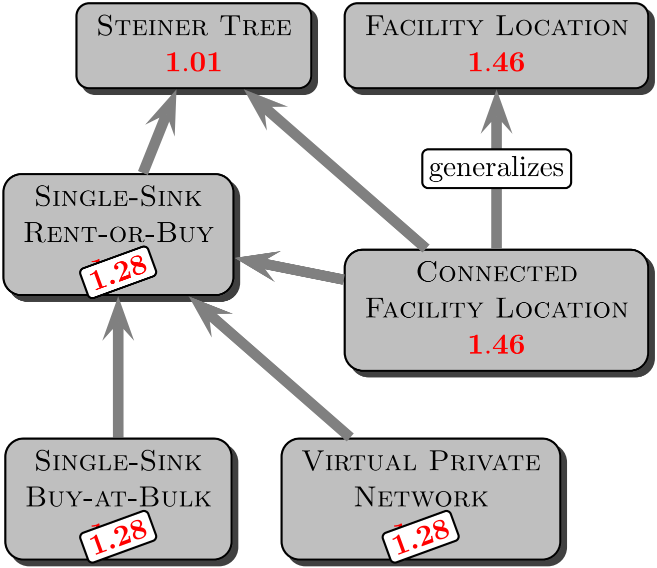

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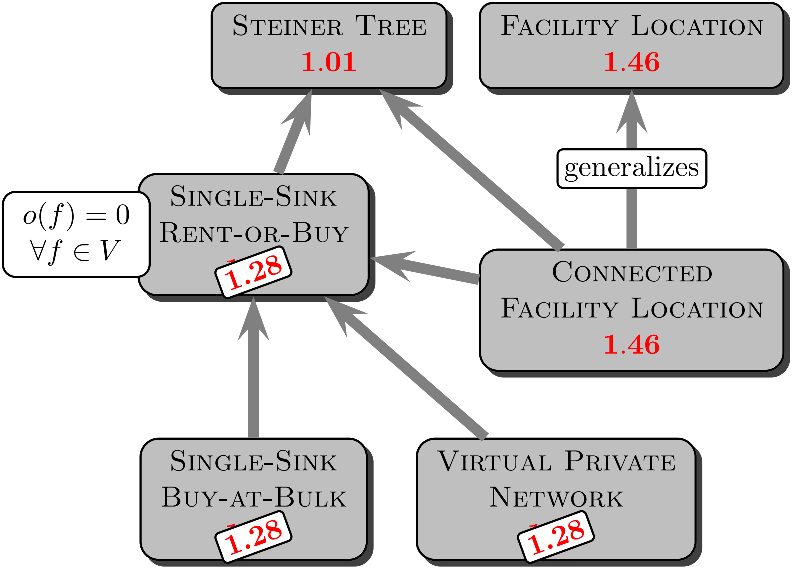
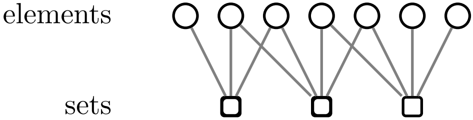


The reduction (1)

- Reduce Set Cover to SROB

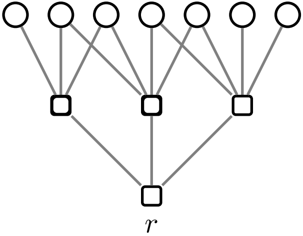


The reduction (1)

- Reduce Set Cover to SROB

clients/elements

facilities/sets



$c(e) \in \{1, 2\}$

$M = \dfrac{0.27n}{OPT_{SC}}$

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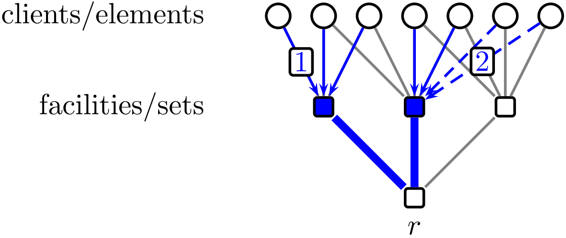

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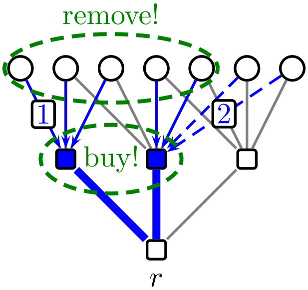


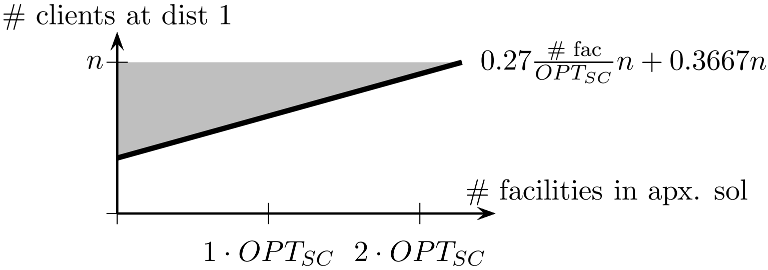
$c(e) \in \{1, 2\}$

$$M = \frac{0.27n}{OPT_{SC}}$$

(1) WHILE not all elements covered DO
   (2) Compute 1.27-apx SROB sol
   (3) Buy facilities/sets in sol. & remove covered elements

The reduction (2)

- Use idea from [Guha & Khuller '99]



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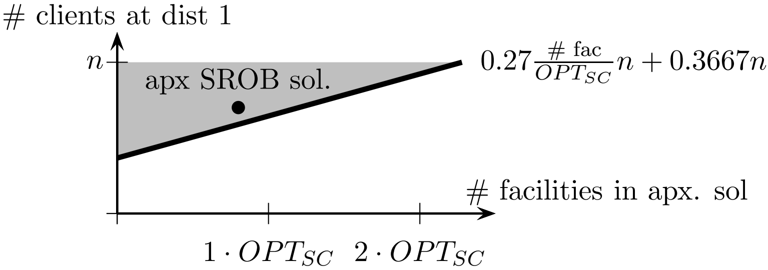

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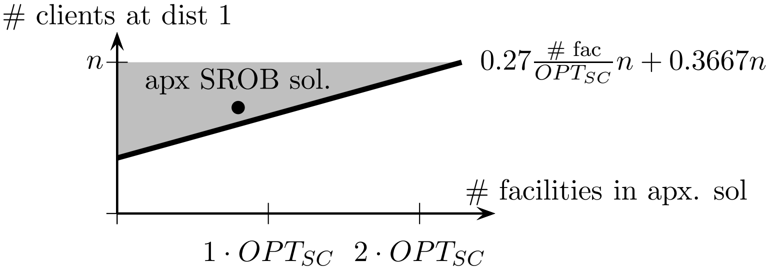

$$\#\text{ needed sets} \quad \leq \quad [\ldots\text{some calc}\ldots] \leq 0.999\ln(n)\cdot OPT_{SC}$$

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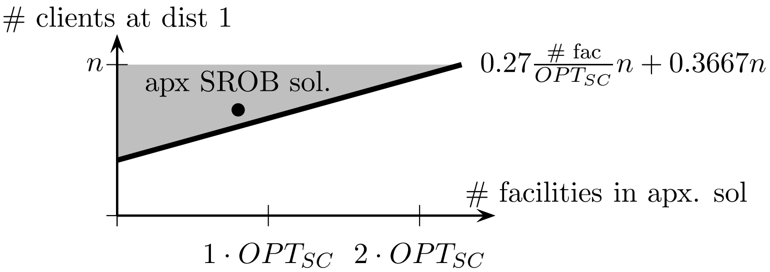

$$\#\text{ needed sets} \quad \leq \quad [\ldots\text{some calc}\ldots] \leq 0.999 \ln(n) \cdot OPT_{SC}$$

- Contradiction!

Theorem (Feige '98)

Unless $\mathbf{NP} \subseteq \mathbf{DTIME}(n^{O(\log\log n)})$, *there is no* $(1-\varepsilon)\cdot\ln(n)$*-apx for* Set Cover.

The end

Thanks for your attention